Motivated by the two basic properties of a differential operator, the notion of derivation was introduced in rings and algebras long back. Many well-known algebraists have done a lot of work in this area which has got tremendous applications in diverse part of Mathematics (for reference see [32], [42]). The notion of derivation was further extended to generalized derivation, higher derivation, generalized higher derivation etc. The idea of generalized derivation was introduced and studied by Brešar [28]. An additive map $F : R \rightarrow R$ is said to be generalized derivation on a ring $R$ if there exists a derivation $d$ (i.e; an additive map $d : R \rightarrow R$ satisfying $d(xy) = d(x)y + xd(y)$) such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. The concept of higher derivation was given by Hasse and Schmidt [43]. Let $D = \{d_n\}_{n \in \mathbb{N}}$ be a family of additive maps $d_n : R \rightarrow R$. Then $D$ is said to be higher derivation on $R$ if $d_0 = I_R$ and $d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$ holds for all $a, b \in R$ and for each $n \in \mathbb{N}$. Various results concerning higher derivations were obtained by Ferrero [38], [39], Haetinger [40], [41], Cortes [34], Jung [53], etc.

The present thesis is a part of the research work carried out by the author during the last five years concerning generalized derivations, higher derivations and their various generalizations mostly in the setting of prime and semiprime rings. Each chapter is subdivided into sections. The definitions, examples, results and remarks etc. have been specified with double decimal numbers. The first figure denotes the number of the chapter, second represents the section in the chapter and third points out the number of the definition, the example, or the remark as the case may be in particular chapter. For example, Theorem 3.2.4 refers to the fourth theorem appearing in the second section of the third chapter.

Chapter 1 contains preliminary notions, basic definitions and some important known results which may be needed for the development of the subject in the subsequent chapters.

Chapter 2 deals with the commutativity of rings with involution. An additive mapping $x \mapsto x^*$ on a ring $R$ is called an involution on $R$ if
(xy)^* = y^*x^* and (x^*)^* = x hold for all \(x, y \in R\). A ring equipped with an involution is called a ring with involution or *-ring. A ring with an involution \('*'\) is said to be *-prime if \(aRb = aRb^* = \{0\}\) or \(a^*Rb = aRb = \{0\}\) implies that either \(a = 0\) or \(b = 0\). A Lie ideal \(U\) (resp. Jordan ideal \(J\)) of \(R\) is said to be *-Lie ideal (resp. *-Jordan ideal) of \(R\) if \(*(U) = U\) (resp. \(*(J) = J\).

In Section 2.2, it is shown that a *-Lie ideal \(U\) of of *-prime rings \(R\) is central if it admits a generalized derivation \(F\) with associated derivation \(d\) satisfying any one of the following identities on a non-zero \(R\):

\[(i)\] \(F[u,v] = [F(u), v]\), \(F(u \circ v) = F(u) \circ v\), \(F(u) = F(u, v) + [d(v), u]\), \(F(u \circ v) = F(u) \circ v + d(v) \circ u\), \(F(u) = (uv) \pm uv = 0\) and \((vi)\) \(d(u)F(v) = uv = 0\) for all \(u, v \in U\).

Section 2.3 deals with the commutativity of *-prime ring \(R\) admitting generalized derivations \(F\) and \(G\) associated with non-zero derivations \(d\) and \(g\) respectively satisfying any one of the following identities on a non-zero *-Jordan ideal \(J\) of \(R\):

\[(i)\] \(F(u, [u, v]) = 0\), \(F[u, v] = u \circ v\), \(F(u \circ v) = [u, v]\), \(F(u) = F(u) \circ F(v)\), \(F(u) = [G(u), v]\), \(F(u)G(v) = uv = 0\) for all \(u, v \in J\).

