CHAPTER 2

Some Properties of Geodesic \(^\prime\)-Preinvex and Semistrictly Geodesic \(^\prime\)-Preinvex Functions

Mathematics is a vast adventure in ideas, its history reflects some of the noblest thoughts of countless generations.

-Dirk Struik

2.1. Introduction

In the recent years several generalizations have been developed for the classical properties of convexity. Preinvex functions are one of the most significant generalizations introduced by Weir and Mond [51] and Weir and Jeyakumar [50]. Mohan and Neogy [32] introduced a condition and proved that a differentiable function which is invex with respect to \(^\prime\) is also preinvex under that condition. Recently, Barani and Pouryayevali [5] introduced the concept of geodesic invex set and preinvex functions on geodesic invex set with respect to the particular maps.

In this chapter we discuss some properties of geodesic \(^\prime\)-preinvex functions, their relationships with \(^\prime\)-invex functions and strictly geodesic \(^\prime\)-preinvex functions. The geodesic \(^\prime\)-pre-pseudo-invex and geodesic \(^\prime\)-pre-quasi-invex functions on the geodesic invex set are introduced and some of their properties are discussed. We also introduce semistrictly geodesic \(^\prime\)-preinvex functions on Riemannian manifolds and study some of their properties.

Let \(M\) be a Riemannian manifold. We consider the function \(^\prime:\) \(M \rightarrow M\) such that \((x; y) \in T_y M\); for every \(y \in M\) and any \(x \in M\). Here the function \(^\prime(x; \cdot)\) assigns to each point \(y \in M\) a tangent vector \(v_y\) to \(M\) at

The contents of this chapter are published in Optimization, 1-6, 2011 and Advances in Operations Research, Hindawi Publishing Corporation, Volume 2009, Article ID 381831.
y so that $\dot{y}(x;\cdot)$ is a vector $\in \mathbb{E}_d$ on $M$, for each $x \in M$. For a differentiable function $f: M \to \mathbb{R}$, Pini [39] defined invexity as follows:

**Definition 2.1.1.** The differentiable function $f$ is said to be $\dot{y}$-invex on $M$ if for any $x; y \in M$

$$f(x) \leq f(y) + df_y(\dot{y}(x; y))$$

where $df_x$ is the differential of the differentiable map from a manifold $M$ to the manifold $N$ at the point $x$.

Mititelu [31] generalized the above definition as follows:

**Definition 2.1.2.** The differentiable function $f$ is said to be $\dot{y}$-pseudoinvex on $M$ if for any $x; y \in M$

$$df_y(\dot{y}(x; y)) \leq 0 \quad \forall f(x), f(y):$$

**Definition 2.1.3.** The differentiable function $f$ is said to be $\dot{y}$-quasiinvex on $M$ if for any $x; y \in M$

$$f(x) \leq f(y) \quad df_y(\dot{y}(x; y)) \leq 0$$

**Remark 2.1.1.** If $f$ is a differentiable and $\dot{y}$-invex function defined on $M$ and $df_y(\dot{y}(x; y)) \leq 0$, for all $x; y \in M$, then $f$ is $\dot{y}$-pseudoinvex on $M$.

In the above definitions

$$df_y(\dot{y}(x; y)) = [df(y)](x; y)$$

If $(M; g)$ is a Riemannian manifold and $f$ is a differentiable map from $M$ to $N$, then

$$df_y(\dot{y}(x; y)) = g_y(\text{grad } f(y); \dot{y}(x; y))$$

where $\text{grad } f(y)$ is the gradient of $f$ at the point $y$.

Barani and Pourayevali [5] defined the geodesic invex set and the invexity of a function $f$ on an open geodesic invex subset of a Riemannian
manifold.

Definition 2.1.4.[5] Let $M$ be a Riemannian manifold and $\gamma: M \to M$ ! $TM$ be a function such that for every $x,y \in M$, $\gamma(x,y) \in T_yM$. A non-empty subset $S$ of $M$ is said to be geodesic invex set with respect to $\gamma$ if for every $x,y \in S$, there exists a unique geodesic $\mathcal{O}_{x,y} : [0;1] \to M$ such that

$$\mathcal{O}_{x,y}(0) = y; \quad \mathcal{Q}_{x,y}(0) = \gamma(x,y); \quad \mathcal{O}_{x,y}(t) \in S; \quad \text{for all } t \in [0;1];$$

