Chapter - IV

Optimization of a Multiple-Vendor Single Buyer Integrated Inventory Model with a Variable Number of Vendors and Quantity Discount Permissible Delay in Payments
CHAPTER – IV

OPTIMIZATION OF A MULTIPLE-VENDOR SINGLE BUYER INTEGRATED INVENTORY MODEL WITH A VARIABLE NUMBER OF VENDORS AND QUANTITY DISCOUNT PERMISSIBLE DELAY IN PAYMENTS

In this model we consider the fuzzy total cost under crisp order quantity or fuzzy order quantity and in order to extend the traditional inventory model to the fuzzy environment. Then we use the Kuhn-Tucker method to find the optimal order quantity of the fuzzy order inventory model. We use Function Principle as arithmetical operations of fuzzy total average individual cost and use the Graded Mean Integration Representation Method to defuzzify the fuzzy total average individual cost. Then we use the Kuhn-Tucker Method to find the optimal order quantity of the fuzzy order inventory model.

SECTION – 1

4.1. A MULTIPLE-VENDOR SINGLE BUYER INTEGRATED INVENTORY MODEL

This model considers a single buyer who orders a single product at multiple vendors. The following assumptions were made in developing the proposed model by Kim, T., Goyal, S.K., [73], Agrawal, K., & Raju, A., [2].

Section 1 of this chapter has been published in the International Journal of Mathematical Sciences and Engineering Applications (IJMSEA), ISSN 0973-9424, Vol.5, No.V (Sep.2011), pp.173-188.
Section 2 of this chapter has been published in the International Journal of Advance in Mathematical Sciences, ISSN 0973-5798, Vol.1, No.61, (Jan-July 2011).
The objective of the model is to minimize the total costs of the system under study.

All parameters are deterministic and constant over time.

The suppliers are heterogeneous, (i.e.) their cost parameters may be different.

The buyer faces a pool of pre-selected suppliers which are capable of delivering the requested product (i.e.) we assume that the buyer has conducted a market study and excluded all supplies from the analysis which do not meet pre-defined selection criteria.

The production capacity of a supplier can be smaller than the demand rate of the buyer. However, the sum of the production capacities of all pre-selected suppliers is larger than the demand rate of the buyer. This is necessary to assure that the demand of the buyer can be satisfied without interruption.

We assume that the unit inventory carrying charges per unit of time at the buyer $h^{(b)}$ are higher than the unit inventory carrying charges per unit of time at one of the vendors due to the value added concept. As a consequence, it is optimal to store batches at the vendors until the inventory of the buyer has been depleted.

A lot produced at one of the vendors is sent in equal-sized batch shipments to the buyer. The number of batch shipments may be different for each vendor.

Shortages are not allowed.
For the sake of brevity, we will use the male gender to refer the actors that could be male or female.

4.1.1. Decision Variables

$\beta_i \to$ Proportion of the order lot size that is produced by vendor $i$ with

$$\sum_{i=1}^{N} \beta_i = 1$$

where $N$ is the number of vendors in the supplier pool.

$Q \to$ Order lot size with $Q = \sum_{i=1}^{N} q_i$ where $q_i$ is the production lot size of supplier $i$.

$m_i \to$ Number of equal-sized batches per lot of vendor $i$ variables that can be derived implicitly by the decision variables.

$\delta_i \to$ A binary variable which is 1 if supplier $i$ is selected and 0 in all other instances, (ie) $\delta_i = \begin{cases} 1 & \text{for } \beta_i > 0 \\ 0 & \text{for } \beta_i = 0 \end{cases}$

$TC^{(b)} \to$ Total costs of the buyer.

$TC_i^{(b)} \to$ Total costs of the vendor $i$.

$TC^{(s)} \to$ Total costs of the system.

$[a]^+ \to$ The largest element of $a$ and 0 i.e., $[a]^+ = \max \{a, 0\}$

$\psi \to$ The set of selected suppliers with $\psi \subseteq \Omega$

i.e., $\psi(\delta) = \{i \in \Omega/\delta_i = 1\}$ with $\delta_i = \{\delta_1, \delta_2, \ldots, \delta_N\}$.

$\Omega \to$ The set of pre-selected suppliers with $\Omega = \{1, 2, 3, \ldots, N\}$. 
4.2. THE EOQ INVENTORY MODEL WITH A VARIABLE NUMBER OF VENDORS

In this section, we develop a sequential optimization method using Kuhn-Tucker Method. We use this method to find the optimal Economic Order Quantity (EOQ) of the fuzzy inventory model by, Aissani, N., Haouari, M., & Hassini, E., [3] and Christoph Glock., [36].

