Chapter - I

INTRODUCTION
1.1 Theory of General Relativity

General relativity is the theory of space, time and gravitation formulated by Einstein in 1915. It is often regarded as a very hard to understand and difficult theory, because of the mathematics required for a precise formulation of the ideas and equations of general relativity. Although it has been universally acknowledged as being a beautiful theory, the potential relevance of general relativity to the rest of physics has not been universally acknowledged and, indeed, probably for this reason, the subject was lain nearly dormant during much of its history.

Einstein arrived at a novel concept which says that gravitation has a basic relationship with the space-time in which it is always present. The theory of general relativity (GR) is a more accurate and comprehensive description of gravitation than the prevailing Newtonian gravitation theory. This theory yields the results to a great degree of accuracy in comparison with experimental results. It is based on Riemannian metric tensor $g_{\mu\nu}$ which describes not only the gravitational field but also the geometry.

There are some non-satisfactory features of the theory and the attempts are made [Kaluza-Klein (1921), Dirac (1938, 73), Rosen (1940, 73, 74, 75, 77), Brans-Dicke (1961), Bergmann-Wagoner (1968, 70), Callan et al. (1970), Nordtvedt (1970), Kobus (1971), Hoyle and Narlikar (1971), Sen-Dunn (1971), Ross (1972), Dunn (1974), Canuto V. et al. (1977), Schmidt et al. (1981), Barber (1982), Wesson (1983), Seaz-Ballester (1985), Logunov and Mestvirshvili (1989), Mottat J. W. (1997) etc.] to establish new theories to generalize the Einstein's theory and to incorporate some overlooks and lapses in the general theory of gravitation. But Einstein theory of gravitation has great formal beauty and mathematical elegance and is found to lead to a
complete theory of gravitational action. In the development of general relativity, Einstein was mainly guided by three general principles: principle of covariance, principle of equivalence and Mach's principle. The importance of Mach's principle is that it can be used to determine the geometry of space-time and thereby the inertial properties of a test particle from the information of density and mass energy distribution in its neighbourhood.

The field equations of general relativity are

\[ G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij} \]  \hspace{1cm} (1.1.1)

where \( G_{ij} \), \( R_{ij} \), \( R \), and \( T_{ij} \) are Einstein tensor, Ricci tensor, curvature invariant and energy momentum tensor respectively describing the matter content of the space-time.

The principle of covariance

The principle of covariance states that the laws of physics must be independent of space-time coordinates, i.e. The laws of nature must retain their original form in all coordinates system.

According to this principle we must express all the physical laws of nature by means of equations in the covariant form, which are independent of coordinate systems. This can be done by expressing the laws of nature in the form of tensor equations, because the tensor equation has exactly the same form in all coordinate system. Therefore, the laws of physics must be expressed in tensorial equations.
e.g. consider the laws of nature in system of variables $x$ is represented by the tensor equation

$$A^i_\nu = B^i_\nu \quad (1.1.2)$$

Then this law when transformed to new system of variables $\bar{x}$ by

$$A^i_\nu = B^i_\nu \equiv \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^j}{\partial \bar{x}^\nu} A^j_\nu - \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^j}{\partial \bar{x}^\nu} B^j_\nu$$

$$= (A^i_\nu - B^i_\nu) \frac{\partial x^j}{\partial \bar{x}^\nu} \frac{\partial \bar{x}^\nu}{\partial \bar{x}^\nu}$$

$$= 0$$

$$\bar{A}^i_\nu = \bar{B}^i_\nu \quad (1.1.3)$$

Hence (1.1.3) has exactly the same nature forming equation (1.1.2). Thus the laws of nature when expressed in the form of tensor equations follow the general covariance principle.

Another implication of covariance principle:

Space-time geometry of special relativity is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + c^2 dt^2 \quad (1.1.4)$$

where $x, y, z$ are spatial cartesian coordinates and $t$ is the time coordinate.

The line element (1.1.4) is not invariant under the general coordinate transformation (it is invariant under Lorentz's transformation).
The generalization of equation (1.1.4) is

\[ ds^2 = g_{ij} \, dx^i \, dx^j \quad (i, j = 1, 2, 3, 4) \] (1.1.5)

where the metric \( g_{ij} \) is symmetric and depends on coordinates.

This form (1.1.5) is invariant under general transformation i.e.

\[ \overline{g}_{ij} = \frac{\partial x^u}{\partial \bar{x}^i} \frac{\partial x^v}{\partial \bar{x}^j} \, g_{uv} \]

This is first implication of covariance principle.

