Chapter – II

Optimization of Fuzzy Integrated Two Stage Vendor-Buyer Inventory Models
CHAPTER - II

OPTIMIZATION OF FUZZY INTEGRATED TWO STAGE VENDOR-BUYER INVENTORY MODELS

This chapter develops an approach to determine the optimum economic order quantity and total annual integrated two stage supply chain for both vendor and buyer under the fuzzy arithmetical operations of function principle. The optimal policy for the fuzzy production inventory model is determined using the algorithm of extension of the Lagrangean method for solving inequality constraint problem and graded mean integration method is used for defuzzifying the fuzzy total annual integrated cost. Numerical examples are provided to highlight the difference between crisp and the fuzzy cases.

SECTION – 1

2.1. FUZZY INTEGRATED VENDOR-BUYER INVENTORY MODELS

In this model, we consider the situation that a vendor and a buyer can invest in reducing the buyer's ordering cost to decrease their joint total cost. We consider a model to determine an optimal integrated vendor-buyer inventory policy under conditions of order processing time reduction and permissible delay in payments. The total annual cost function of the model processes some kinds of convexities. The vendor and buyer usually establish

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Ben-Daya, M., Darwish, M., Ertogral, K., [8, 9] a long term production purchasing agreement before any action is taken and then work together towards maximizing their mutual benefits. This implies that the optimal contract quantity and number of deliveries must be determined at the outset of the contract based on their integrated total cost function. In this model we consider integrated inventory model with a single vendor and single buyer for a single product with fuzzy input parameters. Here demand and cost are represented as a trapezoidal fuzzy number.

2.2. AN INTEGRATED INVENTORY MODEL WITH ORDER PROCESSING COST REDUCTION AND PERMISSIBLE DELAY IN PAYMENTS

2.2.1. Assumptions

1. The integrated inventory model only deals with a single vendor and single buyer for a single product.
2. The demand for the item is constant overtime.
3. Shortages are not allowed.
4. The Lead time \( L \) has mutually independent components.
5. Since the vendor allows the buyer a delay in payment, the cost for improving the annual order processing cost is assigned to the buyer.
6. The time period is infinite.

2.2.2. Mathematical Model

This section examines the cost implications of integrating the lotsizing policies by determining a common economic policy using the total cost for
both parties. Fig 2.1. Shows the behavior of inventory levels for both the vendor and the buyer based on the above notations and assumptions. The annual integrated total cost for the both the vendor and the buyer consist of (I) the vendor's total annual cost, (II) the buyer's total annual cost.

![Inventory Level Diagram](image)

**Fig.2.1. Time weighted inventory for vendor and buyer**

(I) The accumulation and depletion processes of the vendor's inventory for each production cycle are shown in Fig.2.1. according to Cardenas – Barron, L.E., [12-20]. The vendor's holding cost per production cycle is equal
to the unit holding cost times the value of accumulated inventory (bold-lined area) minus the depleted inventory (shaded area).

Therefore the vendor's holding cost per year is given by

\[
\text{Holding cost per year} = \frac{(\text{Bold Lined Area} - \text{Shaded Area}) \times h_v}{\text{Cycle Time}}
\]

\[
= \left[ Q \left( \frac{Q}{R} + \frac{(n-1)Q}{nD} \right) - \frac{Q(Q/R)}{2} \right] - \frac{Q}{nD} \left[ \frac{Q}{n} + \frac{2Q}{n} + \ldots + \frac{(n-1)Q}{n} \right] h_v
\]

\[
\frac{Q}{D}
\]

\[
= \frac{(n-2)Q}{2n} \left( 1 - \frac{D}{R} \right) h_v + \frac{Q}{2n} h_v
\]

After adding the setup cost, the vendor's total annual cost is given by

\[
\text{TC}_v(n, Q) = \frac{DS_x}{Q} + \frac{(n-2)Q}{2n} \left( 1 - \frac{D}{R} \right) h_v + \frac{Q}{2n} h_v
\]

(ii) The buyer's annual inventory cost consists of the cost of placing orders, holding cost excluding interest charges and the cost of interest charges for the items kept in stock during the permissible settlement period. Castello, et al., [21], Weber, C.A., [125], Sarker, B.R., [108], Ronald, R.J., [101] assumed that the delay of \( t \) periods in making the payment to the supplier is equivalent to a price discount. Therefore if \( p \) is the price per unit with permissible delay of \( t \) periods, then the effective price per unit is

\[
\hat{p} = \frac{p}{1 + It}
\]

After adding the transportation cost, order processing cost, carrying cost and interest charges the buyer's annual cost is
The annual integrated total cost for both the vendor and buyer

\[ JTC(n, Q, L_0) = TC_v(n, Q) + TC_b(n, Q, L_0) \]

The objective is to find the optimal shortage quantity and optimal order quantity which minimize the annual integrated total cost by Liao, C.J., Shyu, C.H., [82] and Sana, S.S., Chandhuri, K.S., [106].

The necessary condition for minimum \( \frac{\partial JTC}{\partial Q} = 0 \)

At a particular value of \( n \), let \( JTC(Q) = JTC(n, Q, L_0) \)

\[ Q^* = \sqrt{ \frac{2nD[S_v + n(F + UL_0)]}{(n-2)(1 - \frac{D}{R})h_v + h_v + h_b + \frac{Pl}{1 + lt}}} \]

2.2.3. An Integrated Inventory Model for Crisp Production Quantity

Throughout this model, we use of the following variables in order to simplify the treatment of an integrated inventory models.

\( \bar{D}, \bar{R}, \bar{S}_v, \bar{h}_v, \bar{h}_b, \bar{P}, \bar{F}, \bar{U}, \bar{l} \) are fuzzy parameters.

We introduce an integrated inventory model with fuzzy parameters for crisp production quantity \( JTC(n, Q, L_0) \) as follows.
The annual integrated total inventory cost for both the vendor and buyer

\[ \tilde{JTC}(n, Q, L_0) = \left\{ \frac{D_1 \left[ S_{v_1} + n(F_1 + U_1 L_0) \right]}{Q} + \frac{Q}{2n} \left[ (n-2) \left( 1 - \frac{D_4}{R_4} \right) h_{v_1} + h_{v_1} + h_{b_1} + \frac{P_{l_1}}{1 + l_1 t} \right] \right\} \]

\[ \frac{D_2 \left[ S_{v_2} + n(F_2 + U_2 L_0) \right]}{Q} + \frac{Q}{2n} \left[ (n-2) \left( 1 - \frac{D_3}{R_3} \right) h_{v_2} + h_{v_2} + h_{b_2} + \frac{P_{l_2}}{1 + l_2 t} \right] \]

\[ \frac{D_3 \left[ S_{v_3} + n(F_3 + U_3 L_0) \right]}{Q} + \frac{Q}{2n} \left[ (n-2) \left( 1 - \frac{D_2}{R_2} \right) h_{v_3} + h_{v_3} + h_{b_3} + \frac{P_{l_3}}{1 + l_3 t} \right] \]

\[ \frac{D_4 \left[ S_{v_4} + n(F_4 + U_4 L_0) \right]}{Q} + \frac{Q}{2n} \left[ (n-2) \left( 1 - \frac{D_1}{R_1} \right) h_{v_4} + h_{v_4} + h_{b_4} + \frac{P_{l_4}}{1 + l_4 t} \right] \]

\[ \tilde{JTC}(n, Q, L_0) = \tilde{D} \otimes \left[ \tilde{S}_{v} \otimes n(\tilde{F} \oplus \tilde{U} \otimes L_0) \right] \ominus \tilde{Q} \otimes 2 \otimes n [ (n-2) \otimes (1 - \tilde{D}) \otimes \tilde{R} \otimes h_v \oplus h_v \oplus h_b \oplus P \oplus \tilde{I} \ominus (1 + \tilde{I} \otimes t) ] \]

where \( \ominus, \otimes, \oplus, \ominus \) are the fuzzy arithmetical operators under Function Principle.