4.2.1. Model Formulation

The problem studied in this model consists of an inventory problem, where the lot size Q, the individual production quantities qi and shipment frequency mi have to be calculated with the objective to minimize total system costs, and a combinatorial problem which consists in selecting a set of suppliers ψ from the set of pre-selected suppliers Ω by Ghodsypour, S.H., & O’Brain, C., [44]. The total cost per unit of time of the buyer can be formulated as

$$TC^{(b)} = \sum_{i \in \psi} \frac{q_i^2}{2m_iQ} + \frac{AD}{Q} + R\sum_{i=1}^{N} \delta_i$$

The total costs of vendor i are given as

$$TC_i^{(v)} = q_i^2 \left( \frac{1}{2P_i} + \left( \frac{1}{m_i} \right) \left( \frac{1}{D_i} \right) \right) - \frac{m_i-1}{2m_iD_i} \frac{Dh_i^{(v)}}{Q} + (S_i + F_i m_i + C_i q_i) \frac{D}{Q}$$

The total costs of the system is defined as

$$TC^{(s)} = Q \left( \sum_{i \in \psi} \frac{\beta_i h_i^{(v)}}{2m_i} + \sum_{i \in \psi} \beta_i^2 \lambda_i h_i^{(v)} \right) + \frac{\gamma D}{Q} + \sum_{i \in \psi} C_i \beta_i D + R \sum_{i=1}^{N} \delta_i$$
where \( \lambda_i = \frac{1}{2P_i} + \left[ \left( \frac{1}{m_i} - 1 \right) \left( \frac{1}{D_i} - 1 \right) \right] - \frac{m_i - 1}{2mD} \)

and \( \chi = \sum_{i \in \psi} (S_i + Fm_i) + A \)

Differentiating the total cost with respect to \( Q \), equating to zero and solving for \( Q \) yields the optimal order quantity by Glock, C.H., [45]

\[
Q_{opt} = \sqrt[1]{\frac{D\chi}{\sum_{i \in \psi} \beta_i^2 \left( \frac{h_i^{(b)}}{2m_i} + D\lambda h_i^{(v)} \right)}}
\]

Throughout this, we use the following variables in order to simplify the treatment of the fuzzy inventory models \( \bar{h}_i^{(b)}, \bar{D}, h_i^{(v)}, \bar{R}, \bar{C}_i, \bar{S}, \bar{P} \), \( \bar{S}_i, \bar{\chi}, \bar{\lambda}_i \) are fuzzy parameters.

This fuzzy total cost of the system is

\[
TC_s = \left\{ Q \left( \sum_{i \in \psi} \frac{\beta_i^{(b)} h_i^{(b)}}{2m_i} + \sum_{i \in \psi} \beta_i^2 \lambda_i h_i^{(v)} D_1 \right) + \frac{\chi_1 D_1}{Q} + \sum_{i \in \psi} C_i \beta_i D_1 + R_1 \sum_{i = 1}^N \delta_i, \right. \\
Q \left( \sum_{i \in \psi} \frac{\beta_i^{(b)} h_i^{(b)}}{2m_i} + \sum_{i \in \psi} \beta_i^2 \lambda_i h_i^{(v)} D_2 \right) + \frac{\chi_2 D_2}{Q} + \sum_{i \in \psi} C_i \beta_i D_2 + R_2 \sum_{i = 1}^N \delta_i, \\
Q \left( \sum_{i \in \psi} \frac{\beta_i^{(b)} h_i^{(b)}}{2m_i} + \sum_{i \in \psi} \beta_i^2 \lambda_i h_i^{(v)} D_3 \right) + \frac{\chi_3 D_3}{Q} + \sum_{i \in \psi} C_i \beta_i D_3 + R_3 \sum_{i = 1}^N \delta_i, \\
Q \left( \sum_{i \in \psi} \frac{\beta_i^{(b)} h_i^{(b)}}{2m_i} + \sum_{i \in \psi} \beta_i^2 \lambda_i h_i^{(v)} D_4 \right) + \frac{\chi_4 D_4}{Q} + \sum_{i \in \psi} C_i \beta_i D_4 + R_4 \sum_{i = 1}^N \delta_i \right\}
\]

which implies
\[
\tilde{C}^{(s)} = Q \otimes \left( \sum_{i \in \Psi} \left( \frac{\beta_i^2 \otimes \tilde{h}_i^{(b)}}{2} \otimes m_i \right) \oplus \sum_{i \in \Psi} (\beta_i^2 \otimes \lambda_i \otimes \tilde{h}_i^{(v)} \otimes D) \right) \oplus \left( \tilde{\chi} \otimes \tilde{D} \otimes Q \right)
\]

where \( \otimes, \oplus, \ominus, \oplus \) are the fuzzy arithmetical operations under function principle.

Suppose,
\[
\tilde{A} = (A_1, A_2, A_3, A_4) \quad \tilde{C}_i = (C_{i1}, C_{i2}, C_{i3}, C_{i4})
\]
\[
\tilde{R} = (R_1, R_2, R_3, R_4) \quad \tilde{\lambda}_i = (\lambda_{i1}, \lambda_{i2}, \lambda_{i3}, \lambda_{i4})
\]
\[
\tilde{F}_i = (F_1, F_2, F_3, F_4) \quad \tilde{\chi} = (\chi_1, \chi_2, \chi_3, \chi_4)
\]
\[
\tilde{D} = (D_1, D_2, D_3, D_4) \quad h_i^{(b)} = (h_1^{(b)}, h_2^{(b)}, h_3^{(b)}, h_4^{(b)})
\]
\[
\lambda_i^{(v)} = (\lambda_1^{(v)}, \lambda_2^{(v)}, \lambda_3^{(v)}, \lambda_4^{(v)})
\]

