SYNOPSIS

Numerical analysis is a vast subject with diverse branches. One of these branches is “Numerical Integration”. Due to immense applications in different branches of sciences and technologies, numerical integration itself has grown into a very vast subject and quite a large number of researchers of Mathematics have enriched this subject by their pioneering researches.

In this thesis, some techniques have been developed for evaluating approximately integrals of real and complex valued functions where the integrand is well behaved i. e. without any kind of singularities in the range of integration. Also quadrature rules have been constructed for approximation of integrals in which the integrand has a singularity in the range of integration.

In particular, we have considered approximate evaluation of Cauchy Principal Value of the type:

\[ I(f) = P \int_{-a}^{a} \frac{f(x)}{x} \, dx, \]

along with the numerical evaluation of hyper singular integrals of the type :

\[ \Gamma(f) = H \int_{-a}^{a} \frac{f(x)}{x^2} \, dx; \]

where \( f(x) \) is a real valued function having no singularity of any kind in the range of integration i.e. \([-a, a]\).

In addition to this, rules have been constructed for getting approximate values of the integral:

\[ I(f) = P \int_{z=z_0}^{z=z_0} \frac{f(z)}{z \cdot (z-z_0)^2} \, dz \]

and

\[ I(f) = H \int_{z=z_0}^{z=z_0} \frac{f(z)}{z \cdot (z-z_0)^2} \, dz; \]
where $L$ is a line segment in the complex plane joining the point $z_0 - h$ to $z_0 + h$ and $f(z)$ is analytic on a domain containing the line segment $L$.

The thesis starts with the **first chapter**, entitled “Introduction” where in all basic ideas and concepts of approximate integration and the work done by other researchers have been given. Most of the things discussed in this chapter exist is standard and advanced texts.

Out of the remaining chapters, the very next chapter i.e. **Chapter-II** deals with construction of some quadrature rules of degree of precision seven and nine. These rules have been defined as “**MIXED QUADRATURE RULES**” (Ref. Das and Pradhan [41], Das and Hota[34]) which are formed by **hybridization** of Newton-Cotes type of rules and Gauss-Legendre rules of suitable degree of precision.

The rules which have been combined to produce **MIXED QUADRATURE RULES** are termed as **BASIC RULES**.

The degree of precision of mixed quadrature rule increases by **two** than these of the basic rules employed to construct the same. For example, we state here the mixed quadrature rule:

$$
R_{12}(f) = \frac{1}{623} \left[ 648R_2(f) - 25R_1(f) \right];
$$

where

$$
R_1(f) = \frac{1}{6} \left[ f(-1) + f(1) + 5 \left( f\left( -\frac{1}{\sqrt{5}} \right) + f\left( \frac{1}{\sqrt{5}} \right) \right) \right];
$$

is the **Gauss- Labatto 4- point rule**; and

$$
R_2(f) = \frac{1}{10} \left[ f(-1) + 6f(0) + f(1) + \left( f\left( -\frac{1}{3} \right) + f\left( \frac{1}{3} \right) \right) + 5 \left( f\left( -\frac{2}{3} \right) + f\left( \frac{2}{3} \right) \right) \right];
$$

is popularly known as the **Weddle's rule** for the approximate evaluation of a real definite integral without having any kind of singularities.
It is to be noted that the degree of precision of the mixed quadrature rule $R_{12}(f)$ is seven which is two more than that of each of the basic rules $R_1(f)$ and $R_2(f)$ i.e. five. It may be noted here that the coefficients of $R_1(f)$ and $R_2(f)$ in the mixed quadrature rule $R_{12}(f)$ are simple fractions $\frac{-25}{623} = -0.4$ and $\frac{648}{623} = 1.04$ respectively. So there is no addition of errors like truncation error, round off error or machine error due to finite precision of computing machine, if the integral:

$$I = \int_{-1}^{1} f(x) \, dx$$

is numerically integrated by this rule $R_{12}(f)$ or by any other rules of this class of rules.