Similarly a covariant expression for the trajectory of a free particle is provided by geodesic equations

\[ \left( \frac{dx^i}{ds} \right)_{;j} = 0 \]

i.e.

\[ \frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \quad (i, j, k = 1, 2, 3, 4) \]

where \( \frac{dx^i}{ds} \) is a four velocity and semicolon (\( ; \)) stands for covariant derivative.

This is second implication of the covariance principle.
The principle of equivalence

The principle of equivalence states that the fundamental tensor $g_{ij}$ can be chosen to account for the presence of a gravitational field, i.e. $g_{ij}$ depends on the distribution of matter and the energy in physical space.

According to Newton's theory of gravitation the inertial mass and gravitational mass of the body are equal.

Consider two systems $x$ and $x'$, the latter moving with uniform acceleration $a_0$ relative to the former.

If we consider the first frame $x$ to be inertial, the second frame $x'$ will be non-inertial and vice-versa. Consider a body of inertial mass $m$ not acted by any force is at rest in the inertial system $x$. Then there is no force acting on the body relative to system $x$, but if observed from $x'$, the body appears to experience a force $-ma_0$. Obviously the force $-ma_0$ acting on the body is not real and arises as a result of the acceleration of the non-inertial frame. Such a force which appears only due to acceleration of the non-inertial frame is called the inertial force.

The acceleration which do not correspond to real forces acting on the body, but gives rise to inertial forces are called inertial acceleration when the non-inertial frame rotates uniformly about an axis fixed with respect to the inertial frame, the inertial acceleration is the centrifugal acceleration. Thus in accelerated system we always get inertial force. In order to introduce the effects of gravitational action, Einstein pointed out that the inertial acceleration is similar to gravitational acceleration.
For example a person in an elevator feels to become momentarily heavier when the elevator is accelerated upward and lighter when the elevator is accelerated downward. Einstein pointed out that a gravitational field can be produced by accelerating uniformly an inertial frame of reference. By this analogy Einstein gave the principle of equivalence which states that: "In the neighbourhood of any given point it is not possible to distinguish between the gravitational field produced by the attraction of masses and the field produced by the accelerating of inertial frame of reference". Thus we learn from this principle that the two fields, a gravitational field produced by the attraction of the matter in the universe and other field produced by the accelerating uniformly an inertial frame outside all gravitational field, are identical provided the acceleration of the inertial frame is equal and opposite to the actual gravitational acceleration.

Obviously according to the principle of equivalence the gravitational and the inertial forces are of the same nature and obey the same laws.

The principle of equivalences can be classified into

(i) the strong principle of equivalence,

(ii) the weak principle of equivalence.

The strong principle of equivalence is the principle of equivalence just described above when laws of nature mean all laws of nature.
This principle can also be divided into

(i) a very strong principle of equivalence and

(ii) a medium strong principle of equivalence

The weak principle of equivalence replaces the laws of nature by the laws of motion of freely falling particles i.e. it is a equality of inertial and gravitational mass of closed system.

1.2. Bimetric theory of relativity

Though the general theory of relativity is one of the most beautiful structure in all of theoretical physics, it gives a description of gravitation of phenomena in agreement with observation, and it also provides a conceptual framework for all large scale phenomena. In order to get rid of the singularity problems in general relativity Professor Rosen (1940, 73, 77) proposed a modification in the general relativity which is widely known as Bimetric Relativity (BR).

Rosen (1940) proposed, in the framework of general theory of relativity, two metric tensors viz, $g_{ij}$ the usual Riemannian metric associated with the line element

$$ds^2 = g_{ij} \, dx^i \, dx^j$$  \hspace{1cm} (1.2.1)
and \( f_{ij} \), a metric tensor describing a flat space-time, associated with the line element

\[
\mathrm{d}\sigma^2 = f_{ij} \mathrm{d}x^i \mathrm{d}x^j
\]  

(1.2.2)

at each point of the space-time.

The tensor \( g_{ij} \) describes gravitation and interacts with matter. The background metric \( f_{ij} \) has no direct physical significance but appears in the field equations. Therefore it interacts with \( g_{ij} \) but not directly with matter. One can regard \( f_{ij} \) as giving the geometry that would exist if there were no matter. With the help of \( g_{ij} \) one can define the Christoffel symbols \( \Gamma^p_{ij} \) and hence covariant differentiation (i.e. \( g \)-differentiation), denoted by a semicolon. Then one forms the curvature tensor

\[
R^p_{ij} = \left\{ \frac{s}{i_j} \right\}_p + \left\{ \frac{s}{i_p} \right\}_j + \left\{ \frac{h}{i_j} \right\}_p - \left\{ \frac{s}{h_p} \right\}_i, \quad (1.2.3)
\]

where comma (, ) stands for partial differentiation.

