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INTRODUCTION
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1.1 Introduction

This thesis entitled “SOME DOMINATION RESULTS ON GRAPH VALUED FUNCTIONS” is in the area of Graph Theory which is one of the ever growing branches of mathematics. Graph Theory, in its essence can be described as the study of relations on finite sets, which are visualized with vertices and edges in a two dimensional plane. Graph Theory is intimately related to many branches of mathematics, including group theory, probability, numerical analysis, matrix theory, topology, operational research, combinatorics and many more. This thesis is devoted to introduction of certain new parameters of domination number of  lict graph and to study their relationship with other domination parameters.
1.2 A brief history of Graph Theory and its relevance

The basic ideas of graph theory were introduced in eighteenth century by the great Mathematician Leonard Euler. Since then it has been the source of interest of many researchers and it has achieved remarkable development leading to fruitful generalization and extensions yielding interesting and beautiful combinatorial results.

The past 50 years has been a spectacular growth in pure and applied graph theory due to its wide range of applications in many fields like Engineering, Physical, Social and Biological Sciences, Linguistics, Discrete Optimization Problems, Combinatorial Problems and Classical Algebra Problems. There is virtually no end to list the problems that can be solved using Graph Theory.

Historically, graph theory has been developed as a mathematical discipline. Graph theory is a fascinating subject involving only two elements viz, vertices (or points) and edges (or lines). Its origin is as diverse as its application. One of the most appealing features of graph theory lies in the geometric or pictorial aspect of the subject. Given a graph G consisting of vertices and edges, diagrammatically each vertex is represented by a point in the plane and each edge by a line segment joining appropriate distinguished vertices.

Among the various concepts of Graph Theory, the concept of domination, independence etc., have existed for a long time. The concept of “Domination in graphs” originated in 1850 with the problem of placing minimum number of queens on n x n chess board so as to cover (or dominate) every square. The problem of dominating the squares of a chess board can be stated more generally as a problem
of dominating the vertices of a graph. For the comprehensive survey of the chess N board problems, see Cockayne [9], Cockanyne etal [12], [15], Hedetniemi etal [20], Kraitchik [23], Spencer and Cockayne [31] and Wagner and Geist [33].

The next appearance of domination in the literature was also associated with game applications. In 1953, Von Neumann and Morgenstern [32] in their book on game theory considered domination in digraphs to find solutions (Kernels) for cooperative n-person games.

By Berge [4], [5] domination was formalized as a theoretical area in Graph Theory in 1958. Berge referred to the domination number as the coefficient of external stability. In 1962, Ore [28] was the first to use the term domination for undirected graphs and also he introduced the concepts of minimal and minimum dominating sets of vertices in a graph.

Until 1975, there has not been any considerable progress in this topic. In 1975, Cockayne and Hedetniemi [13] the pioneers of the theory of domination in graphs, published the first paper entitled “Optimal domination in graphs”. Later in 1977, another fundamental paper of the same authors [14] entitled “Towards a theory of domination in graphs” appeared, in which they surveyed all the existing results up to 1977 and considered many problems related to the concepts, viz: colorability, clique size and Nordhaus-Gaddum type results etc. In [21], Heditniemi and Laskar published their “Bibliography on domination in graphs and some basic definitions of domination parameters” in 1990, which was containing about 400
entries. This bibliography has grown to cover 1200 entries at the end of 1997. The publication of the first large two volume text books on domination entitled “FUNDAMENTALS OF DOMINATION IN GRAPHS” and “DOMINATION IN GRAPHS: ADVANCED TOPICS” in 1998, edited by Haynes, Hedetniemi and Slater [19] and [18]. Finally chartrand and Lesniak [8] have included a chapter on domination in their revised book, “Graphs and Digraphs”. During the past 28 years the study of domination in graphs has emerged as a significant area of research not only in graph theory but also in combinatorial optimization and analysis of algorithms.
1.3 Definitions and Terminologies

Here we recall some of the basic definitions and notations, which are needed for the subsequent chapters. However any undefined terms, may be found in Harary[17]. The graphs considered here are finite, undirected, connected, without loops or multiple edges unless otherwise specifically mentioned.