are non-negative trapezoidal fuzzy numbers. Then we solve the optimal order quantity as the following steps. Second we defuzzify the fuzzy total inventory cost, using the Graded Mean Integration Representation Method. The result is

\[
P\left(\tilde{C}^{(s)}\right) = \frac{1}{6} \left\{ Q \left( \sum_{i \in \Psi} \frac{\beta_i^2 h_1^{(b)}}{2m_i} + \sum_{i \in \Psi} \beta_i^2 \lambda_i h_2^{(v)} D_1 \right) + \frac{\chi_2 D_2}{Q} \sum_{i \in \Psi} C_i \beta_i D_i + R_1 \sum_{i = 1}^N \delta_i \right. \\
+ Q \left( \sum_{i \in \Psi} \frac{\beta_i^2 h_2^{(b)}}{2m_i} + \sum_{i \in \Psi} \beta_i^2 \lambda_i h_3^{(v)} D_2 \right) + \frac{\chi_3 D_3}{Q} \sum_{i \in \Psi} C_i \beta_i D_i + R_2 \sum_{i = 1}^N \delta_i \right. \\
+ Q \left( \sum_{i \in \Psi} \frac{\beta_i^2 h_3^{(b)}}{2m_i} + \sum_{i \in \Psi} \beta_i^2 \lambda_i h_4^{(v)} D_3 \right) + \frac{\chi_4 D_4}{Q} \sum_{i \in \Psi} C_i \beta_i D_i + R_3 \sum_{i = 1}^N \delta_i \right. \\
+ Q \left( \sum_{i \in \Psi} \frac{\beta_i^2 h_4^{(b)}}{2m_i} + \sum_{i \in \Psi} \beta_i^2 \lambda_i h_5^{(v)} D_4 \right) + \frac{\chi_5 D_5}{Q} \sum_{i \in \Psi} C_i \beta_i D_i + R_4 \sum_{i = 1}^N \delta_i \left. \right\}
\]
Third, we can get the optimal order quantity $Q_{opt}$, when $P(TC^{(S)})$ is minimization. In order to find the minimization of $P(TC^{(S)})$ the derivative of $P(TC^{(S)})$ with $Q$ is

$$\frac{\partial P(TC^{(S)})}{\partial Q} = 0$$

Hence we find the optimal order quantity $Q_{opt}$

$$Q_{opt} = \sqrt{\frac{D_{1}X_{1} + 2(D_{2}X_{2}) + 2D_{3}X_{3} + D_{4}X_{4}}{\sum_{i \in \Psi} \beta^{2}_{i} \left( \frac{h_{i}^{(b)}}{2m_{i}} + \lambda_{i} h_{i}^{(v)}D_{1} \right) + 2 \sum_{i \in \Psi} \beta^{2}_{i} \left( \frac{h_{i}^{(b)}}{2m_{i}} + \lambda_{i} h_{i}^{(v)}D_{2} \right) + 2 \sum_{i \in \Psi} \beta^{2}_{i} \left( \frac{h_{i}^{(b)}}{2m_{i}} + \lambda_{i} h_{i}^{(v)}D_{3} \right) + \sum_{i \in \Psi} \beta^{2}_{i} \left( \frac{h_{i}^{(b)}}{2m_{i}} + \lambda_{i} h_{i}^{(v)}D_{4} \right)}}$$

### 4.2.2. Fuzzy Inventory EOQ Model with Fuzzy Order Quantity

In this section, we introduce the fuzzy inventory EOQ models by changing the crisp order quantity $Q$ be a trapezoidal fuzzy number $\tilde{Q} = (Q_{1}, Q_{2}, Q_{3}, Q_{4})$ with $0 < Q_{1} \leq Q_{2} \leq Q_{3} \leq Q_{4}$ by Joglekar, P.N. and Tharthare, S., [69], Lu, L., [85], Sarker, B.R. and Coates, E.R., [107]. Then we get the fuzzy total cost function as

$$\tilde{TC}^{(S)} = \left\{ \left( Q_{1} \left( \sum_{i \in \Psi} \frac{\beta^{2}_{i} h_{i}^{(b)}}{2m_{i}} + \sum_{i \in \Psi} \beta^{2}_{i} \lambda_{i} h_{i}^{(v)}D_{1} \right) + \frac{\chi_{1} D_{1}}{Q_{4}} + \sum_{i \in \Psi} C_{1} \beta_{i} D_{1} + R_{1} \sum_{i = 1}^{N} \delta_{i} \right) \right\}$$

$$\left( Q_{2} \left( \sum_{i \in \Psi} \frac{\beta^{2}_{i} h_{i}^{(b)}}{2m_{i}} + \sum_{i \in \Psi} \beta^{2}_{i} \lambda_{i} h_{i}^{(v)}D_{2} \right) + \frac{\chi_{2} D_{2}}{Q_{4}} + \sum_{i \in \Psi} C_{2} \beta_{i} D_{2} + R_{2} \sum_{i = 1}^{N} \delta_{i} \right)$$
Secondly we defuzzify the fuzzy total cost function using the Graded Mean Integration Representation Method with $0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4$.