Remark 2.1.2. Let $M$ be a Cartan-Hadamard manifold (either finite-dimensional or infinite-dimensional). On $M$ there exists a natural map $\gamma$ playing the role of $\gamma$ in the Euclidean space $\mathbb{R}^n$, for every $x,y \in \mathbb{R}^n$. Indeed, we can define the function $\gamma$ as

$$\gamma(x,y) := \mathcal{O}_{x,y}(0); \quad \text{for all } x,y \in M; \quad (2.1.1)$$

where $\mathcal{O}_{x,y}$ is the unique minimal geodesic joining $y$ to $x$ (see [59, p.253]) as follows:

$$\mathcal{O}_{x,y}(t) := \exp_y(t \exp_y^{-1}x); \quad \text{for all } t \in [0;1];$$

Therefore, every geodesic convex set $S \subseteq M$ is a geodesic invex set with respect to $\gamma$ defined in (2.1.1).

However, the converse is not true in general as shown by Barani [5] in the following example.

Example 2.1.1. Let $M$ be a Cartan-Hadamard manifold and $x_0; y_0 \in M$, $x_0 \not\in y_0$. Let $B(x_0;r_1) \setminus B(y_0;r_2) = \emptyset$ for some $0 < r_1; r_2 < \frac{1}{2}d(x_0; y_0)$, where $B(x;r) = \{ y \in M : d(x;y) < r \}$ is an open ball with center $x$ and radius $r$. Let $S$ be defined as:

$$S := B(x_0;r_1) \setminus B(y_0;r_2);$$

Then, $S$ is not a geodesic convex set because every geodesic curve passing through $x_0$ and $y_0$ does not completely lie in $S$. Now, let the function $\gamma: M \to M$ ! $M$ be defined by

$$\gamma \big|_{B(x_0;r_1) \setminus B(y_0;r_2)} := \begin{cases} \exp_y^{-1}x; & x, y \in B(x_0;r_1) \setminus B(y_0;r_2); \\ 0_y; & \text{otherwise}; \end{cases}$$

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For every \( x; y \in M \), consider the geodesic \( \gamma : [0; 1] \to M \) defined by
\[
\gamma(t) := \exp_y(t \cdot (x; y)) \quad \text{for all } t \in [0; 1];
\]
Hence,
\[
\gamma(0) = y; \quad \gamma^0(0) = \gamma(x; y).
\]
Now, we show that \( S \) is a geodesic invex set with respect to \( \gamma \). For, let \( x; y \in B(x_0; r_1) \), since \( B(x_0; r_1) \) is geodesic convex (see [28, p.259]), therefore
\[
\gamma(t) := \exp_y(t \exp_y^{-1} x) \in B(x_0; r_1) \quad \text{for all } t \in [0; 1];
\]
Similarly, for \( x; y \in B(y_0; r_2) \), we have
\[
\gamma(t) \in S \quad \text{for all } t \in [0; 1];
\]
If \( x \in B(x_0; r_1) \) and \( y \in B(y_0; r_2) \) or \( x \in B(y_0; r_2) \) and \( y \in B(x_0; r_1) \) then, we have
\[
\gamma(t) := \exp_y(t0_y) = y \quad \text{for all } t \in [0; 1];
\]
Hence, \( S \) is a geodesic invex set with respect to \( \gamma \).

Definition 2.1.5.5 Let \( M \) be a Riemannian manifold, \( S \) be an open subset of \( M \) which is geodesic invex with respect to \( \gamma : M \to M \), \( TM \) and let \( f \) be a real differentiable function defined on \( S \). Then, \( f \) is said to be \( \gamma \)-invex on \( S \) if for every \( x; y \in S \)
\[
f(x) \leq f(y) + \nabla f(y) \cdot (\gamma(x; y) - y);
\]
Differentiable convex functions (on an open convex subset \( A \)) are invex. Also, the wider class of geodesically convex functions on manifolds is included in the class of invex functions under some additional hypotheses on manifolds [34]. In particular, this is true if the manifold has the property that given any two points \( x; y \) there exists a unique geodesic \( \gamma(t) \) joining them. In this case, a function which is geodesically convex is also geodesic invex with respect to \( \gamma(x; y) = \gamma_0(0) \).
2.2. Some Properties of Geodesic ‘-Preinvex Functions

The concept of preinvex functions on $\mathbb{R}^n$ was initiated and defined by Weir and Mond in [51], (See also [32, 53]) and they also discussed the properties of these functions. Later on Barani and Pouryayevali [5] extended this notion to Riemannian manifolds.

