Chapter 5

Equality Of Strong Domination And Chromatic Strong Domination In Graphs

5.1 Introduction

In any graph $G$, every chromatic strong dominating set is a strong dominating set. But the converse is not true. In some graphs, every $\gamma_s$-set is a chromatic strong dominating set. In some graphs, some $\gamma_s$-set is a chromatic strong dominating set and some $\gamma_s$-set is not a chromatic strong dominating set while in some graphs no $\gamma_s$-set is a chromatic strong dominating set. This observation motivates us to classify all graphs into three categories. We define that, a graph $G$ is in $(\gamma_s \equiv \gamma_c)$-class if every $\gamma_s$-set is a chromatic strong dominating set. If some $\gamma_s$-set is a chromatic strong dominating set and some $\gamma_s$-set is not a chromatic strong dominating set, we say that $G$ is in
(\(\gamma_s \cong \gamma_s^c\))-class. If no \(\gamma_s\)-set is a chromatic strong dominating set, we say that \(G\) is in \((\gamma_s \neq \gamma_s^c)\)-class. For example, \(K_{n,n}\) is in \((\gamma_s \equiv \gamma_s^c)\)-class, \(K_{m,n}, (m \neq n)\) is in \((\gamma_s \cong \gamma_s^c)\)-class and subdivision graph of \(K_{1,n}\) is in \((\gamma_s \neq \gamma_s^c)\)-class.

### 5.2 Equality Of Sets

**Definition 5.2.1** A graph \(G\) is said to be in \((\gamma_s \equiv \gamma_s^c)\)-class if every \(\gamma_s\)-set is a chromatic strong dominating set.

**Example 5.2.2**

\[
G_1:\begin{array}{c}
\includegraphics[width=0.3\textwidth]{example_graph}
\end{array}
\]

\(D_1 = \{u_1, u_4\}; D_2 = \{u_1, u_5\}; D_3 = \{u_1, u_6\}; D_4 = \{u_2, u_4\};
\]
\(D_5 = \{u_2, u_5\}; D_6 = \{u_2, u_6\}; D_7 = \{u_3, u_4\}; D_8 = \{u_3, u_5\}; D_9 = \{u_3, u_6\}.
\]
\(D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8\) and \(D_9\) are \(\gamma_s\)-sets as well as \(\gamma_s^c\)-sets.

**Definition 5.2.3** A graph \(G\) is said to be in \((\gamma_s \cong \gamma_s^c)\)-class if some \(\gamma_s\)-sets are chromatic strong dominating set and some \(\gamma_s\)-sets are not Chromatic Strong dominating set.
Example 5.2.4

\[ G_2 : \]

\[ D_1 = \{u_1, u_2, u_3, u_4, u_5\} \] and \[ D_2 = \{u_1, u_6, u_7, u_8, u_9\} \]. \( D_1 \) and \( D_2 \) are \( \gamma_s \)-sets. But \( D_2 \) is not \( \gamma_s^c \)-set. \( \blacksquare \)

**Definition 5.2.5** A graph \( G \) is said to be in \( (\gamma_s \neq \gamma_s^c) \)-class if no \( \gamma_s \)-set is chromatic strong dominating set.

Example 5.2.6

\[ G_3 : \]

\[ D_1 = \{u_1, u_2, u_3\} \] is the only \( \gamma_s \)-set. But \( D_1 \) is not a \( \gamma_s^c \)-set.

**Remark 5.2.7** Let \( G \) be a \( \chi \)-critical graph with \( |V(G)| > 1 \). Then \( G \) is in \( (\gamma_s \neq \gamma_s^c) \)-class. (Since for such graphs \( \gamma_s(G) < n \) and \( \gamma_s^c(G) = n \))

**Theorem 5.2.8** 1. The path \( P_n \) belongs to \( (\gamma_s \neq \gamma_s^c) \)-class if and only if \( n \equiv 0 \) or \( 2 \) (mod 3).
2. The path $P_n$ belongs to $(\gamma_s \cong \gamma_s^c)$-class if and only if $n \equiv 1 \pmod{3}$

Proof: Let $P_n : u_1, u_2, \ldots, u_n$ be a path on $n$ vertices.

Case(i) Let $n = 3k$. Then $\gamma_s(P_{3k}) = k$ and it has a unique $\gamma_s$-set $D = \{u_2, u_5, u_8, \ldots, u_{3k-1}\}$. Since $D$ is independent, $\chi(<D>) = 1$. But $\chi(G) = 2$. Therefore $D$ is not a chromatic strong dominating set. Hence $P_{3k}$ belongs to $(\gamma_s \not\cong \gamma_s^c)$-class.