Similarly from \( f_{ij} \) one can define the Christoffel symbols \( \Gamma^p_{ij} \) and corresponding covariant differentiation (i.e. \( f \)-differentiation) denoted by a vertical bar (|). One can then form the curvature tensor \( P^p_{ij} \) for the line element (1.2.2) such that

\[
P^p_{ij} = -\Gamma^p_{ij} + \Gamma^p_{i|j} + \Gamma^h_{ij} \Gamma^h_{p|} - \Gamma^h_{ip} \Gamma^h_{j} . \quad (1.2.4)
\]
Defining

\[ \Delta_{ij}^p = \frac{1}{2} g^{pq} (g_{iq,j} + g_{jq,i} - g_{ij,q}) , \]  

we can show that

\[ \Delta_{ij}^p = \left\{ \begin{array}{c} p \\ i, j \end{array} \right\} - \Gamma_{ij}^p \]  

But now \( \Delta_{ij}^p \) is a third rank mixed tensor having the same form as

\[ \left\{ \begin{array}{c} p \\ i, j \end{array} \right\} = \frac{1}{2} g^{pq} (g_{iq,j} + g_{jq,i} - g_{ij,q}) , \]

with the ordinary derivatives replaced by \( f \)-derivatives.

Another interesting relation has the form

\[ K_{up}^s = R_{up}^s - P_{up}^s , \]

where

\[ K_{up}^s = -\Delta_{ij}^s - \Delta_{up}^s + \Delta_{up}^s \Delta_{ij}^s - \Delta_{ij}^s \Delta_{ur}^s \Delta_{uj}^s \]

which has the same form as \( R_{up}^s \) in (1.2.3) but \( \Gamma \) replaced by \( \Delta \) and partial derivatives by \( f \)-derivatives. On contracting it gives

\[ K_{ij} = R_{ij} - P_{ij} , \]

where \( R_{ij} \) is the Ricci tensor for the line element (1.2.1) and \( P_{ij} \) is the Ricci tensor for the background metric \( f_{ij} \) in (1.2.2).
Here

\[ K_y = -\Delta^a_{y,;a} + \Delta^a_{al,;a} - \Delta^a_{ab} \Lambda^b_{y} + \Delta^a_{bi} \Lambda^b_{ai} . \]  

It thus appears that one can rewrite all the quantities occurring in GR so that \( \{ \} \) is to be replaced by \( \Delta \), partial derivative by \( f \)-covariant derivative and \((-g)^{1/2} \) by \( \kappa = (g/f)^{1/2} \),

where

\[ g = \text{det}(g_y), \quad f = \text{det}(f_y). \]

The advantage of this formalism is that it imparts tensor character to the quantities which in the usual form of the theory they do not have. The best example is that of a pseudo-energy tensor of gravitational field in GR — it becomes a gravitational energy - momentum density tensor in the new formalism.

If \( f_y \) describes a flat space-time, then

\[ P^s_{\alpha\beta} = 0 . \]

It follows that

\[ K^s_{\alpha\beta} = R^s_{\alpha\beta} \]

or

\[ K_y = R_y. \]
Then the Einstein field equations

\[ G^g_y = R^g_y = -\frac{1}{2} R g^g_y = -8\pi T^g_y \]  \hspace{1cm} (1.2.12)

can be put in the form

\[ K^g_y = \frac{1}{2} K g^g_y = -8\pi T^g_y \]  \hspace{1cm} (1.2.13)

where the symbols have their conventional meanings.

In this way GR can be converted into BR [See Rosen (1963)].

