CHAPTER - 1
MINIMAL QUOTIENT MAPPINGS

1.1 INTRODUCTION

Njastad [55] introduced the concept of an \( \alpha \)-sets and Mashhour et al [46] introduced \( \alpha \)-continuous mappings in topological spaces. The topological notions of semi-open sets and semi-continuity, and preopen sets and precontinuity were introduced by Levine [37] and Mashhour et al [47] respectively. After advent of these notions, Reilly [73] and Thivagar [36] obtained many interesting and important results on \( \alpha \)-continuity and \( \alpha \)-irresolute mappings in topological spaces. Lellis Thivagar [36] introduced the concepts of \( \alpha \)-quotient mappings and \( \alpha^* \)-quotient mappings in topological spaces. Maki [45] introduced the notion of minimal structures to this research world. The concepts of minimal structures (briefly m-structures) were developed by Popa and Noiri [66] in 2000.

In this chapter, we introduce a new class of minimal mappings called M-\( \alpha \)-continuous mappings and M-\( \alpha^* \)-quotient mappings in minimal spaces. At every places the new notions have been substantiated with suitable examples.

1.2 PRELIMINARIES

Definition 1.2.1 [66]
Let \( X \) be a nonempty set and \( \wp(X) \) the power set of \( X \). A subfamily \( m_x \) of \( \wp(X) \) is called a minimal structure (briefly, m-structure) on \( X \) if \( \emptyset \in m_x \) and \( X \in m_x \).
A set $X$ with an $m$-structure $m_x$ is called an $m$-space and is denoted by $(X, m_x)$. Each member of $m_x$ is said to be $m_x$-open and the complement of an $m_x$-open set is said to be $m_x$-closed.

Throughout this thesis, $(X, m_x)$, $(Y, m_y)$ and $(Z, m_z)$ (or $X$, $Y$ and $Z$) mean minimal spaces.

**Definition 1.2.2 [66]**

Let $X$ be a nonempty set and $m_x$ an $m$-structure on $X$. For a subset $A$ of $X$, the $m_x$-closure of $A$ and the $m_x$-interior of $A$ are defined as follows:

(i) $m_x$-Cl($A$) = $\bigcap\{ F : A \subseteq F, X - F \in m_x \}$,

(ii) $m_x$-Int($A$) = $\bigcup\{ U : U \subseteq A, U \in m_x \}$.

**Lemma 1.2.3 [66]**

Let $X$ be a nonempty set and $m_x$ a minimal structure on $X$. For subsets $A$ and $B$ of $X$, the following properties hold:

(i) $m_x$-Cl($X - A$) = $X - m_x$-Int($A$) and $m_x$-Int($X - A$) = $X - m_x$-Cl($A$),

(ii) If $(X - A) \in m_x$, then $m_x$-Cl($A$) = $A$ and if $A \in m_x$, then $m_x$-Int($A$) = $A$,

(iii) $m_x$-Cl($\emptyset$) = $\emptyset$, $m_x$-Cl($X$) = $X$, $m_x$-Int($\emptyset$) = $\emptyset$ and $m_x$-Int($X$) = $X$,

(iv) If $A \subseteq B$, then $m_x$-Cl($A$) $\subseteq$ $m_x$-Cl($B$) and $m_x$-Int($A$) $\subseteq$ $m_x$-Int($B$),

(v) $A \subseteq m_x$-Cl($A$) and $m_x$-Int($A$) $\subseteq$ $A$. 
(vi) \( m_\kappa\text{-Cl}(m_\kappa\text{-Cl}(A)) = m_\kappa\text{-Cl}(A) \) and \( m_\kappa\text{-Int}(m_\kappa\text{-Int}(A)) = m_\kappa\text{-Int}(A) \).

**Definition 1.2.4 [66]**

A minimal structure \( m_\kappa \) on a nonempty set \( X \) is said to have property \( \mathfrak{B} \) if the union of any family of subsets belonging to \( m_\kappa \) belongs to \( m_\kappa \).

**Lemma 1.2.5 [66]**

The following are equivalent for the minimal space \((X, m_\kappa)\).

(i) \( m_\kappa \) have property \( \mathfrak{B} \).

(ii) If \( m_\kappa\text{-Int}(E) = E \), then \( E \in m_\kappa \).

(iii) If \( m_\kappa\text{-Cl}(F) = F \), then \( F^c \in m_\kappa \).

**Definition 1.2.6**

Let \( S \) be a subset of \( X \). Then \( S \) is said to be

(i) \( m_\kappa\text{-}\alpha\text{-open} \) [51] if \( S \subseteq m_\kappa\text{-Int}(m_\kappa\text{-Cl}(m_\kappa\text{-Int}(S))) \).