Case(ii): Let $n \equiv 1 \pmod{3}$. Let $n = 3k + 1$. Then $\gamma_s(P_{3k+1}) = k + 1$. Let $D_1 = \{u_2, u_5, u_8, \ldots, u_{3k-1}, u_{3k+1}\}$ and $D_2 = \{u_2, u_5, u_8, \ldots, u_{3k-1}, u_{3k}\}$. Then $D_1$ and $D_2$ are $\gamma_s$-sets. But $D_1$ is independent and $D_2$ is not independent. Therefore $D_1$ is not a chromatic strong dominating set and $D_2$ is a $\gamma_s^c$-set.

Thus, $P_{3k+1}$ is in $(\gamma_s \cong \gamma_s^c)$-class.

Case(iii): Let $n \equiv 2 \pmod{3}$. Let $n = 3k + 2$. Then $\gamma_s(P_{3k+2}) = k + 1$. Let $D$ be a $\gamma_s$-set. Suppose $D$ is not independent. Let $v_i, v_{i+1} \in D$.

Sub Case(i): Let $i = 3l$. Consider the path $P_1 : u_1, u_2, \ldots, u_{3l-2}$ and the path $P_2 : u_{3l+3}, \ldots, u_{3k+2}$. Then $\gamma_s(P_1) = \left\lceil \frac{3l-2}{3} \right\rceil = l$ and $\gamma_s(P_2) = \left\lceil \frac{3(k-l)}{3} \right\rceil = k - l$. Therefore $\gamma_s(P_n) = \gamma_s(P_1) + 2 + \gamma_s(P_2) = l + 2 + k - l = k + 2$, which is a contradiction.

Sub Case(ii): Let $i = 3l + 1$. Consider the path $P_1 : u_1, u_2, \ldots, u_{3l-1}$ and the path $P_2 : u_{3l+4}, \ldots, u_{3k+2}$. Then $\gamma_s(P_1) = \left\lceil \frac{3l-1}{3} \right\rceil = l$ and $\gamma_s(P_2) =$
\[
\left\lceil \frac{3(k-l)-1}{3} \right\rceil = k - l.
\]
Therefore \(\gamma_s(P_n) = \gamma_s(P_1) + 2 + \gamma_s(P_2) = l + 2 + k - l = k + 2\), which is a contradiction.

**Sub Case(iii):** Let \(i = 3l + 2\). Consider the path \(P_1 : u_1, u_2, ..., u_{3l}\) and the path \(P_2 : u_{3l+5}, ..., u_{3k+2}\). Then \(\gamma_s(P_1) = \left\lceil \frac{3l}{3} \right\rceil = l\) and \(\gamma_s(P_2) = \left\lceil \frac{3(k-l)-2}{3} \right\rceil = k - l\). Therefore \(\gamma_s(P_n) = \gamma_s(P_1) + 2 + \gamma_s(P_2) = l + 2 + k - l = k + 2\), which is a contradiction. Therefore \(D\) is independent. Hence, \(D\) is not chromatic strong dominating set. Therefore \(P_{3k+2}\) belongs to \((\gamma_s \neq \gamma_s^c)\)-class. Hence, the theorem.

**Theorem 5.2.9** The cycle \(C_n\) of \((\gamma_s \neq \gamma_s^c)\)-class if and only if \(n\) is odd or \(n \equiv 0 \text{ or } 2 \pmod{3}\), when \(n\) is even.

**Proof:** Let \(C_n : u_1, u_2, u_3, ..., u_n\) be a cycle on \(n\)-vertices.

**Case(i):** Let \(n\) be odd. \(\gamma_s(C_n) = \left\lceil \frac{n}{3} \right\rceil < n = \gamma_s^c(C_n)\) when \(n\) is odd. Thus, no \(\gamma_s\)-set is a chromatic strong dominating set. Hence \(C_n\) of \((\gamma_s \neq \gamma_s^c)\)-class.

**Case(ii):** Let \(n\) be even and \(n \equiv 0 \pmod{3}\). Let \(n = 3k\), let \(D\) be a \(\gamma_s\)-set. Then \(D_1 = \{u_2, u_5, u_8, ..., u_{3k-1}\}, D_2 = \{u_3, u_6, u_9, ..., u_{3k}\}\) and \(D_3 = \{u_1, u_4, u_7, ..., u_{3k-2}\}\) are the only \(\gamma_s\)-sets. But all \(D_i\) are independent. Therefore \(\chi(<D_i>) = 1 \neq 2 = \chi(C_n)\) for all \(i\). Thus, \(C_n\) is of \((\gamma_s \neq \gamma_s^c)\)-class.

**Case(iii):** Let \(n\) be even and \(n \equiv 2 \pmod{3}\). Let \(n = 3k+2\). Then \(\gamma_s(C_n) = \)
\[ \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{3k+2}{3} \right\rceil = k + 1. \] Let \( D \) be a \( \gamma_s \)-set. Suppose \( D \) is not independent.