Suppose

\[ \tilde{D} = (D_1, D_2, D_3, D_4) \]
\[ \tilde{S}_{v} = (S_{v_1}, S_{v_2}, S_{v_3}, S_{v_4}) \]
\[ \tilde{F} = (F_1, F_2, F_3, F_4) \]
\[ \tilde{U} = (U_1, U_2, U_3, U_4) \]
\[ \tilde{Q} = (Q_1, Q_2, Q_3, Q_4) \]
\[ \hat{\mathbf{R}} = (R_1, R_2, R_3, R_4) \]
\[ \hat{\mathbf{h}} = (h_{v1}, h_{v2}, h_{v3}, h_{v4}) \]
\[ \hat{\mathbf{h}} = (h_{b1}, h_{b2}, h_{b3}, h_{b4}) \]
\[ \hat{\mathbf{P}} = (P_1, P_2, P_3, P_4) \]
\[ \hat{\mathbf{I}} = (I_1, I_2, I_3, I_4) \]

are nonnegative trapezoidal fuzzy numbers. Then we solve the optimal production quantity of formula as the following steps. Second, we defuzzify the fuzzy total production inventory for the vendor and buyer cost by Chia-Huei Ho, Hung-Chi-Chang., [34]

\[
P\left( \text{JTC}(n, Q, L_0) \right) = \frac{1}{6} \left[ \frac{D_1 [S_{v1} + n(F_1 + U_1 L_0)]}{Q} + \frac{Q}{2n} \left[ \frac{Q}{2n} \left( n - 2 \right) \left( 1 - \frac{D_4}{R_4} \right) h_{v4} + h_{v3} + h_{b4} + \frac{P_{I4}}{1 + I_4 t} \right] \right]
\]

\[
+ \frac{D_2 [S_{v2} + n(F_2 + U_2 L_0)]}{Q} + \frac{Q}{2n} \left[ \frac{Q}{2n} \left( n - 2 \right) \left( 1 - \frac{D_3}{R_3} \right) h_{v2} + h_{v1} + h_{b3} + \frac{P_{I3}}{1 + I_3 t} \right] \]

\[
+ \frac{D_3 [S_{v3} + n(F_3 + U_3 L_0)]}{Q} + \frac{Q}{2n} \left[ \frac{Q}{2n} \left( n - 2 \right) \left( 1 - \frac{D_2}{R_2} \right) h_{v3} + h_{v2} + h_{b2} + \frac{P_{I2}}{1 + I_2 t} \right] \]

\[
+ \frac{D_4 [S_{v4} + n(F_4 + U_4 L_0)]}{Q} + \frac{Q}{2n} \left[ \frac{Q}{2n} \left( n - 2 \right) \left( 1 - \frac{D_1}{R_1} \right) h_{v4} + h_{v3} + h_{b4} + \frac{P_{I4}}{1 + I_4 t} \right] \]

Third, we can get the optimal production quantity \( Q^* \) when

\[
P\left( \text{JTC}(n, Q, L_0) \right) \text{ is minimization.} \]
In order to find the minimization of \( P(J\tilde{T}C(n, Q, L_0)) \), the derivative of

\[
\frac{\partial P(J\tilde{T}C(n, Q, L_0))}{\partial Q} = 0
\]

We find the optimal production quantity

\[
Q = Q^* = \frac{2n}{(n-2)} \left[ \frac{D_1 (S_{v_1} + n(F_1 + U_1 L_0)) + 2D_2 (S_{v_2} + n(F_2 + U_2 L_0))}{1 - \frac{D_3}{R_3}} + \frac{2D_4 (S_{v_4} + n(F_4 + U_4 L_0))}{1 - \frac{D_4}{R_4}} \right] + \frac{P_{1_1}}{1 + I_1 t} + \frac{P_{2_2}}{1 + I_2 t} + \frac{P_{3_3}}{1 + I_3 t} + \frac{P_{4_4}}{1 + I_4 t}
\]

2.2.4. An Integrated Inventory Model for Fuzzy Production Quantity

In this section, we introduce an integrated inventory model by changing the crisp production quantity into fuzzy production quantity by Jaggi, C.K., [67, 68].

Suppose fuzzy production quantity \( \tilde{Q} \) be a trapezoidal fuzzy number \( \tilde{Q} = (Q_1, Q_2, Q_3, Q_4) \) with \( 0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4 \).

Thus we can get the fuzzy total production inventory cost
\[
P(JT\tilde{C}_1(n, Q, L_0)) = \left\{ \begin{array}{l}
\frac{D_1[S_{v_1} + n(F_1 + U_{L_0})]}{Q_4} + \frac{Q_1}{2n} \left( n-2 \left( 1 - \frac{D_4}{R_1} \right) h_1 + h_1 + h_1 + \frac{P_l}{1 + l_1 t} \right), \\
\end{array} \right.
\]

\[
2D_2 \left[ \frac{S_{v_2} + n(F_2 + U_{L_0})}{Q_3} \right] + \frac{2Q_2}{2n} \left( n-2 \left( 1 - \frac{D_2}{R_2} \right) h_2 + h_2 + h_3 + \frac{P_{l_2}}{1 + l_2 t} \right), \\
\end{array} \right.
\]

\[
2D_3 \left[ \frac{S_{v_3} + n(F_3 + U_{L_0})}{Q_2} \right] + \frac{2Q_3}{2n} \left( n-2 \left( 1 - \frac{D_3}{R_3} \right) h_3 + h_3 + h_3 + \frac{P_{l_3}}{1 + l_3 t} \right), \\
\end{array} \right.
\]

\[
D_4 \left[ \frac{S_{v_4} + n(F_4 + U_{L_0})}{Q_1} \right] + \frac{Q_4}{2n} \left( n-2 \left( 1 - \frac{D_4}{R_4} \right) h_4 + h_4 + h_4 + \frac{P_{l_4}}{1 + l_4 t} \right) \}
\]

We can obtain the Graded Mean Integration Representation of

\[
P(JT\tilde{C}_1(n, Q, L_0))
\]

\[
= \frac{1}{6} \left\{ \begin{array}{l}
\frac{D_1[S_{v_1} + n(F_1 + U_{L_0})]}{Q_4} + \frac{Q_1}{2n} \left( n-2 \left( 1 - \frac{D_4}{R_1} \right) h_1 + h_1 + h_1 + \frac{P_l}{1 + l_1 t} \right), \\
\end{array} \right.
\]

\[
+ 2D_2 \left[ \frac{S_{v_2} + n(F_2 + U_{L_0})}{Q_3} \right] + \frac{2Q_2}{2n} \left( n-2 \left( 1 - \frac{D_2}{R_2} \right) h_2 + h_2 + h_3 + \frac{P_{l_2}}{1 + l_2 t} \right), \\
\end{array} \right.
\]

\[
+ 2D_3 \left[ \frac{S_{v_3} + n(F_3 + U_{L_0})}{Q_2} \right] + \frac{2Q_3}{2n} \left( n-2 \left( 1 - \frac{D_3}{R_3} \right) h_3 + h_3 + h_3 + \frac{P_{l_3}}{1 + l_3 t} \right), \\
\end{array} \right.
\]

\[
+ D_4 \left[ \frac{S_{v_4} + n(F_4 + U_{L_0})}{Q_1} \right] + \frac{Q_4}{2n} \left( n-2 \left( 1 - \frac{D_4}{R_4} \right) h_4 + h_4 + h_4 + \frac{P_{l_4}}{1 + l_4 t} \right) \}
\]

with \(0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4\).
It will not change the meaning of formula if we replace inequality conditions $0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4$ into the following inequality

$$Q_2 - Q_1 \geq 0, \quad Q_3 - Q_2 \geq 0, \quad Q_4 - Q_3 \geq 0, \quad Q_1 > 0.$$ 

In the following steps, extension of the Lagrangean method is used to find the solutions of $Q_1, Q_2, Q_3, Q_4$ to minimize $P\left(JT\tilde{C}_1(n, Q, L_0)\right)$.