(ii) \( m_\kappa\text{-semi-open} \) [50] if \( S \subseteq m_\kappa\text{-Cl}(m_\kappa\text{-Int}(S)) \).

(iii) \( m_\kappa\text{-preopen} \) [49] if \( S \subseteq m_\kappa\text{-Int}(m_\kappa\text{-Cl}(S)) \).

The family of all \( m_\kappa\text{-}\alpha\text{-open} \) [resp. \( m_\kappa\text{-semi-open}, m_\kappa\text{-preopen} \)] sets of \( X \) is denoted by \( m_\kappa\text{-}\alpha\text{O}(X) \) [resp. \( m_\kappa\text{-SO}(X), m_\kappa\text{-PO}(X) \)].

**Remark 1.2.7**

(i) Every \( m_\kappa\text{-open} \) set is \( m_\kappa\text{-}\alpha\text{-open} \) but not conversely.

(ii) A \( m_\kappa\text{-semi-open} \) [\( m_\kappa\text{-preopen} \)] set need not be \( m_\kappa\text{-}\alpha\text{-open} \).

**Example 1.2.8**
Let \( Y = \{p, q, r\} \) and \( m_y = \{\phi, Y, \{p\}, \{q\}, \{p, q\}\} \). We have
\[ m_y^{-\alpha}O(Y) = \{\phi, Y, \{p\}, \{q\}, \{p, q\}\}; \]
\[ m_y^{-SO}(Y) = \{\phi, Y, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, r\}\} \]
and \( m_y^{-PO}(Y) = \{\phi, Y, \{p\}, \{q\}, \{p, q\}\} \).

1.3 **M-\( \alpha \)-CONTINUOUS MAPPINGS**

**Definition 1.3.1**

Let \( f : X \rightarrow Y \) be a mapping. Then \( f \) is said to be

(i) \( M^{*} \)-continuous [49] if the inverse image of each \( m_y \)-open set in \( Y \) is \( m_x \)-open set in \( X \).

(ii) \( M-\alpha \)-continuous [resp. \( M \)-semi-continuous, \( M \)-pre-continuous] if the inverse image of each \( m_y \)-open set in \( Y \) is an \( m_x-\alpha \)-open set [resp. \( m_x \)-semi-open set, \( m_x \)-preopen set] in \( X \).

(iii) \( M-\alpha \)-open [resp. \( M \)-semi-open, \( M \)-preopen, \( M \)-open] if the image of each \( m_x \)-open set in \( X \) is an \( m_y-\alpha \)-open [resp. \( m_y \)-semi-open, \( m_y \)-preopen, \( m_y \)-open] set in \( Y \).

**Theorem 1.3.2**

\( A \) is \( m_x \)-semi-open set in \( X \) if and only if \( m_x-\text{Cl}(A) = m_x-\text{Cl}(m_x-\text{Int}(A)) \).

**Proof**
Suppose $A$ is $m_x$-semi-open set. Then $A \subseteq m_x\text{-Cl}(m_x\text{-Int}(A))$ and $m_x\text{-Cl}(A) \subseteq m_x\text{-Cl}(m_x\text{-Int}(A))$. On the other hand, we have $m_x\text{-Int}(A) \subseteq A$ and hence $m_x\text{-Cl}(m_x\text{-Int}(A)) \subseteq m_x\text{-Cl}(A)$.

Conversely, we have $A \subseteq m_x\text{-Cl}(A)$ and $m_x\text{-Cl}(A) = m_x\text{-Cl}(m_x\text{-Int}(A))$. Therefore $A \subseteq m_x\text{-Cl}(m_x\text{-Int}(A))$. Hence $A$ is $m_x$-semi-open set.

**Theorem 1.3.3**

Let $A$ be a subset of $X$. Then $A$ is $m_x$-$\alpha$-open set in $X$ if and only if $A$ is $m_x$-semi-open set and $m_x$-preopen set in $X$.

**Proof**

Let $A \in m_x\text{-}\alpha O(X)$. By the definition of $m_x$-$\alpha$-open set, we have $A \subseteq m_x\text{-Int}(m_x\text{-Cl}(A))$ and $A \subseteq m_x\text{-Cl}(m_x\text{-Int}(A))$. Therefore $A \in m_x\text{-PO}(X)$ and $A \in m_x\text{-SO}(X)$. Hence $A \in m_x\text{-SO}(X) \cap m_x\text{-PO}(X)$.