Let \( v_i, v_{i+1} \in D \). Consider the path \( P_1 : v_{i+3}, v_{i+4}, ..., v_n, v_1, v_2, v_3, ..., v_{i-2} \) of length \( n - 4 \). Therefore \( \gamma_s(P_1) = \left\lceil \frac{n-4}{3} \right\rceil = \left\lceil \frac{3k+2-4}{3} \right\rceil = \left\lceil \frac{3k-2}{3} \right\rceil = k. \)

Therefore \( \gamma_s(C_n) = \gamma_s(P_1) + 2 = k + 2 \), which is a contradiction. Therefore \( D \) is independent. Therefore \( \chi(< D >) = 1 \neq \chi(C_n) = 2 \). Therefore every \( \gamma_s \)-set is not a chromatic strong dominating set. Therefore \( C_n \) is in \( (\gamma_s \neq \gamma_s^c) \)-class.

**Case(iv)**: Let \( n \equiv 1 \pmod{3} \). Let \( n = 3k + 1 \). Then \( \gamma_s(C_{3k+1}) = k + 1 \).

Let \( D_1 = \{u_2, u_5, u_8, ..., u_{3k-1}, u_{3k+1}\} \) and \( D_1 \) is independent.

Thus, \( \chi(< D_1 >) = 1 \). But \( \chi(G) = 2 \). Therefore \( D_1 \) is not chromatic strong dominating set. Let \( D_2 = \{u_2, u_5, ..., u_{3k-1}, u_{3k}\} \) and \( D_2 \) is not independent.

Thus, \( \chi(< D_2 >) = 2 \) and \( \chi(G) = 2 \). Therefore \( D_2 \) is a chromatic strong dominating set. Thus \( C_n \) is in \( (\gamma_s \cong \gamma_s^c) \)-class.

\[ \Box \]

**Corollary 5.2.10** \( C_n \) is of \( (\gamma_s \cong \gamma_s^c) \)-class if and only if \( n \) is even and \( n \equiv 1 \pmod{3} \).

**Proposition 5.2.11** Let \( G = K_{m,n} \), \( m \neq n \). Then \( G \) is of \( (\gamma_s \neq \gamma_s^c) \)-class.

**Proof:** Let \( G = K_{m,n} \) and \( m < n \). Let \( (V_1, V_2) \) be the bipartition of \( V(G) \) and \( |V_1| < |V_2| \). Then \( V_1 \) is the unique \( \gamma_s \)-set in \( G \) and \( V_1 \) is a independent.
Therefore \(\chi(< V_1 >) = 1 \neq \chi(G)\), which means \(V_1\) is not a \(\gamma_s^c\)-set. Thus, \(G\) is in \((\gamma_s \neq \gamma_s^c)\)-class.

\[\text{Proposition 5.2.12}\]

Let \(G = K_{n,n}\) where \(m = n \geq 3\). Then \(G\) is of \((\gamma_s \equiv \gamma_s^c)\)-class.

\[\text{Proof:}\]

Let \(G = K_{n,n}\) where \(n \geq 3\). Let \((V_1, V_2)\) be the bipartition of \(V(G)\) where \(u \in V_1\) and \(v \in V_2\). Then every \(\gamma_s\)-set of \(K_{n,n}, n \geq 3\) is \(\{u, v\}\), where \(uv \in E(K_{n,n})\). Therefore \(\gamma_s = 2\) and \(\chi(< \{u, v\} >) = 2 = \chi(G)\). That is \(\{u, v\}\) is a \(\gamma_s\)-set. Thus every \(\gamma_s\)-set is \(\gamma_s^c\)-set. Therefore \(G\) is in \((\gamma_s \equiv \gamma_s^c)\)-class.

\[\text{Observation 5.2.13}\]

Let \(G = K_{1,n-1}, n \geq 3\) then \(G\) is in \((\gamma_s \neq \gamma_s^c)\)-class.

\[\text{Proof:}\]

Let \(G = K_{1,n-1}\). Then \(\gamma_s(K_1, n - 1) = 1\). Let \(u\) be a full degree vertex of \(G\). Then \(D = \{u\}\) is a unique \(\gamma_s\)-set. But \(\chi(< D >) = 1 \neq 2 = \chi(G)\). Therefore \(D\) is not a chromatic strong dominating set. Therefore \(G\) is in \((\gamma_s \neq \gamma_s^c)\)-class.

\[\text{Proposition 5.2.14}\]

Let \(G = K_n, n \geq 2\). Then \(G\) is in \((\gamma_s \neq \gamma_s^c)\)-class.

\[\text{Proof:}\]

Let \(G = K_n, n \geq 2\). Then \(\gamma_s(K_n) = 1\). Therefore \(\gamma_s\)-sets are singleton sets. Therefore \(\chi(< D >) = 1 \neq n = \chi(G)\) for every \(\gamma_s\)-set. Thus, \(D\) is not a chromatic strong dominating set. Hence \(G\) is in \((\gamma_s \neq \gamma_s^c)\)-class.
Theorem 5.2.15 Let $T$ be a tree. If there exist support vertices $u,v$ such that $d(u_i) = 2$ for all $u_i$ on the path joining $u$ and $v$ and $d(u,v) \equiv 0 \pmod{3}$. Then $u,v$ belong to every $\gamma_s$-set of $T$.