Employing the variation principle, Rosen (1973, 74) has obtained the field equations of bimetric relativity as

\[ N^g_y = \frac{1}{2} N g^g_y = -8\pi \kappa T^g_y \]  \hspace{1cm} (1.2.14)

where

\[ N^i_j = \frac{1}{2} f^a_{\tilde{b}c} (g^{\tilde{b}i} g^{c}_{j\tilde{a}})_{\tilde{b}c} \]  \hspace{1cm} (1.2.15)

For empty space-time, the field equations become

\[ N^g_y = 0 \]  \hspace{1cm} (1.2.16)

which has been used by us in our investigations described in all chapters.
This theory has attracted the attention of good number of researcher who have studied the various aspects of BR. To note a few are Liebscher (1975), Yilmaz (1975, 79), Falik and Opher (1979), Karade and Dhoble (1979, 80), Karade (1980, 81), Khadekar and Karade (1989), Reddy and Venkateswarlu (1989), Grigoryan (1994), Adhav and Karade (1994), Banerjee and Beesham (1996), Reddy and Venkateswara Rao (1998, 99), Karade et al. (2001) etc.

Inspired by their work, we have taken up the study of BR as regard to Bianchi type mesonic cosmological models and plane symmetric space-time filled with perfect fluid and are stored in the content of chapters II, III, IV and V.

3. Higher Dimensions

In the recent years many researchers find interest in higher dimensional space-time models of physical importance. The topic of higher dimensional cosmologies has received much attention Lorenz-Petzold (1984a, 1984b, 1985a, 1985b). The question of the “true” dimensionality of the physical world manifold is of deep interest and corresponds to the fundamental structure of space and time. This question can be traced back right to the origin of philosophy and physical science Jammer (1980), Grünbaum (1973). Higher dimensional spaces was first given by Ehrenfest (1917) on the basis of a generalization of Newton’s theory of gravitation.

This approach was later rediscussed by Tangherlini (1962) in the case of Schwarzschild field in n-dimension on the basis of general theory of gravitation (GRT). The idea that the universe as a whole really does possess more than three spatil dimensions (GRT) has been discussed by Kaluza et al. (1921) and Klein (1925) and later by Einstein and Bergmann (1938). During
the last few years the idea of higher-dimensional theories of Kaluza-Klein type has gained much in popularity and is one of the most attractive ways of unifying gauge theories with gravitation. If the extra dimensions really exist, one must explain how it is that they are not observed today. A possible answer was given by Chodos and Detweiler (1980). These authors show that higher-dimensional vacuum solutions of the anisotropic Kasner type may evolve into effective (1+3)-dimensional models. Thus the extra dimensions are not observed today as a consequence of the inherent dynamics of the expansion of the universe.

The number of researchers have studied various aspects of four and higher dimensional bimetric theory of relativity. Inspired by this work we have taken up the study of bimetric relativity as regard to Bianchi type –I and III models with scalar meson field coupled with perfect fluid distribution and form the content of the chapters II and III. The idea of higher dimensional space-time in bimetric theory of relativity are used in chapters II, III and VI.

1.4. Cosmological models

The aim of cosmology is to study the large scale structure and evolution of the universe. We assume two fundamental principles, given below which greatly simplify the development of mathematical theory in cosmology:

(i) Weyl’s postulate,

(ii) The cosmological principle.
(i) Weyl’s postulate

The universe in principle could be a very complicated system, with various astronomical objects moving in it in a jumbled kind of way. However, in practice it is possible to visualize the simpler picture, in which we have galaxies streaming along regular tracks which do not intersect each other. The galaxy tracks, in either case, are drawn in a space-time diagram.

However, the ‘time’ in the former case has no great physical significance. Einstein’s relativity theory tells us that each galaxy will have its own time-keeping device, and in a jumbled-up case of this type it would not be possible to synchronize clocks on different galaxies. In simpler picture, such a synchronization is possible thanks to the regularity of motions involved. Here we can draw a surface $\Sigma$ intersecting all tracks at right angles, and argue that at all such points of intersection the clocks record the same time. In this way we have a universal or a ‘cosmic’ time which serves as a reference coordinate for the universe as a whole. The Weyl postulate gives a precise mathematical definition of this regularity. With its help the large scale structure of the universe is considerably simplified. Moreover, the cosmologists can speak of a time coordinate or an ‘epoch’ with respect to which any changes in the universe can be described.

(ii) The Cosmological Principle

It states that a typical surface is homogeneous and isotropic. In physical terms this can be interpreted as follows. Suppose we fix the value of the cosmic time $t$ and look at the universe from any of the galaxies which are moving according to the Weyl postulate. Then the universe will look the same – no matter which galaxy we choose to observe it from. Moreover, it will present the same view in all directions from any galaxy.
With the help of these two assumptions the cosmologists are able to simplify the overall problem considerably. In essence the structure of the universe may be described in terms of two physical quantities. One is the ‘sign of curvature’ of the spaces. That is, we may imagine the $\Sigma$-spaces as spaces of positive, negative, or zero curvature. We may attach values $+1$, $-1$ or $0$ to these three possibilities. The second parameter is the ‘expansion factor’. We may denote it by a function $R(t)$ of the cosmic time $t$. In physical terms it describes how the separation between any two galaxies, measured at any cosmic time, changes with the cosmic time. If $R(t)$ increases with $t$, it means that the universe is expanding, if $R(t)$ decreases with $t$, the universe contracts; whereas if $R(t)$ stays unchanged, the universe remains static.

