CHAPTER - 4
RAYLIGH-TAYLOR INSTABILITY OF
STRATIFIED VISCOELASTIC
OLDROYD-B FLUIDS
4.1.1 INTRODUCTION

The character of equilibrium of an inviscid, incompressible fluid having variable density in the vertical direction was investigated by Rayleigh (1883). An experimental demonstration of the development of the Rayleigh-Taylor instability was performed by Taylor (1950). A comprehensive account of the Rayleigh-Taylor instability has been given by Chandrasekhar (1955, 1961) wherein the effects of uniform rotation with an angular velocity $\Omega$ about the vertical and uniform horizontal magnetic field, separately, have also been treated. He demonstrated that the system is stable or unstable according as the density decreases everywhere or increases anywhere inside the fluid. The effect of vertical magnetic field on the development of Rayleigh-Taylor instability was considered by Hide (1955). It has been demonstrated by Furth et al. (1963), Jukes (1963) and many others that the inclusion of finite resistivity modifies the Rayleigh-Taylor problem and makes possible new unstable modes. Zadoff and Begun (1968) have treated the case of two incompressible fluids separated by a horizontal boundary in the presence of a uniform horizontal magnetic field. They have discussed the effects of finite resistivity and viscosity of the medium on the growth rate of Rayleigh-Taylor modes and have shown that a finite resistivity does not affect the growth rate of unstable modes when the wave vector is perpendicular to the magnetic field, but that it does increase the growth rate when the wave vector is parallel to the magnetic field. Sundram (1968) has considered gravitational instability of a fluid with finite resistivity and concluded that the density stratification is unstable for all wavenumbers. Gupta (1963) again studied the stability of a horizontal layer of a perfectly conducting fluid with continuous density and viscosity stratifications in the presence of a horizontal magnetic field (constant as well as variable). The Rayleigh-Taylor instability problem arises in oceanography, limnology and engineering. Generally, magnetic field has a stabilizing effect on the instability, but there are few exceptions also. For example, Kent (1966) has studied the
effect of a horizontal magnetic field which varies in the vertical direction on the
stability of parallel flows and has shown that the system is unstable under
certain conditions, while in the absence of magnetic field the system is known to
be stable. In stellar atmospheres and interiors, the magnetic field may be (and
quite often is) variable and may altogether alter the nature of instability.
Bhatia (1974) has studied the problem for a system of superposed viscous
plasmas in the presence of horizontal magnetic field. The same previous system
in the presence of a horizontal rotation was considered by Bhatia and Chhonkar
(1985) and by Sharma and Chhajlani (1998). The effects of Hall currents and
viscosity on the Rayleigh-Taylor instability of incompressible infinitely
conducting stratified plasma in the presence of horizontal magnetic field were
studied by Ahsen and Bhatia (1993). The effects of Hall currents on the
Rayleigh-Taylor instability of finitely conducting stratified partially ionized
plasma in the presence of horizontal magnetic field is considered by Aiyub and
Bhatia (1993). Inertial confinement fusion (Cook & Zhou 2002) is an example
whose success relies on the control of the famous Rayleigh-Taylor instability
occurring when a heavy, denser, fluid is accelerated into a lighter one. Saffman
(1962) has considered the stability of laminar flow of a dusty gas.
Palaniswamy and Purushotham (1981) have considered the stability of shear
flow of stratified fluids with fine dust and have found the effect of fine dust
(suspended particles) to increase the region of instability. In all the above
studies, the fluid has been assumed to be Newtonian. Our attention here is
focused on Rayleigh Taylor instability with the aim of enhancing the
perturbation growth rate in its early stage of evolution. The idea is to inject
polymers into the incompressible viscoelastic fluid and to study both analytical
and numerical ground how the stability of the resulting viscoelastic fluid is
modified. New polymer liquids Bird et al. (1987) with the growing importance
of non-Newtonian fluids in modern technology and industries, the investigations
of such fluids are desirable. Fredricksen (1964) has given a good review non-
Newtonian fluids whereas Joseph (1976) has also considered the stability of
viscoelastic fluids.
With the growing importance of non-Newtonian fluids in industries, oil
recovery, petroleum refining and chemical technology. Much attention is being
paid to the investigation of such fluids. One such class of viscoelastic fluids is
Oldroyd-B fluids with constitutive equations proposed by Oldroyd (1950), we are interested there in. Attention has recently been drawn by calculations of the rheological behavior of dilute suspensions and emulsions, whose behavior at small variable shear stresses is characterized by Oldroyd-B model. An experimental demonstration by Toms and Strawbridge (1953) reveals that a dilute solution of methyl methacrylate in n-butylacetate agrees well with the theoretical model of the Oldroyd-B fluid.