Definition 2.2.1. [5] Let $M$ be a Riemannian manifold, $S$ be an open subset of $M$ which is geodesic invex with respect to $\cdot : M \succcurlyeq M \succcurlyeq TM$. Then, $f : S \succcurlyeq R$ is said to be geodesic ‘-preinvex on $S$ if

$$f(\circ_{x,y}(t)) \leq tf(x) + (1 \cdot t)f(y);$$

for every $x; y \in S$; $t \in [0; 1]$. If the above inequality is strict, then $f$ is said to be strictly geodesic ‘-preinvex on $S$.

Theorem 2.2.1. Let $M$ be a Riemannian manifold and let $S$ be an open subset of $M$ which is geodesic invex with respect to $\cdot : M \succcurlyeq M \succcurlyeq TM$. Let $f : S \succcurlyeq R$ be a geodesic ‘-preinvex function and let $h : I \succcurlyeq R$ be an increasing convex function such that range $(f) \subseteq I$. Then the composite function $h(f)$ is geodesic ‘-preinvex on $S$.

Proof. Since $f$ is geodesic ‘-preinvex function, we have

$$f(\circ_{x,y}(t)) \leq tf(x) + (1 \cdot t)f(y);$$

Since $h$ is an increasing and convex function, we get

$$h(f(\circ_{x,y}(t))) \leq h(tf(x) + (1 \cdot t)f(y))$$

$$\leq th(f(x)) + (1 \cdot t)h(f(y));$$

Hence, $h(f)$ is geodesic ‘-preinvex on $S$.

Next, we have

Theorem 2.2.2. Let $M$ be a Riemannian manifold and $S$ be an open subset of $M$ which is geodesic invex with respect to $\cdot : M \succcurlyeq M \succcurlyeq TM$. Let
f : S ! R be a geodesic \'preinvex function and h : I ! R be a strictly increasing convex function such that range (f) \subseteq I. Then the composite function h(f) is strictly geodesic \'preinvex on S.

Proof. Since f is geodesic \'preinvex function, we have

\[ f (\circ_{x,y}(t)) \leq tf(x) + (1-i) tf(y) \]

Since h is the strictly increasing and convex function, we get

\[ h[f (\circ_{x,y}(t))] \leq h[tf(x) + (1-i) tf(y)] \leq th(f(x)) + (1-i) th(f(y)) \]

Or

\[ h[f (\circ_{x,y}(t))] \leq th(f(x)) + (1-i) th(f(y)) \]

Hence, h(f) is strictly geodesic \'preinvex on S.

Similarly, we can prove the following result.

Theorem 2.2.3. Let M be a Riemannian manifold, S be an open subset of M which is geodesic invex with respect to \': M \to M ! TM. Let f : S ! R be a geodesic \'preinvex function and h : I ! R be increasing and strictly convex function such that range (f) \subseteq I. Then the composite function h(f) is strictly geodesic \'preinvex on S.

The finite linear combination of geodesic \'preinvex functions is geodesic \'preinvex function as proved in the following theorem:

Theorem 2.2.4. Let S be an open subset of M which is geodesic invex with respect to \': M \to M ! TM. Suppose \( f_i : S ! R, i = 1; 2; \ldots; p \), be geodesic \'preinvex. Then

\[ f = \sum_{i=1}^{p} \lambda_i f_i \] for all \( \lambda_i \geq 0; \lambda_i \neq 0 \); i = 1; 2; \ldots; p

is geodesic \'preinvex function on S.

Proof. By hypothesis we have

\[ f_i (\circ_{x,y}(t)) \leq tf_i(x) + (1-i) tf_i(y) \]
It follows
\[ t \cdot f_i(x, y(t)) \leq t \cdot f_i(x) + (1 - t) \cdot f_i(y); \]
and
\[ \sum_{i=1}^{n} t_{i} \cdot f_i(x, y(t)) \leq \sum_{i=1}^{n} t_{i} \cdot f_i(x) + (1 - t) \cdot \sum_{i=1}^{n} f_i(y). \]
Hence, the result.

Now, we prove the following proposition which guarantees that a differentiable and geodesic \( -\)-prein convex function \( f \) is \( -\)-convex.