1.5. Bianchi Models

Theoretical or mathematical cosmologists, however, pointed out that more general cosmological models which differ significantly from a Friedmann-Robertson-Walker (FRW) models at early times, approach the FRW model very closely for a certain epoch, and may diverge from it again in the future. Clearly, the FRW universes represent only a very special class of viable cosmological models, through the simplest and most suitable interpretations of ‘fuzzy’ cosmological observational data.

Belinsky et al. (1970, 1982) have shown the fact that the presence of matter does not influence the qualitative behaviour of a cosmological model near the singularity.
In gaining an intuition in the analysis of general cosmological singularities, the class of spatially homogeneous anisotropic cosmological models have played a crucial role. These so called Bianchi models admit a simply transitive 3-dimensional homogeneity group. Among the Bianchi vacuum models there are special exact explicit solutions, in particular the Kasner and the Bianchi type - II solutions, which exhibit some aspects of general cosmological singularities. The Bianchi models have had an impact on other issues in general relativity and cosmology.

By definition, the Bianchi models admit a simply transitive 3-dimensional homogeneity group $G_3$. There exist special “Locally Rotationally Symmetric” (LRS) Bianchi models which admit a 4-dimensional isometric group $G_4$ acting on homogeneous spacelike hypersurfaces, but these groups have a simply transitive subgroup $G_3$. Although the 3-dimensional Lie groups which are simply transitive on homogeneous 3-spaces were classified by Bianchi in 1897, the importance of Bianchi’s work for constructing vacuum cosmological models was only discovered by Taub (1951), when the Taub space was first given.

The line element of the Bianchi models can be expressed in the form

$$ds^2 = -dt^2 + g_{ab}(t) \omega^a \omega^b,$$

where the time-independent 1 - forms $\omega^a(= E^a_a \, dx^a)$, $a=1, 2, 3$, are dual to time-independent spatial frame vectors $E_a$ (often an arbitrary time variable $\tilde{t}$ is introduced by $dt = N(\tilde{t}) d\tilde{t}$, $N$ being the usual lapse function). Both $\omega^a$ and $E_a$ are group-invariant, commutating with the three Killing fields which generate the homogeneity group. They satisfy the relations
\[ d\omega^a = -\frac{1}{2} C_{bc}^a \omega^b \wedge \omega^c, \quad (1.5.2) \]

\[ [E_a, E_b] = C_{ab}^c E^c, \quad (1.5.3) \]

where \( d \) is the exterior derivative and \( C_{bc}^a \) are the structure constants of the Lie algebra of the homogeneity group.

The models are classified according to the possible distinct sets of the structure constants. They are first divided into two classes: in class \( A \) the trace \( C_{bc}^a = 0 \), and in class \( B \), \( C_{bc}^a \neq 0 \). In class \( A \) one can choose \( C_{bc}^a = n^{(a)} \varepsilon_{abc} \) (no summation over \( a \)), and classify various symmetry types by parameter \( n^{(a)} \) with values \( 0, \pm 1 \). In class \( B \), in addition to \( n^{(a)} \), one needs the value of a constant scalar \( h \) (related to \( C_{bc}^a \)) to characterize types \( \text{VI}_h \) and \( \text{VII}_h \) [Wainwright et al. (1997)].

The simplest models are the Bianchi I cosmologies in class \( A \) with

\[ n^{(a)} = 0 \quad \text{i.e.} \quad C_{bc}^a = 0, \]

so that all three Killing vectors (the group generators) commute. They contain the standard Einstein-de Sitter model with flat spatial hypersurfaces. In the vacuum case, all Bianchi-I models are given by the well-known 1-parameter family of Kasner metrics

\[ ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2, \quad (1.5.4) \]
where
\[ p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1. \quad (1.5.5) \]

These metrics were first used to investigate various effects in anisotropic cosmological models.