Further, it may be mentioned here that the degree of accuracy to a desired decimal place of the approximate value of the integral by a single quadrature rule can’t be ascertained; but the same may be assured of to some extent from the approximate values obtained by numerically integrating the integral by the two rules and the mixed quadrature rule constructed out of these rules.

Here it is noteworthy that, no additional evaluation of function is required while numerically integrating the integral by a mixed quadrature rule as the functional values already evaluated for the basic rules, used to construct it.

The formulation of mixed quadrature rule from the existing rules of numerical integration is quite simple but yields result of greater accuracy and reliability.

Thus, in this chapter we intend to construct quadrature rules of mixed type out of which three rules of precision seven and four rules of precision nine in the same vein, as it is done by Das and Pradhan [41] and Das and Hota[34] in succeeding articles.

Further, we have made a modest attempt to compare the relative accuracy of mixed quadrature rules with the compound form of Basic Rules which are used to construct the former. It is observed that a desired accuracy to an integral can be achieved by a mixed
quadrature rule of higher precision with as much as less functional evaluations than the compound form of these basic rules, used during the formulation of the mixed quadrature rule. This fact is very much noticed when a set of six definite integrals: $I_1$ to $I_6$ which are listed below (whose exact values are otherwise known) have been numerically integrated by the mixed quadrature rules constructed in this chapter and the compound rules can be formulated from the basic rules which are employed for the construction of the former.

Also, the error in approximation of the integrals by such rules has been determined using the technique due to Lether [96] and rules have been numerically verified by taking suitable test integrals:

\[
I_1 = \int_{-1}^{1} e^x \, dx; \quad I_2 = \int_{0}^{1} e^{-x^2} \, dx \\
I_3 = \int_{0}^{1} e^{x^2} \, dx; \quad I_4 = \int_{1}^{\infty} \frac{\sin^2 x}{x} \, dx \\
I_5 = \int_{0}^{1} \frac{1}{1+e^x} \, dx; \quad I_6 = \int_{0}^{1} \frac{1}{1+x} \, dx.
\]

In Chapter III, we have constructed some Newton-Cotes type of quadrature rules based on three, seven, eleven and fifteen points for numerical integration of real Cauchy-principal value of integral of the type:

\[
I(f) = P \int_{-a}^{a} \frac{f(x)}{x} \, dx;
\]

instead of choosing the range of integration $[-1,1]$; for formulas to be readily available for any range of integration for computation on computers.

To generate $(4n-1)$-point rule, the interval of integration is divided into $(4n-2)$-equal parts by the points:

\[
0, \pm \frac{a}{2n}, \pm \frac{2a}{2n}, \pm \frac{3a}{2n}, \ldots, \pm \frac{(2n-1)a}{2n}.
\]

The propose rule based on these nodes is denoted by $Q_n(f)$ and defined as:
\[ Q_n(f) = w_{n0}f(0) + \sum_{k=1}^{(2n-1)} w_{nk} \left[ f(\frac{ka}{2n}) - f(-\frac{ka}{2n}) \right]; \]

For example the 3-point rule is to be obtained for \( n = 1 \) as:

\[ Q_1(f) = w_{10}f(0) + w_{11}\left[ f\left(\frac{a}{2}\right) - f\left(-\frac{a}{2}\right) \right]; \]

where \( w_{10} \) and \( w_{11} \) are the weights associated with the rule \( Q_1(f) \).

Since the nodes are prefixed, thus it is only remain to determine the coefficients \( w_{n0} \) and \( w_{nk} \) associated with \( f(0) \) and with the block:

\[ \left[ f\left(\frac{ka}{2n}\right) - f\left(-\frac{ka}{2n}\right) \right]; \]

respectively for \( k = 1, 2, 3, \ldots, (2n - 1); \) for a fixed \( n \).

It is pertinent to note that the rule \( Q_n(f) \) in general, integrates exactly all polynomials of degree \( \leq (4n - 2) \). Another advantage of constructing such type of rules for arbitrary \( a \in (0, 1] \) instead of the interval \([-1, 1]\) is, for larger interval the rule may be applied in compound form.