**Proposition 2.2.1.** Let \( M \) be a complete manifold and \( S \subseteq M \) which is geodesic convex with respect to \( -\) : \( M \equiv M \). Let \( f : S \rightarrow R \) be a differentiable function and there exists a sequence \( t_{n} \) of positive real numbers such that \( t_{n} \rightarrow \infty \) as \( n \rightarrow \infty \) and
\[ f(x, y) \leq t_{n} f(x) + (1 - t_{n}) f(y); \]
for every \( x, y \in S \), then \( f \) is \( -\)-convex on \( S \).

**Proof.** We have
\[ \frac{f(x, y) - f(x)}{t_{n}} \leq f(x) - f(y); \]
Since \( f \) is differentiable on \( S \), taking the limit as \( n \rightarrow \infty \) on both sides, we get
\[ \frac{f(x, y) - f(x)}{t_{n}} \leq f(x) - f(y); \]
Therefore,
\[ f(y) \leq f(x) - f(y); \]
Hence, the result.

It is to be noted that the converse of above proposition is not true in general. However, Barani and Pourayevali [5] proved that a \( -\)-pre convex function on \( S \) is geodesic \( -\)-pre convex on \( S \) if \( -\) satisfies the condition (C).

It is revealed in the following proposition that like convex functions are transformed into convex functions, \( -\)-pre convex functions are also transformed
into \( \cdot \)-invex functions by a suitable class of monotone functions.

**Proposition 2.2.2.** Let \( \dot{A} : R \rightarrow R \) be a monotone increasing differentiable convex function. If \( f \) is \( \cdash \)-invex on geodesic invex set \( S \), then the composite function \( \dot{A}(f) \) is \( \cdash \)-invex.

**Proof.** Using the fact that \( \dot{A}(x + h), \dot{A}(x) + \dot{A}^{\circ}(x)h \) for every \( x; h \in R \), we have

\[
\begin{align*}
\dot{A}(f(x)) &= \dot{A}(f(y)) + \dot{A}(f(y))\dot{A}(\dot{f}(x;y)) \\
&= \dot{A}(f(y)) + d(\dot{A}(f))\dot{A}(x;y) \\
&\quad \dot{A}(f(x)) \dot{A}(f(y)) \quad d(\dot{A}(f))\dot{A}(x;y): \\
\end{align*}
\]

Hence \( \dot{A}(f) \) is \( \cdash \)-invex on \( S \).

**2.3. Some Properties of Generalized Geodesic \( \cdash \)-P reinvex Functions**

In [38], Pini introduced the notion of \( \cdash \)-pre-pseudo-invex and \( \cdash \)-pre-quasi-invex functions on an invex set.

Let \( f \) be a real valued function defined on a subset of \( R^n \) and \( \cdot : R^m \rightarrow R^n \). \( R^n \).

**Definition 2.3.1.**[38] A function \( f \) is said to be \( \cdot \)-pre-pseudo-invex on a \( \cdot \)-invex set \( A \) if there exist a function \( \cdot \) and a strictly positive function \( b \) such that

\[
f(x) < f(y) \quad f(y + t\dot{x}(x;y)) \leq f(y) + t(t \dot{1})b(x;y);
\]

for every \( t \in (0;1) \) and \( x; y \in A \).

**Definition 2.3.2.**[38] A function \( f \) is said to be \( \cdot \)-pre-quasi-invex on a \( \cdot \)-
invex set $A$ if there exist a function $\gamma$ such that
\[
    f(y + t\gamma(x; y)) \leq \max_{x \in A} f(x); f(y)g;
\]
for all $x; y \in A$ and for every $t \in [0; 1]$.

We extend these notions to geodesic $\gamma$-pre-pseudo-invexity and geodesic $\gamma$-pre-quasi-invexity on a geodesic invex set $S$ by replacing the line segment with the geodesic.

Let $f$ be a function defined on a geodesic invex subset $S$ of a Riemannian manifold $M$ with respect to $\gamma: M \to M$.

**Definition 2.3.3.** Function $f$ is said to be geodesic $\gamma$-pre-pseudo-invex (p,p.i) on $S$, if there exist a geodesic $\gamma(x,y)(t)$ and a strictly positive function $b: S \to S$ ! $\mathbb{R}^+$ such that
\[
    f(x) < f(y) \implies f(\gamma(x,y)(t)) \leq f(y) + t(t \cdot 1)b(x; y);
\]
for every $t \in (0; 1)$ and $x; y \in S$.