We begin Section 2.4 by defining generalized \((\alpha, \beta)^*\)-derivation and generalized \((\alpha, \beta)^*\)-reverse derivation on a ring \(R\). Let \((\alpha, \beta)\) be the endomorphisms on \(R\). An additive mapping \(F : R \to R\) is called a generalized \((\alpha, \beta)^*\)-derivation (resp. generalized \((\alpha, \beta)^*\)-reverse derivation) if there exists an \((\alpha, \beta)^*\)-derivation (resp. \((\alpha, \beta)^*\)-reverse derivation) \(d\) such that \(F(xy) = F(x)\alpha(y^*) + \beta(x)d(y)\) (resp. \(F(xy) = F(y)\alpha(x^*) + \beta(y)d(x)\)) holds for all \(x, y \in R\). The main result of this section states that if a semiprime ring with involution \('*'\) admits a generalized \((\alpha, \beta)^*\)-derivation (resp. generalized \((\alpha, \beta)^*\)-reverse derivation) \(F\) on \(R\) such that \(\alpha\) (resp. \(\beta\) is surjective, then \(F\) maps \(R\) into \(Z(R)\), the center of \(R\). Various other theorems concerning prime ring are also obtained which extend several results of [2], [3] and [6].

Chapter 3 is devoted to the study of bi-derivation, \(n\)-derivation and generalized \(n\)-derivation. The concept of permuting \(n\)-derivation has been very recently introduced by Park [70]. The motivation behind defining permuting \(n\)-derivations comes from the study of symmetric bi-derivations which was first given by Maksa [60]. A mapping \(B : R^2 \to R\) is said to be symmetric if \(B(x, y) = B(y, x)\) holds for all \(x, y \in R\). A mapping \(\varphi : R \to R\) defined by \(\varphi(x) = B(x, x)\) is called the trace of \(B\). It is obvious that in case \(B\) is a symmetric mapping which is also bi-additive (i.e., additive in both the arguments), the trace of \(B\) satisfies the relation \(\varphi(x + y) = \varphi(x) + \varphi(y) + 2B(x, y)\) for all \(x, y \in R\). Following Maksa, a symmetric bi-additive mapping \(B : R^2 \to R\) is called a symmetric bi-derivation if \(B(xy, z) = B(x, z)y + xB(y, z)\) holds for all \(x, y, z \in R\). Obviously in this case also \(B(x, yz) = B(x, y)z + yB(x, z)\) holds for all \(x, y, z \in R\).
In Section 3.2, the notions of symmetric generalized $(\alpha, \beta)^*$ bi-derivation, symmetric generalized $(\alpha, \beta)^*$ reverse bi-derivation and $\alpha^*$-bi-multipliers have been introduced and some results concerning them have been presented which extend various results of [2], [4] and [6]. A symmetric bi-additive mapping $G : R \times R \to R$ is said to be a symmetric generalized $(\alpha, \beta)^*$-bi-derivation (resp. symmetric generalized $(\alpha, \beta)^*$ reverse bi-derivation) on $R$ if there exists a symmetric $(\alpha, \beta)^*$-bi-derivation (resp. symmetric $(\alpha, \beta)^*$ reverse bi-derivation) $B$ on $R$ such that $G(xy, z) = G(x, z)\alpha(y^*) + \beta(x)B(y, z)$ (resp. $G(xy, z) = G(y, z)\alpha(x^*) + \beta(y)B(x, z)$) holds for all $x, y, z \in R$. Further if a semiprime ring $R$ with involution $^*$ admits a symmetric generalized $(\alpha, \beta)^*$-bi-derivation $G : R \times R \to R$ with associated symmetric $(\alpha, \beta)^*$-bi-derivation $B$ such that $\alpha$ is surjective then $G$ maps $R \times R$ into $Z(R)$. Also, it is shown that if $R$ is a non-commutative prime ring with involution $^*$ admitting a symmetric generalized $(\alpha, \beta)^*$-bi-derivation $G : R \times R \to R$ with associated symmetric $(\alpha, \beta)^*$-bi-derivation $B$ such that $\alpha$ is surjective then $G = 0$. Similar results for symmetric $(\alpha, \beta)^*$ reverse bi-derivation have been also obtained. A symmetric bi-additive mapping $M : R \times R \to R$ is said to be a symmetric left $\alpha^*$-bi-multiplier (resp. symmetric right $\alpha^*$-bi-multiplier) if $M(xy, z) = M(x, z)\alpha(y^*)$ (resp. $M(xy, z) = \alpha(x^*)M(y, z)$) holds for all $x, y, z \in R$. If $M$ is both symmetric left $\alpha^*$-bi-multiplier as well as symmetric right $\alpha^*$-bi-multiplier, then $M$ is called symmetric $\alpha^*$-bi-multiplier. The main result concerning $\alpha^*$-bi-multiplier states that if $R$ is a semiprime ring with involution $^*$ and $\alpha$ is an endomorphism of $R$ such that $\alpha$ is surjective and $M : R \times R \to R$ is a non-zero bi-additive mapping such that $M(xy, z) = M(x, z)\alpha(y^*)$ holds for all $x, y, z \in R$, then $M$ maps $R \times R$ into $Z(R)$.