**Step 1**: Solve the unconstrained problem. Consider $\min P\left(JT\tilde{C}_1(n, Q, L_0)\right)$.

To find the $\min P\left(JT\tilde{C}_1(n, Q, L_0)\right)$, we have to find the derivative of $P\left(JT\tilde{C}_1(n, Q, L_0)\right)$ with respect to $Q_1, Q_2, Q_3, Q_4$.

$$\frac{\partial P}{\partial Q_1} = \frac{1}{6} \left[ \frac{1}{2n} (n-2) \left( 1 - \frac{D_4}{R_1} \right) h_{v_1} + h_{v_2} + h_{b_1} + \frac{Pl_1}{1+I_4} - \frac{D_4 \left( S_{v_1} + n(F_4 + U_4L_0) \right)}{Q_1^2} \right]$$

$$\frac{\partial P}{\partial Q_2} = \frac{1}{6} \left[ \frac{2}{2n} (n-2) \left( 1 - \frac{D_3}{R_2} \right) h_{v_2} + h_{v_3} + h_{b_2} + \frac{Pl_2}{1+I_3} - \frac{2D_3 \left( S_{v_2} + n(F_3 + U_3L_0) \right)}{Q_2^2} \right]$$

$$\frac{\partial P}{\partial Q_3} = \frac{1}{6} \left[ \frac{2}{2n} (n-2) \left( 1 - \frac{D_2}{R_3} \right) h_{v_3} + h_{v_1} + h_{b_3} + \frac{Pl_3}{1+I_2} - \frac{2D_2 \left( S_{v_3} + n(F_2 + U_2L_0) \right)}{Q_3^2} \right]$$

$$\frac{\partial P}{\partial Q_4} = \frac{1}{6} \left[ \frac{2}{2n} (n-2) \left( 1 - \frac{D_1}{R_4} \right) h_{v_1} + h_{v_4} + h_{b_4} + \frac{Pl_4}{1+I_1} - \frac{2D_1 \left( S_{v_4} + n(F_1 + U_1L_0) \right)}{Q_4^2} \right]$$

Let all the above results partial derivatives equal to zero and solve $Q_1, Q_2, Q_3, Q_4$. 

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Let \( \frac{\partial P}{\partial Q_1} = 0 \) then \( Q_1 = \ldots \)

Let \( \frac{\partial P}{\partial Q_2} = 0 \) then \( Q_2 = \ldots \)

and \( \frac{\partial P}{\partial Q_3} = 0 \) then \( Q_3 = \ldots \)

and \( \frac{\partial P}{\partial Q_4} = 0 \) then \( Q_4 = \ldots \)

Because the above show that \( Q_1 > Q_2 > Q_3 > Q_4 \). It does not satisfy the constraint \( 0 < Q_1 < Q_2 < Q_3 < Q_4 \).

Therefore set \( K = 1 \) and go to Step 2.

**Step 2**: Convert the inequality constraint \( Q_2 - Q_1 \geq 0 \) into equality constraint \( Q_2 - Q_1 = 0 \) and optimize \( P(JT \tilde{C}_1(n, Q, \Lambda_0)) \) subject to \( Q_2 - Q_1 = 0 \) by the Lagrangean Method. We have Lagrangean function as

\[
L(Q_1, Q_2, Q_3, Q_4, \lambda) = P(JT \tilde{C}_1(n, Q, \Lambda_0)) - \lambda(Q_2 - Q_1)
\]

Taking the partial derivatives of \( L(Q_1, Q_2, Q_3, Q_4, \lambda) \) with respect to \( Q_1, Q_2, Q_3, Q_4 \) and \( \lambda \) to find the minimization of \( L(Q_1, Q_2, Q_3, Q_4, \lambda) \). Let all the partial derivatives equal to zero and solve \( Q_1, Q_2, Q_3 \) and \( Q_4 \).
Then we get,

\[
\frac{\partial L}{\partial Q_1} = \frac{1}{6} \left\{ \frac{1}{2n} \left[ (n-2) \left( 1 - \frac{D_4}{R_1} \right) h_{v_1} + h_{v_2} + h_{v_3} + \frac{P_{l_1}}{1 + l_4 t} \right] - \frac{D_4 \left( S_{v_4} + n(F_4 + U_4 L_0) \right)}{Q_1^2} \right\} \frac{1}{6} + \lambda \\
\Rightarrow \frac{1}{2n} \left[ (n-2) \left( 1 - \frac{D_4}{R_1} \right) h_{v_1} + h_{v_2} + h_{v_3} + \frac{P_{l_1}}{1 + l_4 t} \right] + \frac{D_4 \left( S_{v_4} + n(F_4 + U_4 L_0) \right)}{Q_1^2} = 6\lambda
\]

\[
\frac{\partial L}{\partial Q_2} = \frac{1}{6} \left\{ \frac{2D_3 \left( S_{v_1} + n(F_3 + U_3 L_0) \right)}{Q_2^2} - \frac{2}{2n} \left[ (n-2) \left( 1 - \frac{D_3}{R_2} \right) h_{v_2} + h_{v_3} + h_{v_4} + \frac{P_{l_2}}{1 + l_4 t} \right] \right\} - \lambda = 0
\]

\[
\frac{\partial L}{\partial Q_3} = \frac{1}{6} \left\{ \frac{2D_2 \left( S_{v_2} + n(F_2 + U_2 L_0) \right)}{Q_2^2} - \frac{2}{2n} \left[ (n-2) \left( 1 - \frac{D_2}{R_3} \right) h_{v_2} + h_{v_3} + h_{v_4} + \frac{P_{l_3}}{1 + l_4 t} \right] \right\} = 0
\]

\[
\frac{\partial L}{\partial Q_4} = \frac{1}{6} \left\{ \frac{D_4 \left( S_{v_4} + n(F_4 + U_4 L_0) \right)}{Q_4^2} + \frac{1}{2n} \left[ (n-2) \left( 1 - \frac{D_4}{R_4} \right) h_{v_4} + h_{v_5} + h_{v_6} + \frac{P_{l_4}}{1 + l_4 t} \right] \right\} = 0
\]

\[
\frac{\partial L}{\partial \lambda} = -(Q_2 - Q_1)
\]
\[ Q_1 = Q_2 = \frac{2nD_4(S_v + n(F_4 + U_4L_0))}{(n-2)\left[1 - \frac{D_4}{R_1}\right]h_v + h_{b_1} + \frac{P_{12}}{1 + l_i t}} + 2(n-2)\left[1 - \frac{D_4}{R_1}\right]h_v + 2h_{b_2} + \frac{2P_{12}}{1 + l_i t} \]

\[ Q_3 = \frac{4nD_2(S_v + n(F_2 + U_2L_0))}{2(n-2)\left[1 - \frac{D_2}{R_2}\right]h_v + 2h_{b_1} + \frac{2P_{31}}{1 + l_i t}} \]

\[ Q_4 = \frac{2nD_1(S_v + n(F_1 + U_1L_0))}{(n-2)\left[1 - \frac{D_1}{R_4}\right]h_v + h_{b_1} + \frac{P_{41}}{1 + l_i t}} \]

Because the above results show that \( Q_3 > Q_4 \), it does not satisfy the constraint \( 0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4 \). Therefore it is not a local optimum. Similarly we can get the same result if we select any other one inequality constraint to be equality constraint, therefore set \( K = 2 \) and go to Step 3.