**Theorem 2.3.1.** Let $M$ be a Riemannian manifold and $S$ be an open subset of $M$ which is geodesic invex with respect to $\gamma: M \to M$ ! $TM$. If $f$ is geodesic $\gamma$-preinvec, then $f$ is geodesic $\gamma$-pre-pseudo-invex for the same geodesic.

**Proof.** If $f(x) < f(y)$, for every $t \in (0; 1)$ and $f$ is geodesic $\gamma$-preinvec, then
\[
    f(\gamma(x,y)(t)) \leq tf(x) + (1 - t)f(y)
    = f(y) + t(f(x) - f(y))
    < f(y) + t(f(x) - f(y)) + t^2(f(x) - f(y))
    = f(y) + t(t \cdot 1)(f(y) - f(x))
    = f(y) + t(t \cdot 1)b(x; y);
\]
where $b(x; y) = f(y) - f(x) > 0$.

**Theorem 2.3.2.** Let $f: S ! R$ be a geodesic $\gamma$-pre-pseudo-invex function on $S$ and $h: I ! R$ be strictly increasing convex function such that range
(f) \subseteq I. Then, the composite function \( h(f) \) is geodesic \(^\prime\)-pre-pseudo-invex on \( S \).

**Proof.** Since \( f \) is geodesic \(^\prime\)-pre-pseudo-invex function on \( S \), we have

\[
f(x) < f(y) \quad f(\circ_{x,y}(t)) \leq f(y) + t(t_i 1)b(x; y);
\]

for every \( t 2 (0; 1) \) and \( x, y 2 S \), where \( b(x; y) \) is strict positive function.

Since \( h \) is strictly increasing convex function, we get

\[
h(f(x)) < h(f(y)) \quad h(f(\circ_{x,y}(t))) < h(f(y) + t(t_i 1)b(x; y))
\]

\[
< h(f(y)) + t(t_i 1)h(b(x; y));
\]

for every \( t 2 (0; 1) \) and \( x, y 2 S \); where \( h(b(x; y)) \) is strict positive function. Which shows that \( h(f) \) is a geodesic \(^\prime\)-pre-pseudo-invex function on \( S \).

**Definition 2.3.4.** Function \( f \) is said to be geodesic \(^\prime\)-pre-quasi-invex (p.q.i) on \( S \) if

\[
f(\circ_{x,y}(t)) \leq \max f(x); f(y)
\]

for all \( x, y 2 S \) and for every \( t 2 [0; 1] \).

Now, we characterize geodesic \(^\prime\)-pre-quasi-invex function in terms of its lower level sets.

**Theorem 2.3.3.** Let \( S \) be a geodesic invex subset of \( M \) and \( f : S ! R \). Then, \( f \) is geodesic \(^\prime\)-pre-quasi-invex on \( S \) if and only if its lower level sets are geodesic invex.

**Proof.** Suppose that \( f \) is geodesic \(^\prime\)-pre-quasi-invex function on \( S \) and \( C(\oplus) := \{ x : f(x) \leq \oplus g \} \) is the subset of \( S \). If \( C(\oplus) \) is empty the result is trivial. If \( C(\oplus) \) is neither empty nor the whole set \( S \), take any two points \( x \) and \( y \) in \( C(\oplus) \). Now, we have to show that the geodesic \( \circ_{x,y}(t) \) is contained in \( C(\oplus) \). Since \( f \) is geodesic \(^\prime\)-pre-quasi-invex function, we have

\[
f(\circ_{x,y}(t)) \leq \max(f(x); f(y)) \leq \oplus
\]

for all \( x, y 2 S \) and for every \( t 2 [0; 1] \). Hence \( C(\oplus) \) is geodesic invex.
Conversely, suppose that for every real number $\circ$, the set $\mathcal{C}(\circ)$ is geodesic invex. Take any two points $x, y \in S$ and suppose that $f(x) \leq f(y)$. Consider the lower level set $\mathcal{C}(f(y))$. Since $\mathcal{C}(\circ)$ is geodesic invex, the geodesic $O_{x,y}(t)$ is contained in $\mathcal{C}(f(y))$. Thus,

$$f(O_{x,y}(t)) \leq f(y) = \max(f(x); f(y))$$

for every $t \in [0; 1]$. The proof is complete.

**Proposition 2.3.1.** Let $f$ be a geodesic $\ast$-pre-quasi-invex function on $S$. Then,

(i) every strict local minimum of $f$ is also a strict global minimum;

(ii) the set of all strict global minimum points is geodesic invex set.