Boffetta et al. (2010) have studied Rayleigh-Taylor instability in a viscoelastic binary fluid and found that in polymer solutions, the growth rate of the instability speeds up with elasticity which is confirmed by the numerical simulations of the viscoelastic binary fluid. Sharma and Devi (2012) have studied the numerical investigations of stability of stratified viscoelastic Oldroyd-B fluid in the presence of variable magnetic field and have found that the critical wavenumber \( k_c \) and \( k_{\text{max}} \) for the stability of the system remains unchanged in the presence of stress-relaxation time parameter, strain-retardation time parameter, kinematic viscosity whereas the critical wavenumber \( k_c \) goes on decreasing with the increase in magnetic field.

Such type of instabilities are important in modern technology and industries, astrophysics, supernova explosions and mantly convection in the ocean, (to study the stability of gravity waves in ocean and that of vapour globe) laser techniques (classical mode of laser and laser driven implosions) etc.

The objective of this section of the chapter is to examine and compute the effects of the stress-relaxation and strain-retardation parameters accounting for viscoelasticity of the fluid and the mass concentration of the suspended particles on the stability of the growth rate of the most unstable mode numerically, which is the extension of the work of Sharma and Devi (2012). The novelty of this work is to propose a complete stability of the dynamical system by the introduction of a small number of parameters.

4.1.2 FORMULATION OF THE PROBLEM AND PERTURBATION EQUATIONS

The initial stationary state whose stability we wish to examine is that of an incompressible heterogeneous infinitely conducting Oldroyd-B fluid (i.e., \( \sigma \to \infty, \eta \to 0 \)) of variable density, kinematic viscosity and magnetic field.
arranged in horizontal strata so that the free surface is almost horizontal. This fluid is acted on by the gravity force \( \mathbf{g}(0,0,-g) \) and a variable horizontal magnetic field \( \mathbf{H}(H_0(z),0,0) \); \( z \)-axis being taken as vertical. This layer is confined between the planes \( z = 0 \) and \( z = d \).

The equations of motion, continuity and Maxwell’s equations for the viscoelastic fluid-particle layer are

\[
\rho \left( 1 + \lambda \frac{\partial}{\partial t} \right) \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = \left( 1 + \lambda \frac{\partial}{\partial t} \right) \left[ -\nabla p + \rho \mathbf{g} + \frac{\mu_e}{4\pi} (\nabla \times \mathbf{H}) \times \mathbf{H} + K'N(q_d - \mathbf{u}) \right] +
\]

\[
(1 + \lambda_0 \frac{\partial}{\partial t}) \mu \nabla^2 \mathbf{u},
\]

\( \nabla \cdot \mathbf{u} = 0, \quad (4.1.2) \)

\( \nabla \cdot \mathbf{H} = 0, \quad (4.1.3) \)

\[
\frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{H}),
\]

\( \text{(4.1.4)} \)

where \( \rho, \mu, \rho, \mu_e, \lambda, \lambda_0 \) and \( \mathbf{u} \) denote, respectively, the density, the viscosity, the pressure, the magnetic permeability, stress-relaxation time parameter, strain-retardation time parameter and the fluid velocity (initially zero). \( q_d \) and \( N(x,t) \) denote the velocity and number density of particles. \( K' = 6\pi \mu \eta' \), \( \eta' \) being the particle radius, is the Stoke’s drag and \( x = (x,y,z) \). Assuming uniform particle size, spherical shape and small relative velocities between the fluid and particles, the net effect of the particles on the fluid is equivalent to an extra body force per unit volume \( K'N(q_d - \mathbf{u}) \).