Conversely, let $A \in m_x\text{-SO}(X)$. Then by Theorem 3.2, $m_x\text{-Cl}(A) = m_x\text{-Cl}(m_x\text{-Int}(A))$. Moreover let $A \in m_x\text{-PO}(X)$. Then $A \subseteq m_x\text{-Int}(m_x\text{-Cl}(A))$. Hence $A \subseteq m_x\text{-Int}(m_x\text{-Cl}(m_x\text{-Int}(A)))$. It shows that $A \in m_x\text{-}\alpha O(X)$.

**Theorem 1.3.4**

The mapping $f : X \to Y$ is $M$-$\alpha$-continuous if and only if it is $M$-semi-continuous and $M$-precontinuous.

**Proof**
Let A be an \( m_y \)-open set in Y. Since f is \( M-\alpha \)-continuous, 
\( f^{-1}(A) \in m_x-\alpha O(X) = m_x-SO(X) \cap m_x-PO(X) \). Since \( f^{-1}(A) \in m_x-SO(X) \) and \( f^{-1}(A) \in m_x-PO(X) \), f is \( M \)-semi-continuous and \( M \)-precontinuous.

Conversely, let f be \( M \)-semi-continuous and \( M \)-precontinuous mapping. Let \( V \) be an \( m_y \)-open set in Y. Then \( f^{-1}(V) \in m_x-SO(X) \) and \( f^{-1}(V) \in m_x-PO(X) \). Therefore \( f^{-1}(V) \in m_x-SO(X) \cap m_x-PO(X) = m_x-\alpha O(X) \). Hence f is \( M-\alpha \)-continuous.

1.4 M-\( \alpha \)-IRRESOLUTE MAPPINGS

Definition 1.4.1

Let \( S \) be a subset of X. Then \( S \) is said to be

(i) \( m_x \)-preclosed if \( m_x-Cl(m_x-Int(S)) \subseteq S \).

(ii) \( m_x \)-\( \alpha \)-closed if \( m_x-Cl(m_x-Int(m_x-Cl(S))) \subseteq S \).

(iii) \( m_x \)-semi-closed if \( m_x-Int(m_x-Cl(S)) \subseteq S \).

The family of all \( m_x \)-\( \alpha \)-closed [resp. \( m_x \)-semi-closed, \( m_x \)-preclosed] sets of X is denoted by \( m_x-\alpha C(X) \) [resp. \( m_x-SC(X) \), \( m_x-PC(X) \)].

The complement of \( m_x \)-\( \alpha \)-open [resp. \( m_x \)-semi-open, \( m_x \)-preopen] set is \( m_x \)-\( \alpha \)-closed [resp. \( m_x \)-semi-closed, \( m_x \)-preclosed].

Definition 1.4.2

Let \( f : X \to Y \) be a mapping. Then f is said to be \( M-\alpha \)-irresolute (resp. \( M \)-semi-irresolute, \( M \)-preirresolute) if the inverse image of every
m_γ-α-open [resp. m_γ-semi-open, m_γ-preopen] set in Y is an m_α-α-open [resp. m_α-semi-open, m_α-preopen] set in X.

**Theorem 1.4.3**

A mapping f : X → Y is M-semi-irresolute if and only if for every m_γ-semi-closed subset A of Y, f^(1)(A) is m_α-semi-closed in X.

**Proof**

If f is M-semi-irresolute, then for every m_γ-semi-open subset B of Y, f^(1)(B) is m_α-semi-open in X. If A is any m_γ-semi-closed subset of Y, then Y – A is m_γ-semi-open. Thus f^(1)(Y – A) is m_α-semi-open but f^(1)(Y – A) = X – f^(1)(A) so that f^(1)(A) is m_α-semi-closed in X.

Conversely, if, for all m_γ-semi-closed subsets A of Y, f^(1)(A) is m_α-semi-closed in X and if B is any m_γ-semi-open subset of Y, then Y – B is m_γ-semi-closed. Also f^(1)(Y – B) = X – f^(1)(B) is m_α-semi-closed. Thus f^(1)(B) is m_α-semi-open in X. Hence f is M-semi-irresolute.

**Theorem 1.4.4**

Let f and g be two mappings. If f : X → Y and g : Y → Z are both M-semi-irresolute then gof : X → Z is M-semi-irresolute.

**Proof**

If A ⊆ Z is m_α-semi-open, then g^(−1)(A) is m_γ-semi-open set in Y because g is M-semi-irresolute. Consequently since f is M-semi-irresolute, f^(1)(g^(−1)(A)) = (gof)^(−1)(A) is m_α-semi-open set in X. Hence gof is M-semi-irresolute mapping.

**Corollary 1.4.5**
If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are both $M$-$\alpha$-irresolute mappings then $gof : X \rightarrow Z$ is $M$-$\alpha$-irresolute.