Section 3.3 opens with the definition of permuting $n$-derivation. Suppose $n$ is a fixed positive integer and $R^n = R \times R \times \cdots \times R$. A mapping $\Delta : R^n \to R$ is said to be permuting if the relation $\Delta(x_1, x_2, \cdots, x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)})$ holds for all $x_i \in R$ and for every permutation $\{\pi(1), \pi(2), \cdots, \pi(n)\}$. Motivated by the study of bi-derivations Park introduced the concept of permuting $n$-derivations in [70]. A permuting map $\Delta : R^n \to R$ is known to be permuting $n$-derivation if $\Delta$ is $n$-additive (i.e. additive in each argument) and $\Delta(x_1, x_2, \cdots, x_i, \cdots, x_n) = x_i\Delta(x_1, x_2, \cdots, x_i', \cdots, x_n) + \Delta(x_1, x_2, \cdots, x_i, \cdots, x_n)x_i'$ holds for all $x_i, x_i' \in R$. A map $\delta : R \to R$ defined by $\delta(x) = \Delta(x, x, \cdots, x)$ for all $x \in R$, where $\Delta : R^n \to R$ is a permuting map, is known as trace of $\Delta$. The main result of this section states that if $m \geq 1, n \geq 2$ are any two fixed positive integers and $R$ is a $(m+n)!$-torsion free non-commutative prime ring admitting a permuting $n$-derivation $\Delta : R^n \to R$ such that the trace $\delta$ of $\Delta$ is $m$-centralising i.e. $[\delta(x), x^m] \in Z(R)$, then $\Delta = 0$. Further, some more related results regarding the trace of permuting $n$-derivation have also been given.

The notions of permuting generalized $n$-derivation and permuting
n-multiplier have been introduced in Section 3.4. Let \( n \geq 1 \) be a fixed positive integer. A permuting \( n \)-additive map \( \Omega : R^n \rightarrow R \) is known to be permuting generalized \( n \)-derivation if there exists a permuting \( n \)-derivation \( \Delta : R^n \rightarrow R \) such that \( \Omega(x_1, x_2, \ldots, x_i, \ldots, x_n) = \Omega(x_1, x_2, \ldots, x_i, \ldots, x_n) \) holds for all \( x_i, x_i' \in R \). Let \( \omega : R \rightarrow R \) such that \( \omega(x) = \Omega(x, x, \ldots, x) \). Then \( \omega(x) \) is known as the trace of \( \Omega \). We now define permuting \( n \)-multiplier on rings. A permuting \( n \)-additive map \( \Lambda : R^n \rightarrow R \) is said to be a permuting left \( n \)-multiplier (resp. permuting right \( n \)-multiplier) if \( \Lambda(x_1, x_2, \ldots, x_i, \ldots, x_n) = \Lambda(x_1, x_2, \ldots, x_i, \ldots, x_n) \) (resp. \( \Lambda(x_1, x_2, \ldots, x_i, \ldots, x_n) = x_i \Lambda(x_1, x_2, \ldots, x_i', \ldots, x_n) \) holds for all \( x_i, x_i' \in R \). If \( \Lambda \) is both permuting left \( n \)-multiplier as well as right \( n \)-multiplier, then \( \Lambda \) is a permuting \( n \)-multiplier. Finally, it is shown that if \( n \geq 2 \) is any fixed positive integer and \( R \) is a \( n! \)-torsion free non-commutative prime ring admitting a permuting generalized \( n \)-derivation \( \Omega : R^n \rightarrow R \) with associated permuting \( n \)-derivation \( \Delta : R^n \rightarrow R \) such that the trace \( \omega \) of \( \Omega \) is commuting on \( R \) then \( \Omega \) acts as a permuting left \( n \)-multiplier.