A graph $G$ consist of a finite non empty set $V=V(G)$ of $p$ vertices (or points) together with prescribed set $E$ of $q$ unordered pairs of distinct vertices of $V$. Each pair $e=\{u,v\}$ of vertices in $E$ is called an edge (or line) and $e$ is said to join $u$ and $v$. We write $e=uv$ and say that $u$ and $v$ are adjacent vertices and we say vertices $u$ and $v$ are incident with an edge $e$. If two distinct edges $x$ and $y$ are incident with a common vertex, then they are adjacent edges.

The cardinality of the vertex set of a graph $G$ is called the order of $G$ and is denoted by $p$. The cardinality of its edge set is called the size of $G$ and is denoted by $q$. A graph with $p$ vertices and $q$ edges is called a $(p,q)$ graph. A $(1,0)$ graph is called a trivial graph.

A graph $G$ is isomorphic to a graph $H$ if there exists a bijection $f$ from $V(G)$ to $V(H)$ such that $\{u,v\} \in E(G)$ if and only if $\{f(u),f(v)\} \in E(H)$. If $G$ is isomorphic to $H$, we write $G \cong H$. A subgraph of a graph $G$ is a graph whose vertex set and the edge set are
the subsets of the vertex set and the edge set of $G$ respectively. A **spanning subgraph** is a subgraph containing all vertices of a graph $G$, while its edge set is a subset of edge set of $G$. For any set $S$ of vertices of a graph $G$, the **induced subgraph** denoted as $\langle S \rangle$ is the maximal subgraph of $G$ with vertex set $S$. Thus two vertices of $S$ are adjacent in $\langle S \rangle$ if and only if they are adjacent in $G$.

The **removal** of a vertex $v$ from a graph $G$ results in the subgraph $G - v$ of $G$ consisting of all vertices of $G$ except $v$ and all edges not incident with $v$. The **removal** of an edge $e$ from $G$ yields the spanning subgraph $G - e$ containing all edges of $G$ except $e$. The **degree** of a vertex $v$ in a graph $G$ is denoted by $\deg v$ is the number of edges incident to $v$. If every edge is incident with two vertices, then it contributes two degrees to the sum of the degrees of the vertices.

The **minimum degree** among the vertices of a graph $G$ is denoted by $\min \deg(G)$ or $\delta(G)$. While the **maximum degree** denoted by $\max \deg(G)$ or $\Delta(G)$ is the largest such number. If $\delta(G) = \Delta(G) = r$, then all vertices of the graph $G$ have the same degree $r$ and $G$ is called $r$-**regular**. In particular, for $r = 2$, $G$ is called $2$-**regular**.

The vertex $v$ is **isolated** if $\deg v = 0$. And it is an **end vertex** if $\deg v = 1$. The edge incident to end vertex is a **pendent edge** (or **end edge**). The **degree** of an edge $e = uv$ of a graph $G$ is defined by
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\[ \deg(e) = \deg(u) + \deg(v) - 2. \]

The minimum and maximum degree of an edge in \( G \) is denoted by \( \delta' \) and \( \Delta' \). A walk of a graph \( G \) is an alternating sequence of vertices and edges, beginning and ending with vertices immediately preceding and following it. The walk \( v_0e_1v_1e_2v_2\ldots e_pv_p \) is a closed walk if \( v_0 = v_p \), otherwise it is an open walk. It is called a trail if all the edges are distinct and it is called a path if all vertices are distinct. A closed walk is called a cycle if its \( p \) vertices are distinct and \( p \geq 3 \). The length of a path or a cycle is the number of edges in it. A graph of order \( p \) which is a path or a cycle is denoted by \( P_p \) or \( C_p \) respectively.

A graph \( G \) without a cycle is called a tree (or acyclic). Therefore a tree is a connected acyclic graph. Any graph without cycles is a forest. An acyclic spanning subgraph of \( G \) is called a spanning tree of \( G \). The distance \( d(u,v) \) between two vertices \( u \) and \( v \) of a graph \( G \) is the length of the shortest path joining them if any. Otherwise \( d(u,v) = \infty \). A shortest \( u-v \) path is often called a geodesic.