**Corollary 1.4.6**

If the mapping $f : X \rightarrow Y$ is $M$-$\alpha$-irresolute and the mapping $g : Y \rightarrow Z$ is $M$-$\alpha$-continuous then $gof : X \rightarrow Z$ is $M$-$\alpha$-continuous mapping.

**Corollary 1.4.7**

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two mappings. Then

(i) if $f$ is $M$-semi-irresolute and $g$ is $M$-semi-continuous, then $gof$ is $M$-semi-continuous mapping.

(ii) if $f$ is $M$-preirresolute and $g$ is $M$-precontinuous, then $gof$ is $M$-precontinuous mapping.

**Theorem 1.4.8**

If the mapping $f : X \rightarrow Y$ is both $M$-semi-irresolute and $M$-preirresolute then $f$ is $M$-$\alpha$-irresolute mapping.

**Proof**

It is obvious.

**1.5 M-$\alpha$-QUOTIENT MAPPINGS**

**Definition 1.5.1**

Let $f : X \rightarrow Y$ be a surjective mapping. Then $f$ is said to be $M$-quotient provided a subset $S$ of $Y$ is $m_y$-open in $Y$ if and only if $f^{-1}(S)$ is $m_x$-open in $X$. 
Definition 1.5.2

Let \( f : X \rightarrow Y \) be a surjective mapping. Then \( f \) is said to be

(i) an \( M\)-\( \alpha \)-quotient if \( f \) is \( M\)-\( \alpha \)-continuous and \( f^{-1}(V) \) is \( m_\kappa \)-open in \( X \) implies \( V \) is an \( m_\gamma \)-\( \alpha \)-open set in \( Y \).

(ii) a \( M\)-semi-quotient if \( f \) is \( M\)-semi-continuous and \( f^{-1}(V) \) is \( m_\kappa \)-open in \( X \) implies \( V \) is a \( m_\gamma \)-semi-open set in \( Y \).

(iii) a \( M\)-prequotient if \( f \) is \( M\)-precontinuous and \( f^{-1}(V) \) is \( m_\kappa \)-open in \( X \) implies \( V \) is a \( m_\gamma \)-preopen set in \( Y \).

Example 1.5.3

Let \( X = \{a, b, c\} \) and \( m_\kappa = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \). We have \( m_\kappa\)-\( \alpha \)O(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}; \( m_\kappa\)-SO(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} \) and \( m_\kappa\)-PO(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}.

Let \( Y = \{p, q, r\} \) and \( m_\gamma = \{\emptyset, Y, \{p\}, \{q\}, \{p, q\}\} \). We have \( m_\gamma\)-\( \alpha \)O(Y) = \{\emptyset, Y, \{p\}, \{q\}, \{p, q\}\}; \( m_\gamma\)-SO(Y) = \{\emptyset, Y, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, r\}\} \) and \( m_\gamma\)-PO(Y) = \{\emptyset, Y, \{p\}, \{q\}, \{p, q\}\}.

Define \( f : X \rightarrow Y \) by \( f(a) = p; f(b) = q; f(c) = r \). Since the inverse image of each \( m_\gamma \)-open in \( Y \) is \( m_\kappa \)-\( \alpha \)-open in \( X \), clearly \( f \) is \( M\)-\( \alpha \)-continuous and an \( M\)-\( \alpha \)-quotient mapping.

Theorem 1.5.4

If the mapping \( f : X \rightarrow Y \) is surjective, \( M\)-\( \alpha \)-continuous and \( M\)-\( \alpha \)-open then \( f \) is an \( M\)-\( \alpha \)-quotient mapping.
Proof

Suppose \( f^{-1}(V) \) is any \( m_x \)-open set in \( X \). Then \( f(f^{-1}(V)) \) is an \( m_y \)-\( \alpha \)-open set in \( Y \) as \( f \) is \( M \)-\( \alpha \)-open. Since \( f \) is surjective, \( f(f^{-1}(V)) = V \). Thus \( V \) is an \( m_y \)-\( \alpha \)-open set in \( Y \). Hence \( f \) is \( M \)-\( \alpha \)-quotient mapping.

Theorem 1.5.5

If the mapping \( f : X \to Y \) is \( M \)-open surjective and \( M \)-\( \alpha \)-irresolute, and the mapping \( g : Y \to Z \) is an \( M \)-\( \alpha \)-quotient then \( g \circ f : X \to Z \) is an \( M \)-\( \alpha \)-quotient mapping.