Proof: Let $u,v$ be a pair of vertices in $T$ such that $d(u,v) \equiv 0 \pmod{3}$ and $d(u_i) = 2$ for all $u_i$ on the path joining $u$ and $v$. Suppose there is a $\gamma_s$-set $D$ such that $u \notin D$ or $v \notin D$. Let $u \notin D$. Then $N(u)$ has exactly one pendent vertex, say $w$. Then $w \in D$. Let $u = u_0, u_1, u_2, u_3, \ldots, u_r = v$. Then $r \equiv 0 \pmod{3}$. Since $D$ is a $\gamma_s$-set and $u_0 = u \notin D$, assume that $u_2, u_5, \ldots, u_{r-1} \in D$. Let $r = 3k$. Then $|\{u_2, u_5, \ldots, u_{r-1}\}| = k$.

If $v \notin D$, then $N(v)$ has exactly one pendent vertex $x$, and $x \in D$. $D_2 = (D - \{x, w, u_2, u_5, \ldots, u_{r-1}\}) \cup \{u, v, u_3, u_6, \ldots, u_{r-3}\}$ is a strong dominating set and $|D_2| = |D| - 1$, which is a contradiction. If $v \in D$, then $N(u)$ has exactly one pendent vertex $x$, and $x \in D$. $D_2 = (D - \{x, w, u_2, u_5, \ldots, u_{r-1}\}) \cup \{u, v, u_3, u_6, \ldots, u_{r-3}\}$ is a strong dominating set and $|D_2| = |D| - 1$, which is a contradiction. Hence, $u \in D$ and $v \in D$. 

Proposition 5.2.16 Let $P$ be the set of all pendent vertices of a tree $T$. If $T$ has two vertices $u,v$ such that $|N(u) \cap P| \geq 2$, $|N(v) \cap P| \geq 2$, and $uv \in E(T)$, Then $T$ belongs to $(\gamma_s \equiv \gamma_c^s)$-class.

Proof: Let $|N(u) \cap P| \geq 2$ and $|N(v) \cap P| \geq 2$. Then $u,v$ belong to every $\gamma_s$-set $D$ of $T$. Since $uv \in E(T)$, $\chi(<D>) = 2$, $D$ is a $\gamma_c^s$-set.
Hence $T$ is in $(\gamma_s \equiv \gamma^c_s)$-class.

**Theorem 5.2.17** Let $T$ be a tree in which $d(u, v) \equiv 0 \pmod{3}$ for some support vertices and $d(u_i) = 2$ for all $u_i$ on the path joining $u$ and $v$. If there exists a support vertex $w$ such that $vw \in E(T)$, and either $|N(w) \cap P| \geq 2$ or for some support vertex $x$, $d(w, x) \equiv 0 \pmod{3}$ and $d(x_i) = 2$ for all vertices $x_i$ on the path joining $w$ and $x$, then $T$ belongs to $(\gamma_s \equiv \gamma^c_s)$-class.

**Proof:** Let $T$ be a tree in which for some support vertices $u, v, w$, $d(u, v) \equiv 0 \pmod{3}$, $d(u_i) = 2$ for all vertices $u_i$ on the path joining $u, v$ and $vw \in E(T)$. Suppose that either $|N(w) \cap P| \geq 2$ or there exists a support vertex $x$ such that $d(w, x) \equiv 0 \pmod{3}$ and $d(x_i) = 2$ for all vertices $x_i$ on the path joining $w$ and $x$. Let $D$ be a $\gamma_s$-set. By Theorem 5.2.16, $u, v \in D$. If $|N(w) \cap P| \geq 2$, then $w \in D$. If there exists a support vertex $x$ such that $d(w, x) \equiv 0 \pmod{3}$ and $d(x_i) = 2$ for all vertices $x_i$ on the path joining $w$ and $x$, then by Theorem 5.2.16, $w, x \in D$. Thus, in both cases, $w \in D$. Since $vw \in E(T)$, $\chi(< D >) = 2$. Thus, $D$ is a $\gamma^c_s$-set. Hence $T$ is in $(\gamma_s \equiv \gamma^c_s)$-class. ■