It will not change the meaning of formula, if we replace inequality conditions $0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4$. In the following inequality constraints $Q_2 - Q_1 > 0$, $Q_3 - Q_2 > 0$, $Q_4 - Q_3 > 0$ and $Q_i > 0$.

Thirdly, the Kuhn - Tucker condition is used to find the solution of $Q_1$, $Q_2$, $Q_3$, $Q_4$ to minimize $P\left(\tilde{T}C^{(S)}\right)$, subject to $Q_2 - Q_1 \geq 0$, $Q_3 - Q_2 \geq 0$, $Q_4 - Q_3 \geq 0$ and $Q_1 \geq 0$. The Kuhn-Tucker conditions are, $\lambda \leq 0$.

$$\nabla f\left(\tilde{T}C^{(S)}(Q_i)\right) - \lambda \nabla g(Q_i) = 0$$

$$\lambda g_i(Q_i) = 0$$

$$g_i(Q_i) \geq 0$$

These conditions simplify to the following $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4 \leq 0$ and

$$\nabla f\left(\tilde{T}C^{(S)}(Q_i)\right) - \lambda \nabla g(Q_i) = 0$$
\[
\Rightarrow \frac{1}{6} \left\{ Q_1 \left( \sum_{i \in \psi} \beta_i^2 h_i^{(b)} \frac{1}{2m_i} + \sum_{i \in \psi} \beta_i^2 \lambda_i h_i^{(v)} D_i \right) + \chi \frac{D_1}{Q_4} + \psi \sum_{i \in \psi} C_i \beta_i D_i + R_i \sum_{i = 1}^N \delta_i \right\} \\
+ 2 \left\{ Q_2 \left( \sum_{i \in \psi} \beta_i^2 h_i^{(b)} \frac{1}{2m_i} + \sum_{i \in \psi} \beta_i^2 \lambda_i h_i^{(v)} D_2 \right) + \chi \frac{D_2}{Q_3} + \sum_{i \in \psi} C_i \beta_i D_2 + R_2 \sum_{i = 1}^N \delta_i \right\} \\
+ 2 \left\{ Q_3 \left( \sum_{i \in \psi} \beta_i^2 h_i^{(b)} \frac{1}{2m_i} + \sum_{i \in \psi} \beta_i^2 \lambda_i h_i^{(v)} D_3 \right) + \chi \frac{D_3}{Q_2} + \sum_{i \in \psi} C_i \beta_i D_3 + R_3 \sum_{i = 1}^N \delta_i \right\} \\
+ \left\{ Q_4 \left( \sum_{i \in \psi} \beta_i^2 h_i^{(b)} \frac{1}{2m_i} + \sum_{i \in \psi} \beta_i^2 \lambda_i h_i^{(v)} D_4 \right) + \chi \frac{D_4}{Q_1} + \sum_{i \in \psi} C_i \beta_i D_4 + R_4 \sum_{i = 1}^N \delta_i \right\}
\]

which implies

\[
\frac{1}{6} \left\{ \sum_{i \in \psi} \beta_i^2 h_i^{(b)} \frac{1}{2m_i} + \sum_{i \in \psi} \beta_i^2 \lambda_i h_i^{(v)} D_1 + \frac{\lambda_4 D_4}{Q_4} \right\} - \lambda_4 + \lambda_1 = 0
\]

\[
\frac{2}{6} \left\{ \sum_{i \in \psi} \beta_i^2 h_i^{(b)} \frac{1}{2m_i} + \sum_{i \in \psi} \beta_i^2 \lambda_i h_i^{(v)} D_2 + \frac{\lambda_3 D_3}{Q_3} \right\} - \lambda_3 + \lambda_2 = 0
\]

\[
\frac{2}{6} \left\{ \sum_{i \in \psi} \beta_i^2 h_i^{(b)} \frac{1}{2m_i} + \sum_{i \in \psi} \beta_i^2 \lambda_i h_i^{(v)} D_3 + \frac{\lambda_2 D_2}{Q_2} \right\} - \lambda_2 + \lambda_3 = 0
\]

\[
\frac{1}{6} \left\{ \sum_{i \in \psi} \beta_i^2 h_i^{(b)} \frac{1}{2m_i} + \sum_{i \in \psi} \beta_i^2 \lambda_i h_i^{(v)} D_4 + \frac{\lambda_1 D_1}{Q_1} \right\} - \lambda_3 = 0
\]

\[
\lambda_1 (Q_2 - Q_1) = 0
\]

\[
\lambda_2 (Q_3 - Q_2) = 0
\]

\[
\lambda_3 (Q_4 - Q_3) = 0
\]

\[
\lambda_4 Q_1 = 0
\]
Because \( Q_1 > 0 \) and \( \lambda_4 Q_1 = 0 \) then \( \lambda_4 = 0 \).

If \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \) then \( Q_4 < Q_3 < Q_2 < Q_1 \), it does not satisfy the constraints 
\( 0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4 \).