**Proof.** (i) Let $y$ be a strict local minimum which is not global; then there exists a point $x^* \in S$, such that $f(x^*) < f(y)$. Since, $f$ is geodesic $\ast$-pre-quasi-invex, we have $f(O_{x^*,y}(t)) \leq f(y)$, which contradicts the hypothesis that $y$ is a strict local minimum.

(ii) If $f$ has no minimum value in $S$, then the set of minimum points is empty and hence geodesic invex. If $f$ has the minimum point $\circ$ on $S$, then the set of minimum points is $S \setminus \mathcal{C}(\circ)$, which is geodesic invex.

The geodesic $\ast$-pre-quasi-invexity is preserved under composition with non decreasing function $\hat{A} : R ! R$ as can be seen below:

**Proposition 2.3.2.** Let $f$ be a geodesic $\ast$-pre-quasi-invex function and $\hat{A} : R ! R$ be a non decreasing function. Then, $\hat{A}(f)$ is geodesic $\ast$-pre-quasi-invex.

**Proof.** Given that $f$ is a geodesic $\ast$-pre-quasi-invex function and $\hat{A}$ is a non decreasing function. Then, we have

$$(\hat{A}(f))(O_{x,y}(t)) \leq \hat{A}(\max(f(x); f(y)))$$

$$= \max(f(\hat{A}(x)); \hat{A}(f(y)))$$

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which shows that the composite function $A(f)$ is geodesic $\cdot$-pre-quasi-invex.

In the following proposition, we show that every geodesic $\cdot$-preinvex function is geodesic $\cdot$-pre-quasi-invex.

**Proposition 2.3.3.** If the function $f$ is geodesic $\cdot$-preinvex on $S$, then $f$ is geodesic $\cdot$-pre-quasi-invex on $S$.

**Proof.** Let $f$ be geodesic $\cdot$-preinvex function on $S$. Then for every $x, y \in S$ and $t \in [0, 1]$, it follows that

$$f(\theta_{x, y}(t)) \leq tf(x) + (1 - t)f(y)$$

$$\leq t \max f(x); f(y)g + (1 - t) \max f(x); f(y)g$$

$$= \max f(x); f(y)g;$$

which shows that $f$ is geodesic $\cdot$-pre-quasi-invex on $S$.

Further, we have

**Proposition 2.3.4.** If $f : S \to R$ is geodesic $\cdot$-pre-pseudo-invex on $S$ then $f$ is geodesic $\cdot$-pre-quasi-invex on $S$.

**Proof.** Let $f(x) < f(y)$. Since $f$ is geodesic $\cdot$-pre-pseudo-invex function on $S$, $\forall x, y \in S$ and $\forall t \in (0, 1)$, we have

$$f(\theta_{x, y}(t)) \leq f(y) + t(1 - t)b(x; y)$$

$$< f(y)$$

$$= \max f(x); f(y)g;$$

Hence, $f$ is geodesic $\cdot$-pre-quasi-invex on $S$.

### 2.4. Semistrictly Geodesic $\cdot$-Preinvex Functions

In this section, we introduce semistrictly geodesic $\cdot$-preinvex functions on Riemannian manifolds and study their properties. A new set named as geodesic G-invex set is also introduced.
Definition 2.4.1. Let $S$ be an open subset of $M$ which is geodesic invex with respect to $\mathcal{F}: M \not\subseteq \mathbb{R}$. A function $f: S \not\subseteq \mathbb{R}$ is said to be semistrictly geodesic $\mathcal{F}$-preinvex if for every $x; y \in S$; $f(x) \in f(y)$, we have

$$f\left(\mathcal{F}_{x,y}(t)\right) < tf(x) + (1 - t)f(y); \quad 0 < t < 1.$$ 

It is to be noted that semistrictly geodesic $\mathcal{F}$-preinvex function need not be geodesic $\mathcal{F}$-preinvex function as illustrated in the following example:

Example 2.4.1. Let $M = \mathbb{R}^2$ and $\mathcal{F}: \mathbb{R}^2 \not\subseteq \mathbb{R}$ be defined by $\mathcal{F}(e^{i\theta}; e^{iA}) = (A; \theta)(\sin A; \cos A)$. Let $x = e^{i\theta}$ and $y = e^{iA}$. Then, we define a geodesic on $M$ as $\mathcal{F}_{x,y}(t) = (\cos((1 - t)A + t\theta); \sin((1 - t)A + t\theta));$ which is obviously a geodesic and satisfies the reperimetrization condition.