The diameter \( \text{diam}(G) \) of a connected graph \( G \) is the length of any longest geodesic. A graph \( G \) is connected if every pair of its vertices is joined by a path. A maximal connected subgraph of \( G \) called a component of \( G \). Therefore a disconnected graph has at least two components. A graph \( G \) is totally disconnected (or trivial) if \( G \) has no edges.
The **Connectivity** $\kappa$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. A vertex $v$ of $G$ is called a **cutvertex** if its removal results in a disconnected graph. That is, $G - v$ has at least two components. Similarly, a **bridge** is an edge of a graph $G$ whose removal disconnects $G$.

The **edge connectivity** $\lambda$ of a graph $G$ is the minimum number of edges whose removal results in a disconnected or trivial graph. A graph $G$ is said to be **separable** if it has at least one cut vertex or $\kappa(G) = 1$. Otherwise $G$ is **nonseparable**. Thus, a nonseparable graph is a connected, nontrivial graph without cutvertices. A **block** of a graph $G$ is a maximal nonseparable subgraph. A block is called an **end block** of $G$ if it contains exactly one cut vertex of $G$.

A graph is **complete** if every pair of its vertices are adjacent. A complete $(p, q)$ graph is therefore a regular graph of degree $p - 1$ having $q = \frac{p(p-1)}{2}$ edges and it is denoted by $K_p$. The **complement** $\overline{G}$ of a graph $G$ has $V(G)$ as its vertex set but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. A graph $G$ is called a self complimentary graph if it is isomorphic to its compliment $\overline{G}$.

A **bipartite graph (or bigraph)** $G$ is a graph whose vertex set $V$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ has one end in $V_1$ and other end in $V_2$. If every vertex of $V_1$
joins every vertex of $V_2$, then the graph $G$ is called a **complete bipartite** graph. If $V_1$ and $V_2$ have $m$ and $n$ vertices respectively, then we write $G = K_{m,n}$. A **star** is a complete bipartite graph with $m = 1$ and it is denoted by $K_{1,n}$.

A wheel $W_p$ is a graph with $p$ vertices such that one of its vertices has degree $p - 1$, while the degree of the other vertices is three and all such vertices lie on a cycle of length $p - 1$.

A graph is acyclic if it has no cycles. A **tree** is a connected acyclic graph. Any graph without cycles is a forest. An acyclic spanning subgraph of $G$ is called a spanning tree of $G$.

The concept of the line graph of a given graph is so natural that it has been independently discovered by many authors giving different names. The **line graph** of a graph $G$, denoted $L(G)$, is the graph whose vertices are the edges of $G$, with two vertices of $L(G)$ adjacent whenever the corresponding edges of $G$ are adjacent.

The **lict graph** of a graph $G$ denoted $n(G)$, is graph whose vertex set is the union of the set of edges and set of cut vertices of $G$ in which two vertices are adjacent if and only if corresponding members are adjacent or incident.

A **square of a graph** $G$, denoted by $G^2$, has a same vertex set as in $G$ and every two vertices $u$ and $v$ are joined in $G^2$, if and only if they are joined in $G$ by the path of length one or two.
A vertex and an edge are said to cover each other if they are incident. A set of vertices which cover all the edges of a graph $G$ is called a **vertex cover** for $G$. The smallest number of vertices in any cover for $G$ is called its **covering number** and is denoted by $\alpha_0$. A set of edges which cover all the vertices of $G$ is called an edge cover of $G$. The smallest number of edges in any edge cover of a graph $G$ is called the **edge covering** number of $G$ and is denoted by $\alpha_1$.

A subset $S$ of the vertex set in a graph $G$ is said to be independent if no two vertices in $S$ are adjacent in $G$. The maximum number of vertices in an independent set is called the **independence number** of $G$ and is denoted by $\beta_0$. A set $S$ of edges in a graph $G$ is said to be independent if no two edges in $S$ are adjacent in $G$. A set $S$ is said to be a maximal independent set provided it is not a proper subset of some other independent set. The maximum cardinality of an edge independent set of $G$ is called the **edge independence** number or matching number of $G$ and is denoted by $\beta_1$.