Proof

Let \( V \) be any \( m_z \)-open set in \( Z \). Since \( g \) is \( M \)-\( \alpha \)-continuous, \( g^{-1}(V) \in m_y \)-\( \alpha \)O(Y). Since \( f \) is \( M \)-\( \alpha \)-irresolute, \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \in m_x \)-\( \alpha \)O(X). Thus \( g \circ f \) is \( M \)-\( \alpha \)-continuous. Also suppose \( f^{-1}(g^{-1}(V)) \) is \( m_x \)-open set in \( X \). Since \( f \) is \( M \)-open, \( f(f^{-1}(g^{-1}(V))) \) is \( m_y \)-open set in \( Y \). Since \( f \) is surjective, \( f(f^{-1}(g^{-1}(V))) = g^{-1}(V) \) and since \( g \) is \( M \)-\( \alpha \)-quotient, \( V \in m_z \)-\( \alpha \)O(Z). Hence \( g \circ f \) is an \( M \)-\( \alpha \)-quotient.

Corollary 1.5.6

If the mapping \( f : X \to Y \) is \( M \)-open surjective and \( M \)-semi-[M-pre] irresolute and the mapping \( g : Y \to Z \) is \( M \)-semi-[M\((1,2)^*\)-pre] quotient then \( g \circ f : X \to Z \) is \( M \)-semi-[M-pre] quotient mapping.

Theorem 1.5.7

A mapping \( f : X \to Y \) is an \( M \)-\( \alpha \)-quotient if and only if it is \( M \)-semi-quotient mapping and \( M \)-prequotient mapping.
Proof

Let $V$ be any $m_y$-open set in $Y$. Since $f$ is $M$-$\alpha$-quotient, $f^{-1}(V) \in m_x\alpha O(X) = m_x SO(X) \cap m_x PO(X)$. Thus $f$ is both $M$-semi-continuous and $M$-precontinuous. Also suppose $f^{-1}(V)$ is an $m_x$-open set in $X$. Since $f$ is $M$-$\alpha$-quotient, $V \in m_y\alpha O(Y) = m_y SO(Y) \cap m_y PO(Y)$. Thus $V$ is both $m_y$-semi-open set and $m_y$-preopen set in $Y$. Hence $f$ is $M$-semi-quotient and $M$-prequotient.

Conversely, since $f$ is $M$-semi-quotient and $M$-prequotient, $f$ is $M$-semi-continuous and $M$-precontinuous. Hence $f$ is $M$-$\alpha$-continuous. Also suppose $f^{-1}(V)$ is an $m_x$-open set in $X$. By Definition 1.5.2, $V \in m_y SO(Y) \cap m_y PO(Y) = m_y\alpha O(Y)$. Thus $f$ is $M$-$\alpha$-quotient mapping.

Definition 1.5.8

(i) Let $f : X \to Y$ be a surjective and $M$-$\alpha$-continuous mapping. Then $f$ is said to be strongly $M$-$\alpha$-quotient provided a subset $S$ of $Y$ is $m_y$-open set in $Y$ if and only if $f^{-1}(S)$ is an $m_x$-$\alpha$-open set in $X$.

(ii) Let $f : X \to Y$ be a surjective and $M$-semi-continuous mapping. Then $f$ is said to be strongly $M$-semi-quotient provided a subset $S$ of $Y$ is $m_y$-open set in $Y$ if and only if $f^{-1}(S)$ is $m_x$-semi-open set in $X$.

(iii) Let $f : X \to Y$ be a surjective and $M$-precontinuous mapping. Then $f$ is said to be strongly $M$-prequotient
provided a subset $S$ of $Y$ is $m_y$-open set in $Y$ if and only if $f^{-1}(S)$ is $m_x$-preopen set in $X$.

**Theorem 1.5.9**

If the mapping $f : X \rightarrow Y$ is strongly $M$-semi-quotient and strongly $M$-prequotient then $f$ is strongly $M$-$\alpha$-quotient mapping.

**Proof**

Since $f$ is $M$-semi-continuous and $M$-precontinuous, by Theorem 1.3.4, $f$ is $M$-$\alpha$-continuous. Also let $V$ be an $m_y$-open set in $Y$. By Definition 1.3.1 and Theorem 1.3.3, $f^{-1}(V) \in m_x$-SO$(X) \cap m_x$-PO$(X) = m_x$-$\alpha$O$(X)$.

Conversely, let $f^{-1}(V) \in m_x$-$\alpha$O$(X)$. Then $m_x$-$\alpha$O$(X) = m_x$-SO$(X) \cap m_x$-PO$(X)$. Since $f$ is strongly $M$-semi-quotient and strongly $M$-prequotient, $V$ is $m_y$-open set in $Y$. Hence $f$ is strongly $M$-$\alpha$-quotient mapping.