The error in approximation of the integrals by such rules has been determined using the technique due to Lether [96]. For numerical verification the following integrals have been taken.

\[
I_1 = P \int_{-0.5}^{0.5} \tan^{-1} \frac{x}{x} \, dx ; \quad I_2 = P \int_{-1}^{1} e^x \, dx ; \quad I_3 = P \int_{-1/4}^{1/4} \frac{x+1}{x(x^2+1)} \, dx ;
\]

\[
I_4 = P \int_{-1/2}^{1/2} \frac{1}{x(x+2)} \, dx ; \quad I_5 = P \int_{-1}^{1} \frac{(x+1) e^x}{x} \, dx ; \quad I_6 = P \int_{-1}^{1} \cos \frac{x}{x} \, dx;
\]

\[
I_7 = P \int_{-1}^{1} \sin \frac{x}{x} \, dx ; \quad I_8 = P \int_{-1/2}^{1/2} \cos^{-1} \frac{x}{x} \, dx \quad \text{and} \quad I_9 = P \int_{-1/2}^{1/2} \sin^{-1} \frac{x}{x} \, dx.
\]

Also, a quadrature scheme has been proposed for quadrature of real Cauchy principal value integral of the type
\[ I = P \int_{-a}^{a} \frac{f(x)}{x - c} \, dx; |c| < 1. \]

However, the quadrature rules meant for the numerical integration of the real CPV integral of the type:

\[ I(f) = P \int_{-a}^{a} \frac{f(x)}{x} \, dx; \]

behave very much unstable when these rules are applied for the approximation of the hyper singular integral of Hadamard type:

\[ \Gamma(f) = H \int_{-a}^{a} \frac{f(x)}{x^2} \, dx; \]

due to the presence of higher order singularity at \( x = 0 \). Researchers like Ramm and Van der Sluis[143], Groetsch[73], Criscuolo[32], Paget[129], Elliott[61] and many more as available in literature have been contributed their work for the approximate evaluation of this type of integrals.

In this chapter we have also proposed a simple scheme for the approximate evaluation of hyper singular integrals of Hadamard finite part type as given above. Further, to verify the accuracy of the scheme, some standard hyper singular integrals

\[
\Gamma_1 = H \int_{-1}^{1} \frac{e^x}{x^2} \, dx; \quad \Gamma_2 = H \int_{-1}^{1} \frac{\cos x}{x^2} \, dx; \quad \Gamma_3 = H \int_{-1}^{1} \frac{e^x \sin x}{x^2} \, dx;
\]

\[
\Gamma_4 = H \int_{-1}^{1} \frac{e^{-x} \cos x}{x^2} \, dx; \quad \Gamma_5 = H \int_{-1/2}^{1/2} \frac{e^{-x^2}}{x^2} \, dx; \quad \Gamma_6 = H \int_{-1/2}^{1/2} \frac{e^{x^2}}{x^2} \, dx;
\]

also have been numerically evaluated.

However, Chapter IV and Chapter V deals with numerical integrations of functions with one complex variable (with and without singularity).

In particular, the Chapter IV is completely devoted into numerical evaluation of integration of an analytic function of one complex variable. A Class of seven point rules of
**precision at least seven**, involving a parameter $\alpha$ have been constructed for the numerical integration of

$$I = \int_L f(z) \, dz$$

of an analytic function $f(z)$ of a complex variable $z$ along a directed line segment $L$ joining the points $z_0 - h$ to $z_0 + h$ in the complex plane $C$ and subsequently a class of rules involving fewer number of nodes than the former have been derived without changing its algebraic degree of precision i.e. seven. Further, four more rules of precision nine have been formulated in this chapter from the rules of precision seven.