The maximum number of mutually adjacent vertices of a graph $G$ is the **clique number** $\omega(G)$ of $G$ and the **edge clique number** $\omega'(G)$ of $G$ is the maximum number of mutually adjacent edges in $G$.

A **coloring** of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The set of all vertices with any one color is independent and is called a color class. An $n$–coloring of a graph $G$ uses $n$ colors; it therefore partitions $V$
into $n$ color classes. The **chromatic number** $\chi(G)$ is defined as the minimum $n$ for which $G$ has an $n$–coloring.

The **open neighborhood** $N(v)$ of a vertex $v$ in a graph $G$ is the set of all vertices adjacent to $v$ in $G$ and $N[v] = N(v) \cup \{v\}$ is called the **closed neighborhood** of $v$. The **open neighborhood** $N(e)$ of an edge $e$ in a graph $G$ is the set of all edges adjacent to $e$ in $G$ and the **closed neighborhood** $N[e] = N(e) \cup \{e\}$. The **open neighborhood** $N(S)$ of a set $S$ of vertices in a graph $G$ is the set of all vertices adjacent to the vertices in $S$ and $N[S] = N(S) \cup \{S\}$ is called the **closed neighborhood** of $S$.

For any real number $x$, $\lceil x \rceil$ denotes the **largest integer less than or equal to** $x$ and $\lfloor x \rfloor$ denotes the **smallest integer greater than or equal to** $x$.

A set $D$ of vertices in a graph $G$ is a **dominating set**, if every vertex in $V - D$ is adjacent to some vertex in $D$. The **domination number** of $G$ is the minimum cardinality taken over all dominating sets in $G$ and is denoted by $\gamma(G)$. For more details on $\gamma(G)$ see [2], [26] and [34].

Samptkumar and Walikar [30] established a new concept of domination called the **connected domination number** of a graph $G$. A dominating set $D$ is a **connected dominating set**, if an induced subgraph $\langle D \rangle$ is connected. The **connected domination number**
\( \gamma_c(G) \) of \( G \) is the minimum cardinality taken over all connected dominating sets of \( G \).

The concept of edge domination number was studied by Gupta [16] and Mitchell and Hedetniemi [27]. A set \( S \) of edges of a graph \( G \) is an edge dominating set of \( G \), if every edge not in \( S \) is adjacent to at least one edge in \( S \). The edge domination number \( \gamma'(G) \) of \( G \) is the minimum cardinality of edges in an edge dominating set of \( G \). Related to this parameter, we refer [3], [22] and [28].

An edge dominating set \( X \) of a graph \( G \) is called a connected edge dominating set if the edge induced subgraph \( \langle X \rangle \) is connected. The connected edge domination number \( \gamma'_c(G) \) of \( G \) is the minimum cardinality taken over all connected edge dominating sets of \( G \).

A set \( D \subseteq V(G^2) \) is a dominating set of \( G^2 \), if every vertex not in \( D \) is adjacent to at least one vertex in \( D \). The minimum cardinality of vertices in such a dominating set in \( G^2 \) is called the domination number of a square graph of \( G \) and is denoted by \( \gamma(G^2) \).

A set \( D \subseteq V(G) \) is a restrained dominating set of \( G \), if every vertex not in \( D \) is adjacent in \( D \) and to a vertex in \( V - D \). The restrained domination number of graph \( G \), denoted by \( \gamma_r(G) \), is the minimum cardinality of a restrained dominating set of \( G \).
Cockayne, Dawes and Hedetniemi [10] introduced the concept of total domination in graphs. A set $D$ of vertices of a graph $G$ is a total dominating set if each vertex in $V$ is adjacent to some vertex in $D$. The total domination number $\gamma_t(G)$ of $G$ is the minimum cardinality of a total dominating set.

A dominating set $D$ of a graph $G=(V,E)$, is a split dominating set if the induced subgraph $\langle V-D \rangle$ is disconnected. The split domination number $\gamma_s(G)$ of $G$ is the minimum cardinality of a split dominating set.