Therefore \( Q_2 = Q_1 \), \( Q_3 = Q_2 \) and \( Q_4 = Q_3 \) i.e., \( Q_1 = Q_2 = Q_3 = Q_4 = Q^* \).

Hence from we find the optimal order quantity \( Q^* \) as

\[
Q^* = \sqrt{\frac{D_1 \chi_1 + 2(D_2 \chi_2) + 2D_3 \chi_3 + D_4 \chi_4}{\sum_{i \in \psi} \beta_i^2 \left( \frac{h_i^{(n)}}{2m_i} + \lambda_i h_i^{(v)} D_i \right) + 2 \sum_{i \in \psi} \beta_i^2 \left( \frac{h_i^{(n)}}{2m_i} + \lambda_i h_i^{(v)} D_2 \right) + 2 \sum_{i \in \psi} \beta_i^2 \left( \frac{h_i^{(n)}}{2m_i} + \lambda_i h_i^{(v)} D_3 \right) + \sum_{i \in \psi} \beta_i^2 \left( \frac{h_i^{(n)}}{2m_i} + \lambda_i h_i^{(v)} D_4 \right)}}
\]

4.2.3. Numerical Example

Consider an inventory system with the following characteristics:

If \( D = 150 \), \( A = 100 \), \( R = 60 \); \( h_i^{(b)} = 5 \); \( i = 1, 2, 3 \).

\( (P_1, P_2, P_3) = (60, 155, 190) \);

\( (h_1^{(v)}, h_2^{(v)}, h_3^{(v)}) = (1, 2, 2) \);

\( (C_1, C_2, C_3) = (1, 2, 3) \);

\( (S_1, S_2, S_3) = (150, 250, 230) \);

\( (F_1, F_2, F_3) = (55, 60, 105) \);

\( (\beta_1, \beta_2, \beta_3) = (0, 1, 0) \);

\( (m_1, m_2, m_3) = (0, 25, 0) \);

then \( Q = 1278.85 \) and \( TC = 926.79 \)

Suppose fuzzy demand rate at the buyer is "more or less than 150"

\( \tilde{D} = (50, 100, 200, 250) \);
Suppose fuzzy relationship management costs per supplier is "more or less than 60"

\[ \tilde{R} = (40, 50, 70, 80) ; \]

Suppose fuzzy ordering costs per order is "more or less than 100"

\[ \tilde{A} = (50, 75, 125, 150) ; \]

Suppose fuzzy unit inventory carrying charges per unit of time at the buyer is "more or less than 5"

\[ \tilde{h}^{(b)} = (3, 4, 6, 7) ; \]

Suppose fuzzy cost per unit ordered at vendor \( i \) (1, 2, 3) is "more or less than (1 2 3)"

\[ \tilde{C}_i = (.5, .75, 1.5, 2 ; 1, 1.5, 2, 2.5 ; 2, 2.5, 3, 3.5) \]

Suppose fuzzy transportation costs per delivery at vendor \( i \) is "more or less than (55 60 105)"

\[ \tilde{F}_i = (53, 54, 56, 57 ; 40, 50, 70, 80 ; 103, 104, 106, 107) \]

Suppose fuzzy setup costs per set up at vendor \( i \) (1, 2, 3) is "more or less than (150 250 230)"

\[ \tilde{S}_i = (148, 149, 151, 152 ; 248, 249, 251, 252 ; 228, 229, 231, 232) \]

Suppose fuzzy unit inventory carrying charges per unit of time at vendor \( i \) (1, 2, 3) is "more or less than (1, 2, 2)"

\[ \tilde{h}_i^{(v)} = (0.8, 0.9, 1.1, 1.2 ; 1.8, 1.9, 2.1, 2.2 ; 1.8, 1.9, 2.1, 2.2) \]

\[ m_i = (0, 25, 0) \quad (i = 1, 2, 3) \]
\[ \beta_i = (0, 1, 0) \quad (i = 1, 2, 3) \]
\[ P_i = (60, 155, 190) \quad (i = 1, 2, 3) \]
\[ \bar{\lambda}_i = (0.00018, 0.00019, 0.00021, 0.00022) \]
\[ \bar{\chi} = (1674, 2452, 2508, 2786) \]

Fuzzy order quantity

\[ Q^* = (1244.59, 1244.59, 1244.59, 1244.59) \]

Optimal total cost

\[ (252.09, 686.01, 1391.27, 1589.46) \]

Here, we considered a single product from multiple heterogeneous suppliers and tackled the supplier selection and lot size decision with the objective to minimize total system costs. It indicates that our solution procedure reduces the total number of supplier combinations that have to be tested for optimality and thus helps to reduce the complexity of the planning problem.