Clearly,

$$\mathcal{F}_{x,y}(0) = y; \quad \mathcal{F}_{y,y}(0) = \mathcal{F}(x; y); \quad \mathcal{F}_{x,y}(t) \not\subseteq M; \quad \text{for all } t \in (0, 1];$$

Hence by Definition 2.1.4, $M$ is a geodesic invex set.

Now, we define $f: M \not\subseteq \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } \theta = 0; \\ 1 & \text{otherwise.} \end{cases}$$

It is obvious that $f$ is semistrictly geodesic $\mathcal{F}$-preinvex function.

Let $\theta = \frac{\pi}{6}$, $\frac{\pi}{6}$ and $t = \frac{1}{2}$, then

$$f\left(\mathcal{F}_{x,y}(t)\right) = f(\cos((1 - t)A + t\theta); \sin((1 - t)A + t\theta)) = f(\cos 0; \sin 0) = 1 \in \frac{1}{2}f\left(\frac{\pi}{6}\right) + \frac{1}{2}f\left(\frac{\pi}{6}\right) = 0;$$

which shows that $f$ is not geodesic $\mathcal{F}$-preinvex function.
Any local optimal solution is a global optimal solution if \( f \) is semistrictly geodesic \( \cdot \)-preinvex function.

**Theorem 2.4.1.** Let \( S \) be a nonempty geodesic invex set in \( M \) with respect to \( \cdot : M \subseteq M ! T M \) and \( f : S \to R \) be a semistrictly geodesic \( \cdot \)-preinvex function. If \( \bar{x} \in S \) is a local optimal solution to the problem

\[
\begin{align*}
(P) \quad & \text{Min } f(x) \\
&s.t.: \quad x \in S;
\end{align*}
\]

then \( \bar{x} \) is a global minimum in \( (P) \).

**Proof.** Suppose that \( \bar{x} \in S \) is a local minimum. Then, there is a neighborhood \( N_x(\bar{x}) \) such that

\[
\begin{align*}
f(\bar{x}) \leq f(x); \quad 8 x \in S \setminus N_x(\bar{x}), \quad (2:4:1)
\end{align*}
\]

If \( \bar{x} \) is not a global minimum of \( f \), then there exists a point \( x^* \in S \) such that

\[
f(x^*) < f(\bar{x});
\]

Since \( S \) is a geodesic invex set with respect to \( \cdot \), there exists exactly one geodesic \( \circ \) such that

\[
\circ(0) = \bar{x}; \quad \circ(0) = \cdot(x^*; \bar{x}); \quad \circ(t) \in S; \quad 8 t \in [0; 1];
\]

If we choose \( \delta > 0 \) small enough such that \( d(\circ(t); \bar{x}) < \delta \), then \( \circ(t) \in N_x(\bar{x}) \).

By the semistrictly geodesic \( \cdot \)-preinvexity of \( f \), we have

\[
f(\circ(t)) < tf(x^*) + (1 - t)f(\bar{x}) < f(\bar{x}); \quad 8 t \in (0; 1);
\]

Therefore, for each \( \circ(t) \in S \setminus N_x(\bar{x}) \), \( f(\circ(t)) < f(\bar{x}) \), which is a contradiction to \( (2:4:1) \). Hence the result follows.

Now, we define geodesic G-invex set by considering \( S \subseteq M \subseteq M \).

**Definition 2.4.2.** Let \( S \subseteq M \subseteq M \), \( S \) is said to be geodesic G-invex set if there exists \( \cdot : M \subseteq M ! T M \) such that for any pair of \( (x; \circ) \) and \( (y; \cdot) \), we have

\[
(\circ_{x,y}(t); t^\circ + (1 - t)^\cdot) \in S; \quad 8 t \in [0; 1];
\]
In the following, we characterize geodesic \( \rightarrow \text{-preinvert} \) function in terms of its epigraph \( E(f) \), which is defined as:

\[
E(f) = f(x; \otimes) : j \leq 2 \ S; \ \otimes 2 \ R; \ f(x) \leq \otimes g:
\]

**Theorem 2.4.2.** Let \( S \subseteq M \) be a geodesic invert set with respect to \( \rightarrow : M \leq M ! \ TM \). Then \( f : S \rightarrow R \) is geodesic \( \rightarrow \text{-preinvert} \) function on \( S \) if and only if \( E(f) \) is a geodesic \( G \)-invert set in \( M \leq R \).