**1.6. $M$-$\alpha^*$-QUOTIENT MAPPINGS**

**Definition 1.6.1**

Let $f : X \rightarrow Y$ be a surjective mapping. Then $f$ is said to be

(i) $M$-$\alpha^*$-quotient if $f$ is $M$-$\alpha$-irresolute and $f^{-1}(S)$ is $m_x$-$\alpha$-open set in $X$ implies $S$ is $m_y$-open set in $Y$.

(ii) $M$-semi-$^*$quotient if $f$ is $M$-semi-irresolute and $f^{-1}(S)$ is $m_x$-semi-open set in $X$ implies $S$ is $m_y$-open set in $Y$. 
(iii) M-pre*quotient if f is M-preirresolute and \( f^{-1}(S) \) is m\(_x\)-preopen set in X implies S is m\(_y\)-open set in Y.

**Definition 1.6.2**

Let \( f : X \rightarrow Y \) be a mapping. Then \( f \) is said to be strongly M-\( \alpha \)-open if the image of every m\(_x\)-\( \alpha \)-open set in X is an m\(_y\)-\( \alpha \)-open set in Y.

**Example 1.6.3**

Consider the Example 1.5.3. Clearly \( f \) is M-\( \alpha \)-irresolute and M-\( \alpha \)*-quotient mapping.

**Example 1.6.4**

Consider the Example 1.5.3. Clearly \( f \) is strongly M-\( \alpha \)-open mapping.

**Theorem 1.6.5**

Let the mapping \( f : X \rightarrow Y \) be surjective strongly M-\( \alpha \)-open and M-\( \alpha \)-irresolute, and the mapping \( g : Y \rightarrow Z \) be an M-\( \alpha \)*-quotient. Then \( gof : X \rightarrow Z \) is an M-\( \alpha \)*-quotient mapping.

**Proof**

Let V be any m\(_x\)-\( \alpha \)-open set in Z. Then \( g^{-1}(V) \) is an m\(_y\)-\( \alpha \)-open set in Y as \( g \) is an M-\( \alpha \)*-quotient mapping. Then \( f^{-1}(g^{-1}(V)) = (gof)^{-1}(V) \) is an m\(_x\)-\( \alpha \)-open set in X as \( f \) is M-\( \alpha \)-irresolute. This shows that \( gof \) is M-\( \alpha \)-irresolute. Also suppose \( (gof)^{-1}(V) = f^{-1}(g^{-1}(V)) \) is an m\(_x\)-\( \alpha \)-open set in X. Since \( f \) is strongly M-\( \alpha \)-open,
f(f^{-1}(g^{-1}(V))) is an \(m_y-\alpha\)-open set in \(Y\). Since \(f\) is surjective, \(f(f^{-1}(g^{-1}(V))) = g^{-1}(V)\) is an \(m_y-\alpha\)-open set in \(Y\). Since \(g\) is an \(M-\alpha^*\)-quotient mapping, \(V\) is \(m_z\)-open in \(Z\). Hence the theorem.

**Theorem 1.6.6**

If the mapping \(f : X \to Y\) is \(M\)-semi-*quotient and \(M\)-prequotient then \(f\) is \(M-\alpha^*\)-quotient mapping.

**Proof**

Since \(f\) is \(M\)-semi-*quotient and \(M\)-pre*quotient, \(f\) is \(M\)-semi-irresolute and \(M\)-preirresolute. By Theorem 1.4.8., \(f\) is \(M-\alpha\)-irresolute. Also suppose \(f^{-1}(V) \in m_x-\alpha O(X)\). Then \(m_x-\alpha O(X) = m_x-SO(X) \cap m_x-PO(X)\). Therefore \(f^{-1}(V)\) is \(m_x\)-semi-open in \(X\) and \(f^{-1}(V)\) is \(m_x\)-preopen in \(X\). Since \(f\) is \(M\)-semi-*quotient and \(M\)-pre*quotient, by Definition 1.6.1, \(V\) is \(m_y\)-open set in \(Y\). Thus \(f\) is \(M-\alpha^*\)-quotient mapping.

**Theorem 1.6.7**

Let \(f : X \to Y\) be a strongly \(M-\alpha\)-quotient and \(M-\alpha\)-irresolute mapping and \(g : Y \to Z\) be an \(M-\alpha^*\)-quotient mapping then \(gof : X \to Z\) is an \(M-\alpha^*\)-quotient mapping.