The additional definitions will be introduced as and when required.

1.4 An outline of the present investigation:

The contents of this thesis are divided into six chapters. The First chapter is introductory in nature and it gives some basic definitions and terminologies in Graph Theory.

In the second chapter we initiated work on “Connected Lict Domination in Graphs”. For any graph $G$, the lict graph $n(G)=J$ of a graph $G$ is the graph whose vertex set is the union of the set of edges and set of cut vertices of $G$ in which two vertices are adjacent if and only if corresponding members are adjacent or incident. A dominating set of $n(G)$ is called connected dominating set of $n(G)$, if the induced subgraph $\langle n(G) \rangle$ is connected the minimum cardinality of
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\(D\) is called connected domination number of \(G\) and is denoted by \(\gamma_{cc}(G)\). In this chapter, many bounds on \(\gamma_{cc}(G)\) were obtained in terms of vertices, edges and other different parameters of \(G\) but not in terms of elements of \(J\). Further we develop its relation with other different domination parameters. Also Nordhaus_Gaddum type result is obtained.

In the third chapter we introduced “\textbf{Edge Lict Domination in Graphs}”. For any graph \(G\), the lict graph \(n(G) = J\) of a graph \(G\) is the graph whose vertex set is the union of the set of edges and set of cut vertices of \(G\) in which two vertices are adjacent if and only if corresponding members are adjacent or incident. A set \(F'\) of edges in a graph \(n(G)\) is called edge dominating set of \(n(G)\) if every edge in \(E[n(G)] - F'\) is adjacent to at least one edge in \(F'\), denoted as \(\gamma'_n(G)\) and is the minimum cardinality of edge dominating set in \(n(G)\). In this chapter, many bounds on \(\gamma'_n(G)\) were obtained in terms of vertices, edges and other different parameters of \(G\) but not in terms of elements of \(J\). Further we develop its relation with other different domination parameters. Also Nordhaus_Gaddum type result is obtained.

In the fourth chapter we focused on “\textbf{Restrained Lict Domination in Graphs}”. A set \(D_r \subseteq V[n(G)]\) is a restrained dominating set of \(n(G)\), if every vertex not in \(D_r\) is adjacent in \(D_r\) and to a vertex in
The restrained domination number of lict graph $n(G)$, denoted by $\gamma_{rn}(G)$, is the minimum cardinality of a restrained dominating set of $n(G)$. In this chapter, we study its exact values for some standard graphs we obtained. Also its relation with other parameters is investigated. Also Nordhaus-Gaddum type result is obtained.

In the fifth chapter we introduce “Total Lict Domination in Graphs”. For any graph $G$, the lict graph $n(G) = J$ of a graph $G$ is the graph whose vertex set is the union of the set of edges and set of cut vertices of $G$ in which two vertices are adjacent if and only if corresponding members are adjacent or incident. A set $D$ is a total dominating set, if $N(D) = V$ or equivalently, if for every vertex $v \in V$, there exists a vertex $u \in S, u \neq v$ such that $u$ is adjacent to $v$. The total domination number $\gamma_t(G)$ equals the minimum cardinality of total dominating set of $G$. A dominating set $D'$ of $J$ is a total dominating set if $N(J) = V[n(G)]$ and the minimum cardinality of $D'$ is total domination number of $n(G)$ and is denoted by $\gamma_{rn}(G)$. In this chapter, many bounds on $\gamma_{rn}(G)$ were obtained in terms of vertices, edges and other different parameters of $G$ but not in terms of elements of $J$. Further we develop its relation with other different domination parameters.

In the sixth chapter we worked on “Split Lict Domination in Graphs”. In this paper, we introduce the new concept in domination theory, a lict dominating set $D \subseteq V[n(G)]$ is a split lict dominating
set, if the subgraph \( \langle V[n(G)]-D \rangle \) is disconnected. The minimum cardinality of vertices in such a set is called split list domination number in \( G \) and is denoted by \( \gamma_{sn}(G) \). We study the graph theoretic properties of \( \gamma_{sn}(G) \) and many bounds are obtained in terms of elements of \( G \) and its relationship with other domination parameters are found.
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