SECTION – 2

4.3. SINGLE SUPPLIER MULTIPLE COOPERATIVE RETAILERS INVENTORY MODEL WITH QUANTITY DISCOUNT AND PERMISSIBLE DELAY IN PAYMENTS

4.3.1. Mathematical Model

(i) Discount

We adopt the purchasing price’s formulation of Campos, L., Verdegay, J.L., [11] stated in terms of the ordered quantity as follows:
This equation reports three expressions where:

- The initial unit purchasing cost \( C \) is the announced price of the supplier in the market.
- \( e \) is a discount rate using in the landing \( 0 < q_i \leq Q_{\text{max}} \) by Wang, F.K., Wu, K.S., [124]
- Beyond \( Q_{\text{max}} \) and no matter how large the order quantity is the supplier charges a fixed minimum price equal to \( C_{\text{p min}} \), \( Q_{\text{max}} \) can be computed, as proposed in through the following equation.

\[
Q_{\text{max}} = \frac{C - C_{\text{p min}}}{e}
\]

(ii) Discount and Delay

We develop the purchasing cost in terms of two main options, namely: the quantity discount and the delay in payments. We start by stating \( C_p \) only with discount and then add the delay option. The general purchasing cost with discount and delay is reported as follows

\[
C_p = \begin{cases} 
C & q_i = 0 \\
(C - e q_i)(1 + a p_i) & 0 < q_i \leq Q_{\text{max}} \\
C_{\text{p min}}(1 + a p_i) & q_i > Q_{\text{max}} 
\end{cases}
\]
4.3.2. The Average Individual Cost Function for Retailer \( i (C_i) \)

If \( 0 < q_i \leq Q_{\text{max}} \) then

\[
C_i = a \frac{d_i}{q_i} + h_i \frac{q_i}{2} + (C - e q_i)(1 + \alpha p_i) d_i, \forall i \in N
\]

We keep the holding and ordering costs and propose a purchasing cost that insets a fractional cost incurred by postponing the payment. Where \( \frac{d_i}{q_i} \) is the number of placed orders, \( \frac{q_i}{2} \) is the average size of inventory and \((C - e q_i)(1 + \alpha p_i) d_i \) is the annual purchasing cost. The objective is to find the optimal order quantity which minimizes the average individual cost. The necessary conditions for minimum

\[
\frac{\partial C_i}{\partial q_i} = 0
\]

Therefore, the optimal order quantity is

\[
q_i^* = \sqrt{\frac{2 a d_i}{h_i - 2 e (1 + \alpha p_i) d_i}}, \forall i \in N
\]

If \( q_i > Q_{\text{max}} \)

\[
C_i = a \frac{d_i}{q_i} + h_i \frac{q_i}{2} + C_p^{\text{min}} (1 + \alpha p_i) d_i, \forall i \in N
\]

The optimal quantity in this case is equal to

\[
q_i^* = \sqrt{\frac{2 a d_i}{h_i}}, \forall i \in N
\]
Throughout this model, we use the following variables in order to simplify the treatment of the fuzzy inventory models. \( \tilde{a}, \tilde{h}_i, \tilde{c}, \tilde{e}, \tilde{a}, \tilde{p}_i \) are fuzzy parameters. The fuzzy average individual cost function for retailer \( i \) is

\[
\tilde{T}_C_i(q_i) = \left\{ \frac{a_i}{q_i} \times d_i + h_i \times \frac{q_i}{2} + (C_i - e_i \times q_i) \left( 1 + a_i \times P_i \right) \times d_i, \right. \\
\left. a_2 \times \frac{d_i}{q_i} + h_2 \times \frac{q_i}{2} + (C_2 - e_2 \times q_i) \left( 1 + a_2 \times P_2 \right) \times d_i, \\
\left. a_3 \times \frac{d_i}{q_i} + h_3 \times \frac{q_i}{2} + (C_3 - e_3 \times q_i) \left( 1 + a_3 \times P_3 \right) \times d_i, \\
\right. \\
\left. a_4 \times \frac{d_i}{q_i} + h_4 \times \frac{q_i}{2} + (C_4 - e_4 \times q_i) \left( 1 + a_4 \times P_4 \right) \times d_i \right\} 
\]

where \( \bigotimes, \bigoplus, \bigodot, \bigoplus \) are the fuzzy arithmetical operations under function principle.

Suppose,

\[
\tilde{a} = (a_1, a_2, a_3, a_4) \quad \tilde{h}_i = (h_{i1}, h_{i2}, h_{i3}, h_{i4})
\]

\[
\tilde{c} = (C_1, C_2, C_3, C_4) \quad \tilde{e} = (e_1, e_2, e_3, e_4)
\]

\[
\tilde{a} = (a_1, a_2, a_3, a_4) \quad \tilde{p}_i = (P_{i1}, P_{i2}, P_{i3}, P_{i4})
\]

are non-negative trapezoidal fuzzy numbers. Then we solve the optimal order quantity as the following steps: Second we defuzzify the fuzzy total inventory cost, using the Graded Mean Integration Representation Method. The result is

\[
P\left( \tilde{T}_C_i(q_i) \right) = \frac{1}{6} \left\{ a_i \times \frac{d_i}{q_i} + h_i \times \frac{q_i}{2} + (C_i - e_i \times q_i) \left( 1 + a_i \times P_i \right) \times d_i, \\
+ 2 \left[ a_2 \times \frac{d_i}{q_i} + h_2 \times \frac{q_i}{2} + (C_2 - e_2 \times q_i) \left( 1 + a_2 \times P_2 \right) \times d_i \right] \right\}
\]
Third, we can get the optimal order quantity \( q^*_i \), when \( P(\tilde{T}C_i(q_i)) \) is minimization. In order to find the minimization of \( P(\tilde{T}C_i(q_i)) \) the derivative of \( P(\tilde{T}C_i(q_i)) \) with \( q_i \) is

\[
\frac{\partial P(\tilde{T}C_i(q_i))}{\partial q_i} = 0
\]