**Proof**

Let \(V \in m_z-\alpha O(Z)\). Since \(g\) is \(M-\alpha\)-irresolute, \(g^{-1}(V) \in m_y-\alpha O(Y)\). Since \(f\) is \(M-\alpha\)-irresolute, \(f^{-1}(g^{-1}(V)) = (gof)^{-1}(V) \in m_x-\alpha O(X)\). Thus \(gof\) is \(M-\alpha\)-irresolute. Also suppose \((gof)^{-1}(V) = f^{-1}(g^{-1}(V)) \in m_x-\alpha O(X)\). Since \(f\) is
strongly M-\(\alpha\)-quotient, \(g^{-1}(V)\) is \(m_\gamma\)-open set in \(Y\). Then \(g^{-1}(V) \in m_\gamma-\alpha O(Y)\). Since \(g\) is M-\(\alpha^*\)-quotient, \(V\) is \(m_\gamma\)-open set in \(Z\). Hence \(g \circ f\) is M-\(\alpha^*\)-quotient mapping.

1.7 COMPARISON

**Theorem 1.7.1**

Let \(f : X \to Y\) be surjective mapping. Then \(f\) is M-\(\alpha^*\)-quotient if and only if it is strongly M-\(\alpha\)-quotient mapping.

**Proof**

Let \(V\) be an \(m_\gamma\)-open set in \(Y\). Then \(V \in m_\gamma-\alpha O(Y)\). Since \(f\) is M-\(\alpha^*\)-quotient, \(f^{-1}(V) \in m_\alpha-\alpha O(X)\). Conversely, let \(f^{-1}(V) \in m_\alpha-\alpha O(X)\). Since \(f\) is M-\(\alpha^*\)-quotient, \(V\) is \(m_\gamma\)-open set in \(Y\). Hence \(f\) is strongly M-\(\alpha\)-quotient mapping.

Conversely, let \(V\) be \(m_\gamma\)-open set in \(Y\). Then \(V \in m_\gamma-\alpha O(Y)\). Since \(f\) is strongly M-\(\alpha\)-quotient, \(f^{-1}(V) \in m_\alpha-\alpha O(X)\). Thus \(f\) is M-\(\alpha\)-irresolute. Also since \(f\) is strongly M-\(\alpha\)-quotient, \(f^{-1}(V) \in m_\alpha-\alpha O(X)\) implies \(V\) is \(m_\gamma\)-open set in \(Y\). Hence \(f\) is M-\(\alpha^*\)-quotient mapping.

**Theorem 1.7.2**

If the mapping \(f : X \to Y\) is M-quotient then it is M-\(\alpha\)-quotient mapping.

**Proof**

Let \(V\) be an \(m_\gamma\)-open set in \(Y\). Since \(f\) is M-quotient, \(f^{-1}(V)\) is \(m_\alpha\)-open set in \(X\) and \(f^{-1}(V) \in m_\alpha-\alpha O(X)\). Hence \(f\) is M-\(\alpha\)-continuous.
Suppose $f^{-1}(V)$ is an $m_x$-open set in $X$. Since $f$ is M-quotient, $V$ is $m_y$-open set in $Y$. Then $V \in m_y$-$\alpha O(Y)$. Hence $f$ is M-$\alpha$-quotient mapping.

**Theorem 1.7.3**

If the mapping $f : X \to Y$ is M-$\alpha$-irresolute then it is M-$\alpha$-continuous mapping.

**Proof**

Let $A$ be $m_y$-open set in $Y$. Then $A \in m_y$-$\alpha O(Y)$. Since $f$ is M-$\alpha$-irresolute, $f^{-1}(A) \in m_x$-$\alpha O(X)$. It shows that $f$ is M-$\alpha$-continuous mapping.

**Theorem 1.7.4**

If the mapping $f : X \to Y$ is M-$\alpha^*$-quotient then it is M-$\alpha$-quotient mapping.

**Proof**

Let $f$ be M-$\alpha^*$-quotient. Then $f$ is M-$\alpha$-irresolute. We have $f$ is M-$\alpha$-continuous. Also suppose $f^{-1}(V)$ is an $m_x$-open in $X$. Then $f^{-1}(V) \in m_x$-$\alpha O(X)$. By assumption, $V$ is $m_y$-open set in $Y$. Therefore $V \in m_y$-$\alpha O(Y)$. Hence $f$ is M-$\alpha$-quotient mapping.

**Theorem 1.7.5**

Every M-$\alpha^*$-quotient mapping is M-$\alpha$-irresolute.

**Proof**

We obtain it from Definition 1.6.1.
Theorem 1.7.6

Every $M$-$\alpha$-quotient mapping is $M$-$\alpha$-continuous.

Proof

We obtain it from Definition 1.5.2.

Remark 1.7.7

The converses of Theorems 1.5.9 and 1.6.6 are not true as per the following example.