**Step 3**: Convert the inequality constraints \( Q_2 - Q_1 \geq 0, Q_3 - Q_2 \geq 0 \), into equality constraints \( Q_2 - Q_1 = 0 \) and \( Q_3 - Q_2 = 0 \). We optimize

\[ P\left(J^T C_1(n, Q, L_0)\right) \]

Subject to \( Q_2 - Q_1 = 0 \) and \( Q_3 - Q_2 = 0 \) by the Lagrangean Method. Then the Lagrangean method is

\[ L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2) = P\left(J^T C_1(n, Q, L_0)\right) - \lambda_1(Q_2 - Q_1) - \lambda_2(Q_3 - Q_2) \]
In order to find the minimization of $L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2)$, we take the partial derivatives of $L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2)$ with respect to $Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2$ and let all the partial derivatives equal to zero and solve $Q_1, Q_2, Q_3$ and $Q_4$.

\[
\frac{\partial L}{\partial Q_1} = \frac{1}{6} \left[ \frac{1}{2n} \left( n - 2 \right) \left( 1 - \frac{D_4}{R_1} \right) h_{v_1} + h_{a_1} + \frac{P_{I_1}}{1+I_1t} \right] - \frac{D_4 \left( S_{v_4} + n(F_4 + U_4L_0) \right)}{Q_1^2} + \lambda_1 = 0
\]

\[
\frac{\partial L}{\partial Q_2} = \frac{1}{6} \left[ \frac{2}{2n} \left( n - 2 \right) \left( 1 - \frac{D_3}{R_2} \right) h_{v_2} + h_{a_2} + \frac{P_{I_2}}{1+I_2t} \right] - \frac{2D_3 \left( S_{v_3} + n(F_3 + U_3L_0) \right)}{Q_2^2} + \lambda_2 - \lambda_1 = 0
\]

\[
\frac{\partial L}{\partial Q_3} = \frac{1}{6} \left[ \frac{2}{2n} \left( n - 2 \right) \left( 1 - \frac{D_2}{R_3} \right) h_{v_3} + h_{a_3} + \frac{P_{I_3}}{1+I_3t} \right] - \frac{2D_2 \left( S_{v_2} + n(F_2 + U_2L_0) \right)}{Q_3^2} - \lambda_2 = 0
\]

\[
\frac{\partial L}{\partial Q_4} = \frac{1}{6} \left[ \frac{1}{2n} \left( n - 2 \right) \left( 1 - \frac{D_1}{R_4} \right) h_{v_4} + h_{a_4} + \frac{P_{I_4}}{1+I_4t} \right] - \frac{D_1 \left( S_{v_1} + n(F_1 + U_1L_0) \right)}{Q_4^2} = 0
\]

\[
\frac{\partial L}{\partial \lambda_1} = -(Q_2 - Q_1)
\]

\[
\frac{\partial L}{\partial \lambda_2} = -(Q_3 - Q_2)
\]
\[ Q_1 = Q_2 = Q_3 = \]
\[
\frac{2n}{2} \left[ \frac{2D_2 \left( S_{v_2} + n(F_2 + U_2 L_0) \right) + 2D_3 \left( S_{v_3} + n(F_3 + U_3 L_0) \right)}{n} + D_4 \left( S_{v_4} + n(F_4 + U_4 L_0) \right) + 2\left( 1 - \frac{D_2}{R_2} \right) h_{v_2} + h_{v_2} + \frac{P_{I_2}}{1 + I_2 t} \right] \\
+ 2 \left( 1 - \frac{D_3}{R_3} \right) h_{v_3} + 2h_{v_3} + 2h_{v_3} + 2\left( 1 - \frac{D_3}{R_3} \right) h_{v_3} + h_{v_3} + \frac{P_{I_3}}{1 + I_3 t} \right] \\
+ \left[ (n-2) \left( 1 - \frac{D_4}{R_4} \right) h_{v_4} + h_{v_4} + h_{v_4} + \frac{P_{I_4}}{1 + I_4 t} \right] \\
\]

\[ Q_4 = \]
\[
\frac{2nD_1 \left( S_{v_1} + n(F_1 + U_1 L_0) \right)}{(n-2) \left[ 1 - \frac{D_1}{R_1} \right] h_{v_1} + 2h_{v_1} + 2h_{v_1} + \frac{2P_{I_1}}{1 + I_1 t} \right] \\
\]

The above results \( Q_1 > Q_4 \), does not satisfy the constraint \( 0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4 \). Therefore it is not a local optimum. Similarly we can get the same result if we select any other two inequality constraints to be equality constraint, therefore set \( K = 3 \) and go to Step 4.

**Step 4**: Convert the inequality constraints \( Q_2 - Q_1 \geq 0, Q_3 - Q_2 \geq 0 \) and \( Q_4 - Q_3 \geq 0 \) into equality constraints \( Q_2 - Q_1 = 0, Q_3 - Q_2 = 0 \) and \( Q_4 - Q_3 = 0 \).

We optimize \( P \left( JTC_1(n, Q, L_0) \right) \) Subject to \( Q_2 - Q_1 = 0, Q_3 - Q_2 = 0 \) and \( Q_4 - Q_3 = 0 \) by the Lagrangean Method. The Lagrangean function is given by

\[
L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2, \lambda_3) = P \left( JTC_1(n, Q, L_0) \right) - \lambda_1(Q_2 - Q_1) - \lambda_2(Q_3 - Q_2) - \lambda_3(Q_4 - Q_3) 
\]

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In order to find the minimization of \( L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2, \lambda_3) \), we take the partial derivatives of \( L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2, \lambda_3) \) with respect to \( Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2, \lambda_3 \) and let all the partial derivatives equal to zero and solve \( Q_1, Q_2, Q_3 \) and \( Q_4 \). Then we get

\[
\frac{\partial L}{\partial Q_1} = \frac{1}{6} \frac{1}{2n} \left[ (n-2) \left( 1 - \frac{D_4}{R_1} \right) h_{v_1} + h_{b_1} + h_{b_2} + \frac{P_{l_1}}{1+l_1 t} \right] \]

\[
\frac{D_4 \left( S_{v_1} + n (F_4 + U_4 L_0) \right)}{Q_1^2} + \lambda_1 = 0
\]

\[
\frac{\partial L}{\partial Q_2} = \frac{1}{6} \frac{2}{2n} \left[ (n-2) \left( 1 - \frac{D_3}{R_2} \right) h_{v_2} + h_{v_2} + h_{b_2} + \frac{P_{l_2}}{1+l_2 t} \right] \]

\[
\frac{2D_3 \left( S_{v_2} + n (F_3 + U_3 L_0) \right)}{Q_2^2} - \lambda_1 + \lambda_2 = 0
\]

\[
\frac{\partial L}{\partial Q_3} = \frac{1}{6} \frac{2}{2n} \left[ (n-2) \left( 1 - \frac{D_2}{R_3} \right) h_{v_3} + h_{v_3} + h_{b_3} + \frac{P_{l_3}}{1+l_3 t} \right] \]

\[
\frac{2D_2 \left( S_{v_3} + n (F_2 + U_2 L_0) \right)}{Q_3^2} - \lambda_1 + \lambda_2 = 0
\]

\[
\frac{\partial L}{\partial Q_4} = \frac{1}{6} \frac{1}{2n} \left[ (n-2) \left( 1 - \frac{D_4}{R_4} \right) h_{v_4} + h_{v_4} + h_{b_4} + \frac{P_{l_4}}{1+l_4 t} \right] \]

\[
\frac{-D_4 \left( S_{v_4} + n (F_4 + U_4 L_0) \right)}{Q_4^2} - \lambda_3 = 0
\]

\[
\frac{\partial L}{\partial \lambda_1} = -(Q_2 - Q_1)
\]
\[
\frac{\partial L}{\partial \lambda_2} = -(Q_3 - Q_2)
\]
\[
\frac{\partial L}{\partial \lambda_3} = -(Q_4 - Q_3)
\]

\[
Q_1 = Q_2 = Q_3 = Q_4
\]