The general vacuum Bianchi type-II cosmologies (with one \( n^{(a)} = +1 \), and the other two vanishing), discovered by Taub (1951), contain two free parameters:
\[ ds^2 = -A^2 dt^2 + A^{-2}t^{2 p_1}(dx + 4 p_1 bz \ dy)^2 + A^2 (t^{2 p_2} dy^2 + t^{2 p_3} dz^2), \quad (1.5.6) \]

where
\[ A^2 = 1 + b^2 t^{4 p_1}, \quad p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1. \quad (1.5.7) \]

If we put the parameter \( b = 0 \), the metric (1.5.6) become the Kasner solutions (1.5.4). Near the big bang, the general Bianchi type-II solution is asymptotic to a Kasner model. In the future it is asymptotic again to a Kasner model, but with different values of parameter \( p_i \) [Wainwright et al. (1997)].

The general Bianchi type-V vacuum solutions are also known. These are given by the 1 - parameter family of Joseph solutions [Wainwright et al. (1997)]. The type V models are the simplest metrics in class B and are the simplest Bianchi models which contain the standard FRW open universes \( (k = -1) \). The Joseph solutions are asymptotic to the specific Kasner solution in the past, and tend to the “isotropic Milne model” in the future. This is intuitively understandable since open FRW models, as they expand
indeﬁnitely into the future with matter density decreasing, also approach the Milne model. As is well known, the Milne model is just an empty ﬂat (Minkowski) space-time in coordinates adapted to homogeneous spacelike hypersurfaces, with expanding normals [Peebles (1993)]:

\[ ds^2 = -d\tau^2 + \tau^2 [(1 - \rho^2)^{-1} d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)] \tag{1.5.8} \]

with \( \tau = t(1 - u^2)^{1/2}, \quad \rho = u(1 - u^2), \quad u = \frac{r}{t} < 1 \), where \( t, r, \theta, \phi \) are standard Minkowski (spherical) coordinates. Because of its signiﬁcance as an asymptotic solution and its simplicity, the Milne model has been used frequently in pedagogical expositions of relativistic cosmology [Peebles (1993), Rindler (1977)] as well as in cosmological perturbation theory and quantization [Tanaka et al. (1997)]. The Milne universe is also an asymptotic state of other Bianchi models such as, for example, the intriguing Lukash vacuum type VIIa solution Lukash (1975), which can be interpreted as two monochromatic, circularly polarized waves of time-dependent amplitude traveling in opposite directions on a FRW background, with flat or negative curvature spacelike sections.

Shri Ram (1987) have derived an algorithm for constructing spatially-homogeneous perfect ﬂuid solutions of Einstein ﬁeld equations for locally rotationally – symmetric (LRS) Bianchi type-I space-times and obtained an solution and discussed its physical and kinematic properties.

Venkateswarlu and Reddy (1991a) have obtained an exact solutions of Bianchi type II, VIII and IX cosmological models in a conformally-invariant scalar-ﬁeld with trace-free electromagnetic energy-momentum tensor and Venkateswarlu et al. (1991 b) have obtained
spatially-homogeneous and anisotropic Bianchi type-II, VII and IX stiff-fluid cosmological models in the presence of source-free electromagnetic fields in Lyra's manifold and discussed some properties of the model.

Banerjee et al. (2001) have obtained an exact solution for a special case of locally rotationally symmetric Bianchi type-I model which consist of a dilation scalar field and a Liouville type dilatonic potential interacting with electromagnetic field coupled with gravity and discussed the corresponding properties of the model.

Raj Bali and Gokhroo (2001), have investigated Bianchi type – I magnetized cosmological model for perfect fluid distribution and discussed the behaviour of the model in presence and absence of magnetic field.

Raj Bali et al. (2002), have investigated Bianchi type – I inflationary universe in the presence of massless scalar field with a flat potential and also discussed the inflationary scenario of the model in detail.

Mohanty and Mishra (2002), have studied the feasibility of scalar-invariant theory in Bianchi type VIh space-time dependent gauge function (Dirac gauge) and a matter field in the form of a perfect fluid with isotropic matter pressure and observed that the Bianchi type VIh (h = 1) space-time is feasible where Bianchi type VIh (h = -1) and VIh (h = 0) space-times are not feasible and constructed a non-singular model for the universe filled with disorder radiation and studied some physical behaviour of the model.
Khadekar and Karde (1989), have established the non-existence of higher dimensional axially symmetric massive scalar field and massive complex scalar field coupled with electromagnetic field in bimetric relativity.