Example 1.7.8

Let $X=\{a, b, c\}$, $m_x=\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $Y=\{p, q, r\}$ and $m_y=\{\emptyset, Y, \{p\}, \{q\}, \{p, q\}\}$. Define $f : X \to Y$ by $f(a) = p; f(b) = q$ and $f(c) = r$. Clearly $f$ is $M$-$\alpha$-continuous and strongly $M$-$\alpha$-quotient mapping. Since $f^{-1}(\{p, r\}) = \{a, c\} \in m_x$-$SO(X)$ and $\{p, r\}$ is not $m_y$-open set in $Y$, $f$ is not strongly $M$-semi-quotient mapping. Moreover $f$ is $M$-$\alpha$-irresolute, $M$-$\alpha^*$-quotient and $M$-semi-irresolute mapping. Since $f^{-1}(\{q, r\}) = \{b, c\} \in m_x$-$SO(X)$ and $\{q, r\}$ is not $m_y$-open set in $Y$, $f$ is not $M$-semi-$\alpha^*$-quotient mapping.

Remark 1.7.9

The converses of Theorems 1.7.4 and 1.7.5 are not true as per the following example.

Example 1.7.10

Let $X = \{a, b, c\}$, $m_x = \{\emptyset, X, \{b\}, \{b, c\}\}$, $Y = \{p, q, r\}$ and $m_y = \{\emptyset, Y, \{q\}, \{q, r\}\}$. Define $f : X \to Y$ by $f(a) = p; f(b) = q$ and $f(c) = r.$
Clearly f is M-\(\alpha\)- irresolute and M-\(\alpha\)-quotient mapping. Since \(f^{-1}([p, q]) = \{a, b\} \in m_x-\alpha O(X)\) and \([p, q]\) is not \(m_y\)-open set in \(Y\), f is neither strongly M-\(\alpha\)-quotient nor M-\(\alpha^*\)-quotient mapping.

**Remark 1.7.11**

The converse of Theorem 1.7.2 is not true and A strongly M-\(\alpha\)-quotient mapping need not be M-quotient as per the following example.

**Example 1.7.12**

Let \(X = \{a, b, c\}\), \(m_x = \{\emptyset, X, \{a\}\}\), \(Y = \{p, q, r\}\) and \(m_y = \{\emptyset, Y, \{p\}, \{p, q\}, \{p, r\}\}\). Define \(f : X \to Y\) by \(f(a) = p\), \(f(b) = q\) and \(f(c) = r\). Clearly f is M-\(\alpha\)-quotient and strongly M-\(\alpha\)-quotient mapping. Since \(f^{-1}([p, q]) = \{a, b\}\) is not \(m_x\)-open where \([p, q]\) is \(m_y\)-open, f is not M-quotient mapping.

**Remark 1.7.13**

A M-quotient mapping need not be strongly M-\(\alpha\)-quotient as per the following example.

**Example 1.7.14**

Let \(X = \{a, b, c\}\), \(m_x = \{\emptyset, X, \{a\}, \{a, b\}\}\), \(Y = \{p, q, r\}\) and \(m_y = \{\emptyset, Y, \{p\}, \{p, q\}\}\). Define \(f : X \to Y\) by \(f(a) = p\); \(f(b) = q\) and \(f(c) = r\). Clearly f is M-quotient but not strongly M-\(\alpha\)-quotient mapping.

**Remark 1.7.15**

The converses of Theorems 1.7.3 and 1.7.6 are not true as per the following example.
Example 1.7.16

Let $X = \{a, b, c\}$, $m_x = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $Y = \{p, q, r\}$ and $m_y = \{\emptyset, Y, \{p\}\}$. Define $f : X \to Y$ by $f(a) = p$, $f(b) = q$ and $f(c) = r$. Clearly $f$ is $M$-$\alpha$-continuous. Since $f^{-1}(\{p, r\}) = \{a, c\} \not\in m_x$-$\alpha$O$(X)$ where $\emptyset \not\in m_y$-$\alpha$O$(Y)$, $f$ is not $M$-$\alpha$-irresolute. Also, since $f^{-1}(\{q\}) = \{b\}$ is $m_x$-open in $X$ where $\emptyset \not\in m_y$-$\alpha$O$(Y)$, $f$ is not $M$-$\alpha$-quotient mapping.

Remark 1.7.17

We obtain the following diagram from the above discussions.

\[ \text{Diagram} \]

Where $A \leftrightarrow B$ means that $A$ does not necessarily imply $B$ and, moreover,

\begin{align*}
\text{(1)} & \quad M$-$\alpha$-irresolute mapping. \\
\text{(2)} & \quad M$-$\alpha^*$-quotient mapping. \\
\text{(3)} & \quad \text{Strongly } M$-$\alpha$-quotient mapping. \\
\text{(4)} & \quad M$-$\alpha$-continuous mapping. \\
\text{(5)} & \quad M$-$\alpha$-quotient mapping. \\
\text{(6)} & \quad M$-quotient mapping. \]