Hence we find the optimal order quantity \( q^*_i \)

\[
q^*_i = \frac{2[(a_1 \times \bar{d}_i) + 2(a_2 \times \bar{d}_i) + 2(a_3 \times \bar{d}_i) + (a_4 \times \bar{d}_i)]}{(h_i + 2h_a + 2h_b + h_c) - 2[e_4 \times (1 + \alpha_i \times P_i) \times \bar{d}_i] + 2[e_3 \times (1 + \alpha_i \times P_i) \times \bar{d}_i] + [e_i \times (1 + \alpha_i \times P_i) \times \bar{d}_i]}
\]

### 4.3.3. Fuzzy Inventory EOQ Model with Fuzzy Order Quantity

In this section, we introduce the fuzzy inventory EOQ models by changing the crisp order quantity \( q_i \) be a trapezoidal fuzzy number \( \bar{q}_i = (q_{i1}, q_{i2}, q_{i3}, q_{i4}) \) with \( 0 < q_{i1} \leq q_{i2} \leq q_{i3} \leq q_{i4} \). Then we get the fuzzy average individual cost function for retailer as
\[ \tilde{T}\mathcal{C}_i(q_i) = \begin{cases} a_1 \frac{d}{q_i} + h_1 \frac{q_i}{2} + (C_1 - e_1 \times q_i)(1 + \alpha_i \times P_i) \times d_i, \\ a_2 \frac{d}{q_2} + h_2 \frac{q_2}{2} + (C_2 - e_2 \times q_2)(1 + \alpha_2 \times P_2) \times d_2, \\ a_3 \frac{d}{q_3} + h_3 \frac{q_3}{2} + (C_3 - e_2 \times q_3)(1 + \alpha_3 \times P_3) \times d_3, \\ a_4 \frac{d}{q_4} + h_4 \frac{q_4}{2} + (C_4 - e_1 \times q_4)(1 + \alpha_4 \times P_4) \times d_4 \end{cases} \]

Secondly we defuzzify the fuzzy average individual cost function for retailer using the Graded Mean Integration Representation Method. The result is

\[ P(\tilde{T}\mathcal{C}_i(q_i)) = \frac{1}{6} \begin{cases} a_1 \frac{d}{q_i} + h_1 \frac{q_i}{2} + (C_1 - e_1 \times q_i)(1 + \alpha_i \times P_i) \times d_i \\ + 2 \left[ a_2 \frac{d}{q_2} + h_2 \frac{q_2}{2} + (C_2 - e_2 \times q_2)(1 + \alpha_2 \times P_2) \times d_2 \right] \\ + 2 \left[ a_3 \frac{d}{q_3} + h_3 \frac{q_3}{2} + (C_3 - e_2 \times q_3)(1 + \alpha_3 \times P_3) \times d_3 \right] \\ + 2 \left[ a_4 \frac{d}{q_4} + h_4 \frac{q_4}{2} + (C_4 - e_1 \times q_4)(1 + \alpha_4 \times P_4) \times d_4 \right] \end{cases} \]

with \( 0 < q_i \leq q_i \leq q_i \leq q_i \).

It will not change the meaning of formula, if we replace inequality conditions \( 0 < q_i \leq q_i \leq q_i \) into the following inequality constraints

\[ q_i - q_i \geq 0, q_i - q_i \geq 0, q_i - q_i \geq 0 \text{ and } q_i > 0. \]
Thirdly, the Kuhn-Tucker condition is used to find the solution of $q_i$, $q_i$, $q_i$, $q_i$ to minimize $P(T_{C_i}(q_i))$, subject to $q_i - q_i \geq 0$, $q_i - q_i \geq 0$, $q_i - q_i \geq 0$ and $q_i > 0$.

And $\forall f(T_{C_i}(q_i)) - \lambda \nabla g(q_i) = 0$

\[
\Rightarrow \frac{1}{6} \left\{ a_i \times \frac{d_i}{q_i} + h_i \times \frac{q_i}{2} + (C_i - e_4 \times q_i) \left( 1 + \alpha_i \times P_i \right) \times d_i \right. \\
+ 2 \left[ a_i \times \frac{d_i}{q_i} + h_i \times \frac{q_i}{2} + (C_3 - e_2 \times q_i) \left( 1 + \alpha_3 \times P_3 \right) \times d_3 \right] \\
+ 2 \left[ a_4 \times \frac{d_4}{q_4} + h_4 \times \frac{q_4}{2} + (C_4 - e_4 \times q_4) \left( 1 + \alpha_4 \times P_4 \right) \times d_4 \right] \\
\left. - \lambda_1 (q_i - q_i) - \lambda_2 (q_i - q_i) - \lambda_3 (q_i - q_i) - \lambda_4 q_i = 0 \right\}
\]