Chapter 4 deals with the study of higher derivations, generalized higher derivations, \((\sigma, \tau)\)-higher derivation, generalized \((\sigma, \tau)\)-higher derivation. Motivated by the existence of \((\sigma, \tau)\)-derivation in rings the notions of \((\sigma, \tau)\)-higher derivation and Jordan \((\sigma, \tau)\)-higher derivation have been introduced in Section 4.2. Let \( R \) be a ring, \( \sigma, \tau \) be the endomorphisms of \( R \). Suppose \( D = \{d_n\}_{n \in \mathbb{N}} \) is a family of mappings \( d_n : R \rightarrow R \). Then \( D \) is said to be a \((\sigma, \tau)\)-higher derivation (resp. Jordan \((\sigma, \tau)\)-higher derivation) if, \( d_0 = I_R \), \( d_n(a + b) = d_n(a) + d_n(b) \) and \( d_n(ab) = \sum_{i+j=n} d_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(b)) \) (resp. \( d_n(a^2) = \sum_{i+j=n} d_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(a)) \), for all \( a, b \in R \) and for each \( n \in \mathbb{N} \). A classical result due to Herstein states that on a prime ring of characteristic different from two every Jordan derivation is a derivation. This result was further generalized for semi-prime rings by Bresar [24]. Further Haetinger [40] obtained Bresar’s result for higher derivations. He established that on a 2-torsion free semiprime ring every Jordan higher derivation is a higher derivation. In this section we generalize the above mentioned results for \((\sigma, \tau)\)-higher derivations. It is shown that if \( R \) is a 2-torsion free non-commutative semi-prime ring and \( \sigma, \tau \) be endomorphisms of \( R \) such that \( \tau \sigma = \sigma \tau \) and \( \tau \) is one-one and onto then every Jordan \((\sigma, \tau)\)-higher derivation on \( R \) is a \((\sigma, \tau)\)-higher derivations on \( R \).

In Section 4.3 the notion of generalized \((\sigma, \tau)\)-higher derivation and generalized Jordan \((\sigma, \tau)\)-higher derivation on a ring \( R \) have been introduced. A family \( F = \{f_n\}_{n \in \mathbb{N}} \) of additive maps \( f_n : R \rightarrow R \) is said to be generalized \((\sigma, \tau)\)-higher derivation (resp. generalized Jordan \((\sigma, \tau)\)-higher derivation) of \( R \) if there exists a \((\sigma, \tau)\)-higher derivation \( D = \{d_n\}_{n \in \mathbb{N}} \) of
Let $U$ be a Lie ideal of $R$. Then $F$ is said to be a generalized $(\sigma, \tau)$-higher derivation (resp. generalized Jordan $(\sigma, \tau)$-higher derivation) of $U$ into $R$ if the above corresponding conditions are satisfied for all $a, b \in U$. Finally, it is shown that if $R$ is a prime ring of characteristic different from 2, $U$ a square closed Lie ideal of $R$ and $\sigma, \tau$ endomorphisms of $R$ such that $\sigma \tau = \tau \sigma$ and $\tau$ is one-one \& onto, then every generalized Jordan $(\sigma, \tau)$-higher derivation of $U$ into $R$ is a generalized $(\sigma, \tau)$-higher derivation of $U$ into $R$. In case of arbitrary ring $R$ it is proved that if $R$ is a 2-torsion-free ring and $\sigma, \tau$ be endomorphisms of $R$ such that $\sigma \tau = \tau \sigma$ and $\tau$ is one-one \& onto and $U$ has a commutator which is not a right zero divisor, then every Jordan $(\sigma, \tau)$-higher derivation on $U$ into $R$ is a $(\sigma, \tau)$-higher derivation on $U$ into $R$. In fact, our results of this section generalize many well known results namely, ([10], Theorem) ([41], Theorem 2.1), ([8], Main Theorem), ([34], Theorem 1.3) etc. to mention a few only.