Mohanty and Sahoo (2002), have shown that Bianchi type-I cosmological model representing meson field and coupled with perfect fluid do not exist in bimetric theory of gravitation. In chapter – II, we have extended the work of Mohanty et al. (2002) in bimetric theory of relativity and it is observed that the five dimensional Bianchi type-I cosmological model does not exist in a case of scalar meson field and mesonic perfect fluid. Hence only vacuum model is obtained.

In chapter-III, we have continue the study of chapter-II and obtained the non-existence of four and higher dimensional Bianchi type-III cosmological model in the case of meson field and mesonic perfect fluid. It has been found that the Bianchi type-III cosmological model in bimetric theory of gravitation does not accommodate meson field and mesonic perfect fluid. Hence vacuum model is obtained.

Chapter-IV is devoted to the study of non-existence of plane symmetric cosmological model representing a perfect fluid distribution. It is interesting to note that there results in bimetric relativity contrasts sharply with results in general relativity obtained by Singh and Abdussattar (1973).
1.6. Plane symmetry

The field equations of general relativity are nonlinear in ten unknowns \( g_{ij} \) and such as it is very difficult to obtain their exact solutions. The involvement of symmetry – may be spherical, cylindrical or plane – does reduce the number of gravitational potentials \( g_{ij} \) and thus helps one in simplifying the field equations to some extent, in the case of plane symmetry the number of unknown \( g_{ij} \) reduces to five only. From the work of Taub (1951) it is gathered that the space-times with plane symmetry are quite similar to those with spherical symmetry.

Reddy and Innaiah (1985), have obtained the plane symmetric non-static solutions for Zeldovich fluid distribution in general relativity and discussed some geometrical and physical features of the model.

The plane symmetric non-static cosmological model in bimetric theory has been derived by Karade et al. (2000) in the presence of fluid distribution obeying certain equation of state and studied various kinematical parameters and some physical features of the model.

Karade et al. (2001) have investigated some inhomogeneous non-static plane symmetric perfect fluid solutions in bimetric theory of gravitation and discussed some physical and kinematical properties.

Mohanty and Sahoo (2002), have shown that the non-static plane symmetric cosmological model exist in case of scalar meson field where the scalar field becomes constant. Further it is found that in case of mesonic perfect fluid, the bimetric theory does not admit perfect fluid but allows only mesonic scalar field with constant scalar field. In both the cases a cosmological model with constant scalar field is obtained.
In Chapter-V, we have considered the non-static plane symmetric space-time in four dimension and obtained some exact solutions for material and electromagnetic distribution in general and bimetric theory of relativity.

1.7. Takeno's Exposition of Plane Gravitational Waves

The pioneer work of Einstein (1916, 18) and Rosen (1937) forms the cornerstone of the investigations of plane gravitational waves in general relativity.

They have introduced the concept of plane wave by assuming

\[ g_{\theta} = \eta_{ij} + h_{ij} \],

where \( \eta_{ij} \) is the fundamental tensor of Minkowski space-time and \( h_{ij} \) is everywhere sufficiently smooth i.e. small departures from \( \eta_{ij} \) with \( |h_{ij}| \ll 1 \).

Taub (1951), Bondi (1957), Bonnor (1957), Bondi-Piranni-Robinson (1959) and Takeno (1961) studied rigorously for plane wave solutions.

According to Takeno (1961) a four dimensional Riemannian space of signature \((-2)\) is plane symmetric if it admits the three parameter group of transformations composed of

\[ R_1 = z \partial y - y \partial z \quad R_2 = \partial y \quad R_3 = \partial z \]

as a subgroup of motions. Here \( x, y, z, t \) are coordinates and \( t \) corresponds to time. The wave solutions of field equations obtained in the space-time having plane symmetry are called plane wave solutions.
Takeno has put forth his own way of investigating the plane gravitational waves which is mainly mathematical. Takeno defines plane wave as follows:

**Definition**: A plane wave \( g_{ij} \) is a non-flat solution of the field equations

\[
R_{ij} = 0, \quad i, j = 1, 2, 3, 4
\]

in an empty region of space-time such that

\[
g_{ij} = g_{ij}(Z), \quad Z = Z(x^i), \quad x^i = x, y, z, t
\]

in some suitable coordinate system with

\[
g^{ij} Z_{,i} Z_{,j} = 0, \quad Z_{,i} = \frac{\partial Z}{\partial x^i},
\]

\[
Z = Z(z, t), \quad Z_3 \neq 0, \quad Z_4 \neq 0.
\]