The asymptotic error estimates of all the rules constructed have been obtained and the rules have been numerical verified with suitable integral, such as:

$$I_1 = \int_{(l+i)/\sqrt{2}}^{(l+i)/\sqrt{2}} z e^z \, dz; \quad I_2 = \int_{-l}^{l} e^z \, dz$$

and

$$I_3 = \int_{-l}^{l} \cos z \, dz$$

Finally, in the last chapter (i.e.) **Chapter V**, we have constructed some quadrature rules for numerical computation of complex Cauchy Principal Value of integrals of the type:

$$I(f) = P \int_{z_0}^{z_0+h} \frac{f(z)}{z-z_0} \, dz;$$

along the directed line segment $L$, from the point $z_0 - h$ to $z_0 + h$ and $f(z)$ is assumed to be an analytic function in a domain $\Omega$ containing $L$, occur very often in contour integration, which in turn, is an essential tool in applied mathematics.

As far as it is known, the numerical evaluation of the complex CPV integrals is still getting sufficient attention of many researchers. Acharya and Mahapatra [3], Acharya and Das [6,7],
Das and Hota[36], Das, Hota and Bej[37], Milovanovic, Acharya and Pattanaik [120] etc. has given significant contributions for the same.

However, the rules developed by them require evaluation of the derivative of the function \( f(z) \) at \( z = z_0 \) which is, as a result not suitable for evaluation over computer.

Thus, the objective of this chapter is to obtain some interpolatory type of rules not involving derivative of the function for the numerical approximation of the complex Cauchy Principal Value of integrals as stated above, which numerically integrates the complex CPV integrals more accurately than by the existing rules found in the literature with higher degree of precision.

Keeping this in mind, in this chapter a class of **three point rules of precision at least three**, involving a parameter \( \alpha \) have been constructed for the numerical computation of complex Cauchy Principal value of integrals and subsequently a class of rules of precision at **least four** involving two parameters \( \alpha_1 \) and \( \alpha_2 \) have been derived. Then **families of rules of precision six and eight** have been developed.

The error in approximation of the integrals by such rules has been determined using the technique due to Lether [96] and rules have been numerically verified by taking suitable test integrals:

\[
I_1 = P \int_{-i}^{i} \frac{1 + z}{z} e^z dz, \quad I_2 = P \int_{-i}^{i} \frac{1 + z \cos z}{z} dz,
\]

\[
I_3 = P \int_{\frac{(1+i)/4}{(1+i)/4}}^{\frac{(1+i)/4}{(1+i)/4}} \tan^{-1} \frac{z}{z} dz, \quad I_4 = P \int_{\frac{(1+i)/2}{(1+i)/2}}^{\frac{(1+i)/2}{(1+i)/2}} \frac{\sin z}{z - (1 + i)} dz,
\]

\[
I_5 = P \int_{-i}^{i} \frac{e^z}{z} dz, \quad I_6 = P \int_{-\frac{i}{2}}^{\frac{i}{2}} \frac{\sin^{-1} z}{z} dz
\]
and

\[ I_7 = P \int_{-i/2}^{i/2} \frac{\cos^{-1} z}{z} \, dz. \]

It is to be claimed here that the rules constructed in this chapter integrates all the above test integrals (usually considered in this line) accurately up to seven decimal places.

In addition to this, further

(a) an interval selective adaptive scheme has been proposed for the approximate evaluation of complex CPV integrals as stated above, and

(b) following to the scheme as suggested in Chapter-III, a numerical scheme also has been proposed in this chapter for the approximate evaluation of the complex hyper singular integral of the type:

\[ I(f) = \int_{z_0}^{z_0+h} \frac{f(z)}{(z-z_0)^2} \, dz; \]

and the scheme has been successfully verified by integrating some standard integrals.

It is pertinent to note that; the formula constructed for numerical integration of complex Cauchy Principal Value integrals may be fruitfully used for numerical evaluation of real Cauchy Principal Value integrals and for the evaluation of Real as well as Complex definite integrals.

Finally at the end of this thesis a list of books and articles which have been referred to in course of preparation of this dissertation, is provided.