In Chapter 5 the notion of generalized Jordan triple $(\sigma, \tau)$-higher derivation on ring $R$ has been introduced and results concerning semiprime and prime rings have been obtained which extend, generalize and compliment various results. Bresar [26] termed any additive mapping $d : R \to R$ satisfying $d(aba) = d(a)ba + ad(b)a + abd(a)$ for all $a, b \in R$ as a Jordan triple derivation and further concluded that every Jordan triple derivation of a 2-torsion free semi-prime ring is a derivation (see Theorem 4.3 of [26]).

One natural generalization of Jordan triple derivation is of generalized Jordan triple derivation given by Jing and Lu [52]. They also showed that every generalized Jordan triple derivation on a prime ring of characteristic different from two is a generalized derivation. Further, Liu \& Shiue [58] extended this result for generalized Jordan triple $(\sigma, \tau)$-derivation. Liu \& Shiue proved that on a 2-torsion free semi-prime ring every generalized Jordan triple $(\sigma, \tau)$-derivation is a generalized $(\sigma, \tau)$-derivation.

Inspired by the concepts of Jordan triple derivation, generalized derivation and $(\sigma, \tau)$-higher derivation, in Section 5.2 we introduce generalized Jordan triple $(\sigma, \tau)$-higher derivation as follows: a family of additive mappings $F = \{f_n\}_{n \in \mathbb{N}}$ of $R$ is said to be a generalized Jordan triple $(\sigma, \tau)$-higher derivation if there exists a $(\sigma, \tau)$-higher derivation $D = \{d_n\}_{n \in \mathbb{N}}$ of $R$ such that $f_0 = I_R$ and $f_n(aba) = \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))d_j(\tau^{n-j}(b))d_k(\tau^{n-k}(a))$ holds for all $a, b \in R$ and every $n \in \mathbb{N}$. Consequently, in the above definition for $f_i = d_i$ we get the notion of Jordan triple $(\sigma, \tau)$-higher derivation. Here we establish that every generalized Jordan triple $(\sigma, \tau)$-higher derivation of a 2-torsion free semi-prime ring $R$ is a generalized $(\sigma, \tau)$-higher derivation of
Further, in Section 5.3 we shall consider generalized Jordan triple \( (\sigma, \tau) \)-higher derivations on a Lie ideal \( U \) of \( R \). In this section we define generalized Jordan triple \( (\sigma, \tau) \)-higher derivation on a Lie ideal \( U \) of \( R \). A family of additive mappings \( F = \{f_n\}_{n \in \mathbb{N}} \) of \( R \) is said to be a generalized Jordan triple \( (\sigma, \tau) \)-higher derivation of \( U \) into \( R \) if there exist a \( (\sigma, \tau) \)-higher derivation \( D = \{d_n\}_{n \in \mathbb{N}} \) of \( U \) into \( R \) such that \( f_0 = I_R \) and for all \( a, b \in U \) and every \( n \in \mathbb{N} \), it holds that

\[
    f_n(aba) = \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))d_j(\sigma^{k}\tau^{i}(b))d_k(\tau^{n-k}(a))
\]

It can be easily seen that on a 2-torsion free ring \( R \), every generalized \( (\sigma, \tau) \)-higher derivation of a square closed Lie ideal \( U \) into \( R \) is a generalized Jordan triple \( (\sigma, \tau) \)-higher derivation of \( U \) into \( R \). But the converse need not be true in general. Hence, we explore the converse part of this problem and obtain the following: let \( R \) be a prime ring of characteristic different from 2 and \( U \) be a noncentral square closed Lie ideal of \( R \). Suppose \( \sigma, \tau \) are the endomorphisms on \( R \) such that \( \sigma \tau = \tau \sigma \) and \( \tau \) is one-one and onto then every generalized Jordan triple \( (\sigma, \tau) \)-higher derivation of \( U \) into \( R \) is a generalized \( (\sigma, \tau) \)-higher derivation of \( U \) into \( R \). The results of this chapter easily reduces to ([26], Theorem 4.3), ([38], Theorem 1.2), ([53], Main Theorem), ([52], Theorem 3.5) etc.

At the end, an extensive bibliography of the existing literature related to the subject matter is included.