\[
\begin{align*}
Q^* &= \frac{2n}{(n-2)} \left[ D_1 (S_{x_1} + n(F_1 + U_1 L_0)) + 2D_2 (S_{x_2} + n(F_2 + U_2 L_0)) \right. \\
&\quad + 2D_3 (S_{x_3} + n(F_3 + U_3 L_0)) + D_4 (S_{x_4} + n(F_4 + U_4 L_0)) \\
&\quad + 2 \left( \frac{1-D_1}{R_1} \right) h_{x_1} + h_{x_1} + h_{b_1} + \frac{P_{l_1}}{1+I_{l_1}} + 1+I_{l_1} \\
&\left. + 2 \left( \frac{1-D_2}{R_2} \right) h_{x_2} + h_{x_2} + h_{b_2} + \frac{2P_{l_2}}{1+I_{l_2}} + 1+I_{l_2} \\
&\quad + 2 \left( \frac{1-D_3}{R_3} \right) h_{x_3} + h_{x_3} + h_{b_3} + \frac{2P_{l_3}}{1+I_{l_3}} + 1+I_{l_3} \\
&\quad + 2 \left( \frac{1-D_4}{R_4} \right) h_{x_4} + h_{x_4} + h_{b_4} + \frac{2P_{l_4}}{1+I_{l_4}} + 1+I_{l_4} \right]
\end{align*}
\]

Because the above solution \( \hat{Q} = (Q_1, Q_2, Q_3, Q_4) \) satisfies all inequality constraints, the procedure terminates with \( \hat{Q} \) as a local optimum solution to the problem. Since the above local optimum solution is the only one feasible solution. So it is an optimum solution of the inventory model with fuzzy production quantity according to extension of the Lagrangean Method.

Let \( Q_1 = Q_2 = Q_3 = Q_4 = \hat{Q}^* \). Then the optimal fuzzy production quantity is

\[
\begin{align*}
Q^* &= \frac{2n}{(n-2)} \left[ D_1 (S_{x_1} + n(F_1 + U_1 L_0)) + 2D_2 (S_{x_2} + n(F_2 + U_2 L_0)) \right. \\
&\quad + 2D_3 (S_{x_3} + n(F_3 + U_3 L_0)) + D_4 (S_{x_4} + n(F_4 + U_4 L_0)) \\
&\quad + (h_{x_1} + 2h_{v_2} + 2h_{v_3} + h_{x_4}) + (h_{b_1} + 2h_{b_2} + 2h_{b_3} + h_{b_4}) \\
&\quad + \frac{P_{l_1}}{1+I_{l_1}} + \frac{2P_{l_2}}{1+I_{l_2}} + \frac{2P_{l_3}}{1+I_{l_3}} + \frac{P_{l_4}}{1+I_{l_4}} \right]
\end{align*}
\]
2.2.5. Numerical Example

To illustrate the results obtained in this model, the proposed analytic solution method is applied to efficiently solve the following numerical example.

Consider an inventory system with the following characteristics. \( D = 2700, \) \( R = 9000, \) \( h_b = 5.00, \) \( h_v = 2.00, \) \( U = 1400, \) \( S_v = 200, \) \( P = 10, \) \( F = 300, \) \( t = 0.25, \) \( I = 0.15, \) \( n = 2, \) \( Q^* = 1182.77, \) \( L_0 = 0.105, \) \( JTC (n, Q, L_0) = 5144.71. \) In this example can be transferred into the fuzzy parameters as follows:

Consider any problem in which an annual demand is more or less than 2700 units, production rate is more or less than 9000, unit stock-holding cost is more or less than 5.00 per item per year for the buyer, unit stock-holding cost is more or less than 2.00 cost per item per year for the vendor. Order processing cost is more or less than 1400 cost per unit time for the buyer, set up cost is more or less than 200 per production run for the vendor, purchase price is more or less than 10 per units, Fixed transportation cost is more or less than 300 per shipment, permissible delay is more or less than 0.25 in settling amounts, Carrying cost is more or less than 0.15 per dollar per year.

Determine the optimum integrated total cost?

Here we represent the case of value, "more or less than \( Y \)" as the type of trapezoidal fuzzy number.

Suppose Fuzzy annual demand is "more or less than 2700"

\[
\hat{D} = (D_1, D_2, D_3, D_4) = (2550, 2725, 2730, 2740)
\]
Fuzzy production rate is "more or less than 9000"

\[ \tilde{R} = (R_1, R_2, R_3, R_4) = (8850, 9025, 9030, 9040) \]

Fuzzy unit stock holding cost is "more or less than 5.00" per item per year for the buyer

\[ \tilde{h}_b = (h_{b_1}, h_{b_2}, h_{b_3}, h_{b_4}) = (4.8, 4.9, 5.1, 5.2) \]

Fuzzy unit stock holding cost is "more or less than 2.00" cost per item per year for the buyer

\[ \tilde{h}_v = (h_{v_1}, h_{v_2}, h_{v_3}, h_{v_4}) = (1.8, 1.9, 2.1, 2.2) \]

Fuzzy order processing cost is "more or less than 1400" Cost unit per time for the buyer

\[ \tilde{U} = (U_1, U_2, U_3, U_4) = (1380, 1390, 1410, 1420) \]

Fuzzy setup cost is "more or less than 200" per production run for the vendor

\[ \tilde{S}_v = (S_{v_1}, S_{v_2}, S_{v_3}, S_{v_4}) = (180, 190, 210, 220) \]

Fuzzy purchase price is "more or less than 10" per units

\[ \tilde{P} = (P_1, P_2, P_3, P_4) = (9.8, 9.8, 10.1, 10.2) \]

Fuzzy fixed transportation cost is "more or less than 300" per shipment

\[ \tilde{F} = (F_1, F_2, F_3, F_4) = (280, 290, 310, 320) \]

Fuzzy carrying cost is "more or less than 0.15" per dollar per year

\[ \tilde{T} = (T_1, T_2, T_3, T_4) = (0.13, 0.14, 0.16, 0.17) \]

Fuzzy production quantity

\[ \tilde{Q} = (Q_1, Q_2, Q_3, Q_4) \text{ with } 0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4 \]
Replace the above fuzzy parameter values into formula, we find the optimal fuzzy production quantity,

\[ \tilde{Q}^* = (1274.07, 1229.67, 1149.48, 1075.56) \]
\[ Q^* = 1184.66 \]
\[ Q^* = 1183.08 \]

The minimization fuzzy total production inventory cost for both the vendor and buyer is

\[ \tilde{JTC}(n, Q, L_0)^* = (4533.13, 4918.92, 5122.23, 5232.53) \]

This numerical example is used to show the feasibility of the proposed integration models. We observe that by increasing the demand fuzziness, the optimum order quantity and total production inventory cost for both the vendor and the buyer value increases. The reason is that, the more demand uncertainty causes less inventory overage and inventory underage which lead to less total cost.

SECTION – 2

2.3. FUZZY INTEGRATED TWO-STAGE VENDOR-BUYER INVENTORY SYSTEM

In today’s competitive markets, close cooperation between the vendor and the buyer is Sphicas, G.P., [114] necessary to reduce the joint inventory cost and the response time of the vendor-buyer system. The successful experiences of National Semiconductor, Teng, J.T., Chang, C.T., [116-120] and Tsao, Y.C., Sheen, G.J., [121] have demonstrated that integrating the
supply chain has significantly influenced the company's performance and market share.

2.3.1. The Vendor-Buyer Integrated Inventory Model

In this section, we develop the vendor-buyer integrated inventory optimization method by Drake, M.J., Pentico, D.W., Toews, C., [41] and Garcia-Laguna, J., [43], Huang, C.K., [58]. In this, buyer's lot size $Q$ is optimized using a concept of Lagrangean Method.