which implies

\[
\frac{1}{6} \left\{ h_i \times \frac{a_i x d_i}{q_i^2} - e_1 \times \left( 1 + \alpha_4 \times P_i \right) \times d_i \right\} + \lambda_1 - \lambda_4 = 0
\]

\[
\frac{2}{6} \left\{ h_i \times \frac{a_3 x d_3}{q_3^2} - e_2 \times \left( 1 + \alpha_3 \times P_3 \right) \times d_3 \right\} - \lambda_1 + \lambda_2 = 0
\]

\[
\frac{2}{6} \left\{ h_i \times \frac{a_2 x d_2}{q_2^2} - e_3 \times \left( 1 + \alpha_2 \times P_2 \right) \times d_2 \right\} - \lambda_2 + \lambda_3 = 0
\]

\[
\frac{1}{6} \left\{ h_i \times \frac{a_i x d_i}{q_i^2} - e_1 \times \left( 1 + \alpha_i \times P_i \right) \times d_i \right\} - \lambda_3 = 0
\]
\( \lambda_1(q_i - q_n) = 0 \)

\( \lambda_2(q_i - q_n) = 0 \)

\( \lambda_3(q_i - q_n) = 0 \)

\( \lambda_4 q_i = 0 \)

\( q_i - q_n \geq 0 \)

\( q_n - q_i \geq 0 \)

\( q_i - q_n \geq 0 \)

\( q_i > 0 \)

Because \( q_i > 0 \) and \( \lambda_4 q_i = 0 \) then \( \lambda_4 = 0 \).

If \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \) then \( q_i < q_n < q_i < q_n \), it does not satisfy the constraints

\[ 0 < q_n \leq q_i \leq q_n \leq q_n. \]

Therefore \( q_i = q_1, q_2 = q_2 \) and \( q_i = q_n \) (ie) \( q_i = q_i = q_n = q_n = q_i \).

Hence we find the optimal order quantity \( q_i^* \) as

\[
q_i^* = \frac{2\left[ (a_i x d_i) + 2(a_2 x d_2) + 2(a_3 x d_3) + (a_4 x d_4) \right]}{(h_i + 2h_i + 2h_i + h_i) - 2 \left[ e_i x (1 + a_i x P_i) x d_i \right] + 2 \left[ e_2 x (1 + a_2 x P_2) x d_2 \right] + 2 \left[ e_3 x (1 + a_3 x P_3) x d_3 \right] + \left[ e_4 x (1 + a_4 x P_4) x d_4 \right]}
\]

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4.3.4. Numerical Example

Consider an inventory system with the following characteristics:

Retailer's data and costs for $n = 1$, $a = 60$, $c = 0.05$, $\alpha = 0.01$, $d_1 = 500$, $h_1 = 15$ ; $P_1 = 3$ ; $m_1 = 7$.

$q^* = 78.047$

$C_i = 26554.384$

Suppose fuzzy initial unit purchasing cost "more or less than 50"

$\tilde{C} = (C_1, C_2, C_3, C_4) = (40, 45, 55, 60)$ ;

Fuzzy annual demand for retailer ($i = 1$) is "more or less than 500"

$\tilde{d}_i = (d_{i1}, d_{i2}, d_{i3}, d_{i4}) = (460, 480, 520, 540)$ ;

Fuzzy holding cost per unit and for a unit time for retailer ($i = 1$) is "more or less than 15"

$\tilde{h}_i = (h_{i1}, h_{i2}, h_{i3}, h_{i4}) = (13, 14, 16, 17)$ ;

Fuzzy delay period for retailer ($i = 1$) is "more or less than 3"

$\tilde{P}_i = (P_{i1}, P_{i2}, P_{i3}, P_{i4}) = (1, 2, 4, 5)$ ;

Fuzzy number of orders per period for retailer ($i = 1$) is "more or less than 7"

$\tilde{m}_i = (m_{i1}, m_{i2}, m_{i3}, m_{i4}) = (5, 6, 8, 9)$ ;

Fuzzy order cost is "more or less than 60"

$\tilde{a} = (a_1, a_2, a_3, a_4) = (40, 50, 70, 80)$ ;

Fuzzy discount quantity rate is "more or less than 0.005"

$\tilde{e} = (e_1, e_2, e_3, e_4) = (0.003, 0.004, 0.006, 0.007)$ ;
Fuzzy payment rate fixed by the supplier is "more or less than 0.01"

\[ \bar{a} = (a_1, a_2, a_3, a_4) = (0.008, 0.009, 0.011, 0.012) \]

Fuzzy order quantity

\[ q_i^* = (77.01, 77.01, 77.01, 77.01) \]

Optimal average individual cost function for retailer \( i(i = 1) \)

(19036.31, 22614.02, 31091.67, 35427.91)

In this example, one interesting observation is that the impact of fuzziness is more when each of the cost components is allowed fuzziness. Hence it is advised to be more careful in accounting flexibility in the ordering and holding cost.