**Proof.** Let \( f \) be a geodesic \( \rightarrow \text{-preinvert} \) function on \( S \) and \( (x; \otimes); (y; \rightarrow) \leq 2 \ E(f) \). Then,

\[
f(x) \leq \otimes, \ f(y) \leq \rightarrow:
\]

Since \( f \) is geodesic \( \rightarrow \text{-preinvert} \) function on \( S \), for \( t \leq 2 [0; 1] \), we have

\[
f(\langle x, y(t) \rangle) \leq tf(x) + (1 + t)f(y) \leq t\otimes + (1 + t)\rightarrow:
\]

Hence,

\[
(\langle x, y(t); t\otimes + (1 + t)\rightarrow \rangle \leq 2 \ E(f) ; \ 8 t \leq 2 [0; 1]):
\]

Thus, \( E(f) \) is a geodesic \( G \)-invert set.

Conversely, suppose that \( E(f) \) is a geodesic \( G \)-invert set in \( M \leq R \) and \( x; y \leq 2 S \). Then,

\[
(x; f(x)) \leq 2 \ E(f); \ (y; f(y)) \leq 2 \ E(f):
\]

Thus, for \( t \leq 2 [0; 1] \), we have

\[
(\langle x, y(t); t\otimes + (1 + t)\rightarrow \rangle \leq 2 \ E(f);)
\]

which implies that

\[
f(\langle x, y(t) \rangle) \leq tf(x) + (1 + t)f(y) ; \ 8 t \leq 2 [0; 1]:
\]

Hence, \( f \) is a geodesic \( \rightarrow \text{-preinvert} \) function on \( S \).

In the following theorem we show that any \( \rightarrow \) nite intersection of geodesic \( G \)-invert sets in \( M \leq R \) is also a geodesic \( G \)-invert set.

**Theorem 2.4.3.** Let \( S_i, \) where \( i \leq 2 I \), be a family of geodesic \( G \)-invert sets in \( M \leq R \) with respect to the same function \( \rightarrow : M \leq M ! \ TM \). Then, their intersection \( \bigcap_{i \leq 2 I} S_i \) is also a geodesic \( G \)-invert set.
Proof. Let \((x; \mathcal{R}); (y; t) \in T S_i\). Then, for each \(i \in I\), \((x; \mathcal{R}); (y; t) \in T S_i\).

But \(S_i\) is a geodesic \(G\)-invex set for each \(i \in I\), it follows that

\[
(\hat{\text{e}}_{x,y}(t); t \mathcal{R} + (1; t)) \in T S_i; \quad 8 t \in [0; 1]:
\]

Thus,

\[
(\hat{\text{e}}_{x,y}(t); t \mathcal{R} + (1; t)) \in T S_i; \quad 8 t \in [0; 1]:
\]

Hence, the result follows.

Theorem 2.4.4. Let \(S \subseteq M\) be a geodesic invex set with respect to \(\hat{\text{e}}\) : \(M \subseteq M \subseteq T M\) and let \(f_i; i \in I\) be a family of real valued functions which are geodesic \(\hat{\text{e}}\)-preinvex for the same function \(\hat{\text{e}}\) and bounded from above on \(S\); then function \(f(x) = \sup_{i \in I} f_i(x)\) is a geodesic \(\hat{\text{e}}\)-preinvex function on \(S\).

Proof. Given that each \(f_i\) is a geodesic \(\hat{\text{e}}\)-preinvex function for the same \(\hat{\text{e}}\) on \(S\), its epigraph

\[
E(f_i) = f(x; \mathcal{R}) \times 2 S; \mathcal{R} 2 R; f_i(x) \leq \mathcal{R}
\]

is a geodesic \(G\)-invex set in \(M \subseteq R\), it follows from Theorem 2.4.2. Therefore, their intersection

\[
\bigcap_{i \in I} E(f_i) = f(x; \mathcal{R}) \times 2 S; \mathcal{R} 2 R; f_i(x) \leq \mathcal{R}; i \in I
\]

is also a geodesic \(G\)-invex set in \(M \subseteq R\), it follows from Theorem 2.4.3. We can see that this intersection is the epigraph of \(f\). Hence, it follows from Theorem 2.4.2 that \(f\) is a geodesic \(\hat{\text{e}}\)-preinvex function on \(S\).