2.3.2. Mathematical Model

The total cost of the integrated two-stage inventory system for the relationship of vendor-buyer is

$$TC(Q, n) = \frac{DA}{Q} + \frac{rQC}{2} + \frac{DS}{nQ} + \frac{rQC}{2} \left[ (n-1) \left(1 - \frac{D}{P}\right) + \frac{D}{P} \right]$$

The objective is to find the optimal lot size $Q$. The necessary conditions for minimum

$$\frac{\partial TC(Q, n)}{\partial Q} = 0$$

Therefore the optimal lot size $(Q)$ is

$$Q^* = \sqrt{\frac{2D \left( A + \frac{S}{n} \right)}{\left( \frac{C_v}{\bar{C}_v} + \frac{2D}{P} - 1 + n \left( 1 - \frac{D}{P} \right) \right)}}$$

Throughout this model, we use the following variables in order to simplify the treatment of the fuzzy inventory models $\tilde{S}, \tilde{A}, \tilde{C}_v, \tilde{C}_b, \bar{r}, \bar{P}, \bar{D}$ are
fuzzy parameters. The fuzzy total cost of the integrated two stage inventory system for the relationship of vendor-buyer by Huang, Y.F., [60, 61, 62]

\[
\tilde{TC}(Q, n) = \left\{ \frac{D_1 A_1}{Q} + \frac{r_1 QC_{b_1}}{2nQ} + \frac{D_1 S_1}{nQ} + \frac{r_1 QC_{v_1}}{2} \left[ (n - 1) \left( 1 - \frac{D_2}{P_1} \right) + \frac{D_1}{P_1} \right],
\right.
\]

\[
\frac{D_2 A_2}{Q} + \frac{r_2 QC_{b_2}}{2nQ} + \frac{D_2 S_2}{nQ} + \frac{r_2 QC_{v_2}}{2} \left[ (n - 1) \left( 1 - \frac{D_3}{P_2} \right) + \frac{D_3}{P_2} \right],
\]

\[
\frac{D_3 A_3}{Q} + \frac{r_3 QC_{b_3}}{2nQ} + \frac{D_3 S_3}{nQ} + \frac{r_3 QC_{v_3}}{2} \left[ (n - 1) \left( 1 - \frac{D_4}{P_3} \right) + \frac{D_4}{P_3} \right],
\]

\[
\frac{D_4 A_4}{Q} + \frac{r_4 QC_{b_4}}{2nQ} + \frac{D_4 S_4}{nQ} + \frac{r_4 QC_{v_4}}{2} \left[ (n - 1) \left( 1 - \frac{D_4}{P_4} \right) + \frac{D_4}{P_4} \right],
\]

where \( \odot, \boxtimes, \oplus, \ominus \) are the arithmetical operations under function principle,

Suppose \( \tilde{S} = (S_1, S_2, S_3, S_4) ; \quad \tilde{A} = (A_1, A_2, A_3, A_4) \)

\( \tilde{C_v} = (C_{v_1}, C_{v_2}, C_{v_3}, C_{v_4}) ; \quad \tilde{C_b} = (C_{b_1}, C_{b_2}, C_{b_3}, C_{b_4}) \)

\( \tilde{r} = (r_1, r_2, r_3, r_4) ; \quad \tilde{P} = (P_1, P_2, P_3, P_4) \)

\( \tilde{D} = (D_1, D_2, D_3, D_4) \)

are nonnegative trapezoidal fuzzy numbers. Then we solve the optimal lot size formula as the following steps. Second, we defuzzify the fuzzy total cost of the integrated two stage inventory system.

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Graded mean integration representation of $\tilde{T}C(Q, n)$ is

$$P(\tilde{T}C(Q, n)) = \frac{1}{6} \left[ \frac{D_1 A_1}{Q} + \frac{r_1 Q C_{b_1}}{2} + \frac{D_1 S_1}{n Q} + \frac{r_1 Q C_{v_1}}{2} \left( n - 1 \right) \left( 1 - \frac{D_4}{P_1} \right) + \frac{D_1}{P_1} \right]$$

$$+ 2 \left[ \frac{D_2 A_2}{Q} + \frac{r_2 Q C_{b_2}}{2} + \frac{D_2 S_2}{n Q} + \frac{r_2 Q C_{v_2}}{2} \left( n - 1 \right) \left( 1 - \frac{D_3}{P_2} \right) + \frac{D_2}{P_2} \right]$$

$$+ 2 \left[ \frac{D_3 A_3}{Q} + \frac{r_3 Q C_{b_3}}{2} + \frac{D_3 S_3}{n Q} + \frac{r_3 Q C_{v_3}}{2} \left( n - 1 \right) \left( 1 - \frac{D_2}{P_3} \right) + \frac{D_3}{P_3} \right]$$

$$+ 2 \left[ \frac{D_4 A_4}{Q} + \frac{r_4 Q C_{b_4}}{2} + \frac{D_4 S_4}{n Q} + \frac{r_4 Q C_{v_4}}{2} \left( n - 1 \right) \left( 1 - \frac{D_1}{P_4} \right) + \frac{D_4}{P_4} \right]$$

Third, we can get the optimal lot size $Q^*$ when $P(\tilde{T}C(Q, n))$ is minimization. In order to find the minimization of $P(\tilde{T}C(Q, n))$ the derivative of $P(\tilde{T}C(Q, n))$ with $Q$ is

$$\frac{\partial P(\tilde{T}C(Q, n))}{\partial Q} = 0$$

We find the optimal lot size $Q$ as $Q^*$

$$Q^* =$$

$$\left[ 2 \left[ A_1 d_1 + 2 A_2 d_2 + 2 A_3 d_3 + A_4 d_4 \right] + \frac{D_1 S_1}{n} + \frac{2 D_2 S_2}{n} + \frac{2 D_3 S_3}{n} + \frac{D_4 S_4}{n} \right]$$

$$+ \left( r_1 C_{b_1} + 2 r_2 C_{b_2} + 2 r_3 C_{b_3} + r_4 C_{b_4} \right) + \left( r_1 C_{v_1} \left( \frac{2(D_4 + D_1)}{P_1} - 1 + n \left( 1 - \frac{D_4}{P_1} \right) \right) \right)$$

$$+ \left( 2 r_2 C_{v_2} \left( \frac{2(D_3 + D_2)}{P_2} - 1 + n \left( 1 - \frac{D_3}{P_2} \right) \right) + \left( 2 r_3 C_{v_3} \left( \frac{2(D_3 + D_2)}{P_3} - 1 + n \left( 1 - \frac{D_2}{P_3} \right) \right) \right)$$

$$+ \left( r_4 C_{v_4} \left( \frac{2(D_1 + D_4)}{P_4} - 1 + n \left( 1 - \frac{D_1}{P_4} \right) \right) \right)$$

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2.3.3. Fuzzy Integrated Two Stage Vendor-Buyer Inventory Model with Crisp Number of Deliveries and Fuzzy Lot Size

In this section, we introduce the fuzzy inventory integrated models by changing the crisp lot size into fuzzy lot size. Suppose fuzzy lot size \( \tilde{Q} \) be the trapezoidal fuzzy number, \( \tilde{Q} = (Q_1, Q_2, Q_3, Q_4) \) with \( 0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4 \). Then we get the fuzzy total inventory cost of the integrated two stage inventory system,

\[
P(\tilde{T}_C(\tilde{Q}, n)) = \left\{ \left( \frac{D_1A_1}{Q_4} + \frac{r_1Q_1C_{b_1}}{2} + \frac{D_3S_1}{nQ_4} + \frac{r_1Q_1C_{v_1}}{2} \right) \right\} \left( n - 1 \right) \left( 1 - \frac{D_2}{P_1} \right) \left( 1 - \frac{D_3}{P_2} \right) \left( 1 - \frac{D_4}{P_4} \right)
\]

We can apply the Graded Mean Integration Representation if \( P(\tilde{T}_C(\tilde{Q}, n)) \) with \( 0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4 \). It will not change the meaning of formula if we replace inequality conditions \( 0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4 \) into the following inequality \( Q_2 - Q_1 \geq 0, Q_3 - Q_2 \geq 0, Q_4 - Q_3 \geq 0, Q_1 > 0 \). In the following steps, extension of the Lagrangean Method is used to find the solutions of \( Q_1, Q_2, Q_3 \) and \( Q_4 \) to minimize \( P(\tilde{T}_C(\tilde{Q}, n)) \).