The signature convention adopted is

\[
g_{\alpha\alpha} < 0, \quad \begin{vmatrix} g_{\alpha\alpha} & g_{\alpha\beta} \\ g_{\beta\alpha} & g_{\beta\beta} \end{vmatrix} > 0, \quad \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} < 0, \quad g_{44} > 0.
\]

Here no summation for \( \alpha \) and \( \beta \) \((\alpha, \beta = 1, 2, 3)\) and accordingly

\[
g = \det(g_{ij}) < 0
\]
Takeno (1961) defines the plane wave as a non-flat solution of the field equations

\[ R_{ij} = 0. \]

He has organized the solutions of the field equations in the format:

\[ \rho_a = 0 , \quad R_{a\alpha} = 0 , \quad a = 1, 2, \quad \alpha = 3, 4 \]

and

\[ N \rho_{\alpha\beta} + M \sigma_{\alpha\beta} = 0 , \]

which can be put as

\[ \bar{\omega} \rho_{\alpha\beta} + \bar{\phi} \sigma_{\alpha\beta} = 0 = \bar{\phi} \rho_{\alpha\beta} + \phi \sigma_{\alpha\beta} . \]

Here

\[ \omega = t + z\phi , \quad \phi = \frac{Z_3}{Z_4} , \quad M = \bar{\omega} - z\bar{\phi} , \quad N = \bar{\omega} - z\bar{\phi} \]

and a bar (−) over letters denotes derivatie with respect to \( Z \).

Thengane et al. (2000) have generalized Takeno’s (1961) definition in the series that in addition to \( Z_3 \neq 0, \quad Z_4 \neq 0 \), they adopt \( Z_{1} \neq 0, \quad Z_{2} \neq 0 \) with same signature convention and obtain the wave solutions of field equations \( R_{ij} = 0 \) in the form

\[ Q \gamma_{ij} = \rho \in_{ij} = 0 . \]
where
\[ \gamma_{ij} = \frac{1}{2} [\phi_i \gamma_j + \phi_j \gamma_i] - L_2 \phi_i \phi_j \]

and
\[ \varepsilon_{ij} = -\overline{\gamma}_{ij} + \frac{1}{4} [\phi_i \phi_j L_1 - 2L_2 (\phi_j \gamma_i + \phi_i \gamma_j) + 2 \gamma_i \gamma_j] \quad \text{with } i \leq j, \]
\[ \gamma_i = \overline{g}_{ij} \omega^j, \quad \omega^j = \phi_i g^{ij}, \quad L_1 = g^{ij} g_{ab} \overline{g}_{ia} \overline{g}_{jb}, \quad L_2 = \log \sqrt{-g}. \]

These equations can be put in the form
\[ \overline{\omega}_a \gamma_{ij} + \overline{\omega}_a \varepsilon_{ij} = 0 = \overline{\phi}_a \gamma_{ij} + \overline{\phi}_a \varepsilon_{ij}. \]

The work of Takeno (1961) easily follows from these solutions by assuming \( Z \) as independent of \( x \) and \( y \).

Adhav and Karade (1994) has extended the work of Takeno (1961) and investigated \((z-t)\) - type and \((t/z)\) - type plane gravitational waves in six dimensional bimetric theory of relativity.

Hongya and Wanzhong (1988), have obtained the exact solution of plane-electromagnetic gravitational waves in five-dimensional Kaluza-Klein theory and found that the five-dimensional harmonic condition is reduced to the usual four-dimensional harmonic condition plus the Lorentz condition.

In the last chapter-VI we have extended the Thengane’s (2000) work in general theory to bimetric theory of relativity.
This chapter of the thesis is devoted to the study of higher dimensional plane wave solutions in bimetric theory and obtained the plane gravitational waves $g_{ij}$ given by

$$N \rho_i^j + M \sigma_i^j = 0,$$

where

$$\rho_i^j = (\phi^2 - 1) g^{hj} \overline{g}_{hi}, \quad \sigma_i^j = \frac{d}{dz} [(1 - \phi^2) g^{hj} \overline{g}_{hi}],$$

and

$$\overline{\omega} \rho_i^j + \overline{\omega} \sigma_i^j = 0 = \overline{\phi} \rho_i^j + \overline{\phi} \sigma_i^j.$$

For $n = 4$ and for $n = 5, 6$ respectively the results in the work of Adhav and Karade (1994) and Thengane et al. (2000) can be obtained regarding the plane gravitational waves in bimetric theory of relativity.