**Step 1**: Solve the unconstraint problem. To find the \( \min [P(\tilde{T}_C(\tilde{Q}, n))] \), we have to find the derivative of \( \min [P(\tilde{T}_C(\tilde{Q}, n))] \) with respect to \( Q_1, Q_2, Q_3, Q_4 \).
\[
\begin{align*}
\frac{\partial P}{\partial Q_1} &= \frac{1}{6} \left[ \frac{r_1 C_{b_1}}{2} + \frac{r_1 C_{v_1}}{2} \left( n - 1 \right) \left( 1 - \frac{D_4}{P_1} \right) - \frac{D_4 A_4}{Q_1^2} - \frac{D_4 S_4}{nQ_1^2} \right] \\
\frac{\partial P}{\partial Q_2} &= \frac{2}{6} \left[ \frac{r_2 C_{b_2}}{2} + \frac{r_2 C_{v_2}}{2} \left( n - 1 \right) \left( 1 - \frac{D_2}{P_2} \right) + \frac{D_2 A_2}{Q_2^2} - \frac{D_2 S_2}{nQ_2^2} \right] \\
\frac{\partial P}{\partial Q_3} &= \frac{2}{6} \left[ \frac{r_3 C_{b_3}}{2} + \frac{r_3 C_{v_3}}{2} \left( n - 1 \right) \left( 1 - \frac{D_3}{P_3} \right) + \frac{D_3 A_3}{Q_3^2} - \frac{D_3 S_3}{nQ_3^2} \right] \\
\frac{\partial P}{\partial Q_4} &= \frac{1}{6} \left[ \frac{r_4 C_{b_4}}{2} + \frac{r_4 C_{v_4}}{2} \left( n - 1 \right) \left( 1 - \frac{D_1}{P_4} \right) + \frac{D_4 A_4}{Q_4^2} - \frac{D_4 S_4}{nQ_4^2} \right]
\end{align*}
\]

Let all the above partial derivatives equal to zero and solve \( Q_1, Q_2, Q_3, Q_4 \).

\begin{align*}
Q_1 &= \frac{2 \left[ A_4 d_4 + \frac{D_4 S_4}{n} \right]}{r_1 C_{b_1} + r_1 C_{v_1} \left[ n \left( 1 - \frac{D_4}{P_1} \right) + \frac{(D_1 + D_4)}{P_1} \right] - 1} \\
Q_2 &= \frac{2.2 \left[ A_3 d_3 + \frac{D_3 S_3}{n} \right]}{2r_2 C_{b_2} + 2r_2 C_{v_2} \left[ n \left( 1 - \frac{D_3}{P_2} \right) + \frac{(D_3 + D_2)}{P_2} \right] - 1} \\
Q_3 &= \frac{2.2 \left[ A_2 d_2 + \frac{D_2 S_2}{n} \right]}{2r_3 C_{b_3} + 2r_3 C_{v_3} \left[ n \left( 1 - \frac{D_2}{P_3} \right) + \frac{(D_2 + D_3)}{P_3} \right] - 1} \\
Q_4 &= \frac{2 \left[ A_4 d_4 + \frac{D_4 S_4}{n} \right]}{r_4 C_{b_4} + r_4 C_{v_4} \left[ n \left( 1 - \frac{D_1}{P_4} \right) + \frac{(D_1 + D_4)}{P_4} \right] - 1}
\end{align*}
Therefore set K = 1 and go to step 2.

**Step 2:** Convert the inequality constraint $Q_2 - Q_1 \geq 0$ into equality constraint $Q_2 - Q_1 = 0$ and optimize $P(\bar{T}\bar{C}(\bar{Q}, n))$ subject to $Q_2 - Q_1 = 0$ by the Lagrangean Method.

$$L(Q_1, Q_2, Q_3, Q_4, \lambda) = P(\bar{T}\bar{C}(\bar{Q}, n)) - \lambda(Q_2 - Q_1).$$

Taking the partial derivatives of $L(Q_1, Q_2, Q_3, Q_4, \lambda)$ with respect to $Q_1, Q_2, Q_3, Q_4$ and $\lambda$ to find the minimization of $L(Q_1, Q_2, Q_3, Q_4, \lambda)$.

Let all the above partial derivatives $\frac{\partial L}{\partial Q_1}, \frac{\partial L}{\partial Q_2}, \frac{\partial L}{\partial Q_3}, \frac{\partial L}{\partial Q_4}, \frac{\partial L}{\partial \lambda}$ equal to zero and solve to $Q_1, Q_2, Q_3, Q_4$. Then we get

$$Q_1 = Q_2 = \frac{2 \left[ (D_4A_4 + 2D_3A_3) + \left( \frac{D_4S_4 + 2D_3S_3}{n} \right) \right]}{\left( r_tC_{b_1} + 2r_tC_{b_2} \right) + r_tC_{v_1} \left[ n \left( 1 - \frac{D_4}{P_1} \right) + \left( \frac{D_4 + D_2}{P_1} \right) - 1 \right]} ;$$

$$Q_3 = \frac{2 \cdot 2 \cdot \left[ D_2A_2 + \frac{D_2S_2}{n} \right]}{2r_3C_{b_3} + 2r_3C_{v_3} \left[ n \left( 1 - \frac{D_2}{P_3} \right) + \left( \frac{D_2 + D_3}{P_3} \right) - 1 \right]} ;$$

$$Q_4 = \frac{2 \left[ D_1A_1 + \frac{D_1S_1}{n} \right]}{r_4C_{b_4} + 2r_4C_{v_4} \left[ n \left( 1 - \frac{D_1}{P_4} \right) + \left( \frac{D_1 + D_4}{P_4} \right) - 1 \right]} .$$
Because the above show that \( Q_3 > Q_4 \) it does not satisfy the constraint \( 0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4 \). Therefore it is not a local optimum, set \( K = 2 \) and go to step 3.

**Step 3:** Convert the inequality constraint \( Q_2 - Q_1 \geq 0, Q_3 - Q_2 \geq 0 \) into equality constraints \( Q_2 - Q_1 = 0 \) and \( Q_3 - Q_2 = 0 \). We optimize \( P(\bar{TC}(\bar{Q}, n)) \) subject to \( Q_2 - Q_1 = 0 \) and \( Q_3 - Q_2 = 0 \) by the Lagrangean Method. Then the Lagrangean Method is

\[
L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2) = P(\bar{TC}(\bar{Q}, n)) - \lambda_1 (Q_2 - Q_1) - \lambda_2 (Q_3 - Q_2).
\]

In order to find the minimization of \( L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2) \). We take the partial derivatives of \( L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2) \) with respect to \( Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2 \) and let all the partial derivatives equal to zero and to solve \( Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2 \).

\[
Q_1 = Q_2 = Q_3 = Q_4 = \frac{2 \left[ (D_4A_4 + 2D_3A_3 + 2D_2A_2) + \left( \frac{D_4A_4 + 2D_3S_3 + 2D_2S_2}{n} \right) \right]}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3}) + r_1C_{v_1} \left[ n \left( \frac{1 - D_4}{P_1} \right) + \left( \frac{D_4 + D_3}{P_1} \right) - 1 \right] + 2r_2C_{v_2} \left[ n \left( \frac{1 - D_3}{P_2} \right) + \left( \frac{D_3 + D_2}{P_2} \right) - 1 \right] + 2r_3C_{v_3} \left[ n \left( \frac{1 - D_2}{P_3} \right) + \left( \frac{D_2 + D_3}{P_3} \right) - 1 \right]}
\]
The above result $Q_1 > Q_4$ does not satisfy the constraint $0 < Q_i \leq Q_2 \leq Q_3 \leq Q_4$.

Therefore set $K = 3$ and go to step 4.

**Step 4**: Convert the inequality constraint $Q_2 - Q_1 \geq 0$, $Q_3 - Q_2 \geq 0$ and $Q_4 - Q_3 \geq 0$ into equality constraints $Q_2 - Q_1 = 0$, $Q_3 - Q_2 = 0$, $Q_4 - Q_3 = 0$. We optimize $P(TC(Q, n))$ subject to $Q_2 - Q_1 = 0$, $Q_3 - Q_2 = 0$, $Q_4 - Q_3 = 0$ by the Lagrangean Method. The Lagrangean Function is given by

$$L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2, \lambda_3)$$

$$= P(TC(Q, n)) - \lambda_1 (Q_2 - Q_1) - \lambda_2 (Q_3 - Q_2) - \lambda_3 (Q_4 - Q_3)$$

In order to find the minimization of $L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2, \lambda_3)$. We take the partial derivatives of $L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2, \lambda_3)$ with respect to $Q_1$, $Q_2$, $Q_3$, $Q_4$, $\lambda_1$, $\lambda_2$ and $\lambda_3$. Let all the partial derivatives equal to zero.

$$Q^* = Q_1 = Q_2 = Q_3 = Q_4 =$$

$$\frac{2}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3} + r_4C_{v_1}) + r_1C_{v_1}} \left[ n \left( 1 - \frac{D_2}{P_2} \right) + \frac{(D_2 + D_3)}{P_3} - 1 \right]$$

$$+ 2r_2C_{v_1} \left[ n \left( 1 - \frac{D_3}{P_3} \right) + \frac{(D_3 + D_2)}{P_2} - 1 \right] + 2r_3C_{v_1} \left[ n \left( 1 - \frac{D_4}{P_1} \right) + \frac{(D_1 + D_4)}{P_1} - 1 \right]$$

$$+ r_4C_{v_1} \left[ n \left( 1 - \frac{D_1}{P_4} \right) + \frac{(D_1 + D_4)}{P_4} - 1 \right]$$

$$+ \left[ \frac{2}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3} + r_4C_{v_1}) + r_1C_{v_1}} \left[ n \left( 1 - \frac{D_2}{P_2} \right) + \frac{(D_2 + D_3)}{P_3} - 1 \right] \right]$$

$$+ \left[ \frac{2}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3} + r_4C_{v_1}) + r_1C_{v_1}} \left[ n \left( 1 - \frac{D_3}{P_3} \right) + \frac{(D_3 + D_2)}{P_2} - 1 \right] \right]$$

$$+ \left[ \frac{2}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3} + r_4C_{v_1}) + r_1C_{v_1}} \left[ n \left( 1 - \frac{D_4}{P_1} \right) + \frac{(D_1 + D_4)}{P_1} - 1 \right] \right]$$

$$+ \left[ \frac{2}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3} + r_4C_{v_1}) + r_1C_{v_1}} \left[ n \left( 1 - \frac{D_1}{P_4} \right) + \frac{(D_1 + D_4)}{P_4} - 1 \right] \right]$$

$$\cdot \frac{2}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3} + r_4C_{v_1}) + r_1C_{v_1}} \left[ n \left( 1 - \frac{D_2}{P_2} \right) + \frac{(D_2 + D_3)}{P_3} - 1 \right]$$

$$+ \left[ \frac{2}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3} + r_4C_{v_1}) + r_1C_{v_1}} \left[ n \left( 1 - \frac{D_3}{P_3} \right) + \frac{(D_3 + D_2)}{P_2} - 1 \right] \right]$$

$$+ \left[ \frac{2}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3} + r_4C_{v_1}) + r_1C_{v_1}} \left[ n \left( 1 - \frac{D_4}{P_1} \right) + \frac{(D_1 + D_4)}{P_1} - 1 \right] \right]$$

$$+ \left[ \frac{2}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3} + r_4C_{v_1}) + r_1C_{v_1}} \left[ n \left( 1 - \frac{D_1}{P_4} \right) + \frac{(D_1 + D_4)}{P_4} - 1 \right] \right]$$

$$\cdot \frac{2}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3} + r_4C_{v_1}) + r_1C_{v_1}} \left[ n \left( 1 - \frac{D_2}{P_2} \right) + \frac{(D_2 + D_3)}{P_3} - 1 \right]$$

$$+ \left[ \frac{2}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3} + r_4C_{v_1}) + r_1C_{v_1}} \left[ n \left( 1 - \frac{D_3}{P_3} \right) + \frac{(D_3 + D_2)}{P_2} - 1 \right] \right]$$

$$+ \left[ \frac{2}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3} + r_4C_{v_1}) + r_1C_{v_1}} \left[ n \left( 1 - \frac{D_4}{P_1} \right) + \frac{(D_1 + D_4)}{P_1} - 1 \right] \right]$$

$$+ \left[ \frac{2}{(r_1C_{b_1} + 2r_2C_{b_2} + 2r_3C_{b_3} + r_4C_{v_1}) + r_1C_{v_1}} \left[ n \left( 1 - \frac{D_1}{P_4} \right) + \frac{(D_1 + D_4)}{P_4} - 1 \right] \right]$$
2.3.4. Numerical Example

Consider an integrated inventory system with the following characteristics.

\[ \begin{align*}
D &= 1000 \text{ units per year} & P &= 2000 \text{ units per year} \\
r &= 0.2 \text{ per unit per year} & C_v &= $20 \text{ per unit} & C_b &= $25 \text{ per unit} \\
A &= $25 \text{ per order} & S &= $4000 \text{ per setup} & n &= 20 \text{ derivatives} \\
Q^* &= 100 \text{ units} & TC(100, 20) &= 4500 \text{ $ per year}
\end{align*} \]

Suppose Fuzzy annual demand is "more or less than 100"

\[ \tilde{D} = (D_1, D_2, D_3, D_4) = (800, 900, 1100, 1200) \]

Fuzzy annual production is "more or less than 2000"

\[ \tilde{P} = (P_1, P_2, P_3, P_4) = (1800, 1900, 2100, 2200) \]

Fuzzy annual inventory carrying cost is "more or less than 0.2"

\[ \tilde{r} = (r_1, r_2, r_3, r_4) = (0.1, 0.15, 0.25, 0.3) \]

Fuzzy vendor's unit production cost is "more or less than 20"

\[ \tilde{C}_v = (C_{v1}, C_{v2}, C_{v3}, C_{v4}) = (18, 19, 21, 22) \]

Fuzzy unit purchase cost is "more or less than 25"

\[ \tilde{C}_b = (C_{b1}, C_{b2}, C_{b3}, C_{b4}) = (23, 24, 26, 27) \]

Fuzzy buyer's ordering cost is "more or less than 25"

\[ \tilde{A} = (A_1, A_2, A_3, A_4) = (23, 24, 26, 27) \]

Fuzzy vendor's setup cost is "more or less than 4000"

\[ \tilde{S} = (S_1, S_2, S_3, S_4) = (3800, 3900, 4100, 4200) \]

Fuzzy lot size

\[ \tilde{Q} = (Q_1, Q_2, Q_3, Q_4) \text{ with } 0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4. \]
\( Q' = 92.5 \)

Optimal total cost of the integrated two stage inventory system for the relationship of vendor-buyer

\[ \hat{\text{TC}}(\bar{Q}, B) = (3383.68, 4429.38, 4429.32, 4221.46) \]

In this example, it is important to point out that the total cost function of the integrated vendor-buyer inventory system contains two decision variables \( n \) and \( Q \). We used differential calculus to determine the optimal solution for both variables. The fuzzy EOQ obtained by the Lagrangean Method is closer to crisp EOQ and EOQ is more sensitive towards demand. It provides the effect of changes in the production on fuzzy optimal solution.