Since the density of a fluid-particle moving with the fluid remains unchanged, we have

\[
\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla) \rho = 0. \quad (4.1.5)
\]

If \( mN \) is the mass of particles per unit volume, then the equations of motion and continuity for the particles are

\[
mN \left( \frac{\partial q_d}{\partial t} + (q_d \cdot \nabla) q_d \right) = K'N(u - q_d), \quad (4.1.6)
\]

\[
\frac{\partial N}{\partial t} + \nabla \cdot (N \cdot q_d) = 0. \quad (4.1.7)
\]

The presence of particles adds an extra force term, proportional to the velocity difference between particles and fluid which appears in equations of motion \( (4.1.1) \). Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion for the particles. The buoyancy force on the particles is neglected. Interparticle
reactions are also not considered, for we assume that the distances between particles are quite large compared with their diameter. These assumptions have been used in writing the equations of motion for the particles in equation (4.1.6).

Since the equilibrium state under consideration is quiescent one, (i.e. there are no settling of particles) the steady state solutions are given by

\[
\begin{align*}
\mathbf{u} &= (0,0,0) \\
\rho &= \rho(z) \\
p &= p(z) \\
H &= (H_0(z), 0, 0) \\
N &= N_0 \text{ (a constant)}
\end{align*}
\]

(4.1.8)

The character of the equilibrium of this stationary state can be determined by disturbing the system slightly and then, following its further evolution.

Let \( \delta \rho, \delta p, \mathbf{u}(u,v,w) \) and \( \mathbf{h}(h_x, h_y, h_z) \) denote, respectively, the perturbations in density \( \rho(z) \), pressure \( p(z) \), fluid velocity \( (0,0,0) \), particles velocity and horizontal magnetic field \( H = (H_0(z), 0, 0) \). Then using linear theory, the linearized perturbation equations of the fluid-particle layer become

\[
\begin{align*}
\rho \left(1 + \lambda \frac{\partial}{\partial t}\right) \frac{\partial \mathbf{u}}{\partial t} &= \left(1 + \lambda \frac{\partial}{\partial t}\right) [-\nabla \delta p + g \delta \rho + \frac{\mu_\pi}{4\pi} ((\nabla \times h) \times H + (\nabla \times H) \times h) + \\
K' N(q_d - \mathbf{u}) + (1 + \lambda_0 \frac{\partial}{\partial t}) \mu \nabla^2 \mathbf{u},
\end{align*}
\]

(4.1.9)

\[
\nabla \cdot \mathbf{u} = 0, \quad (4.1.10)
\]

\[
\frac{\partial}{\partial z} \delta \rho = -w \frac{\partial \rho}{\partial z}, \quad (4.1.11)
\]

\[
\nabla \cdot \mathbf{h} = 0, \quad (4.1.12)
\]

\[
\frac{\partial \mathbf{h}}{\partial t} = (H \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{H}, \quad (4.1.13)
\]

\[
\left(\frac{m}{\kappa} \left(\frac{\partial}{\partial t}\right) + 1\right) q_d = \mathbf{u}. \quad (4.1.14)
\]

Operating equation (4.1.9) by \( \left(\frac{m}{\kappa} \left(\frac{\partial}{\partial t}\right) + 1\right) \) to eliminate \( q_d \) between equations (4.1.10) and (4.1.14), we get

\[
\begin{align*}
\left(1 + \lambda \frac{\partial}{\partial t}\right) \left[\rho \frac{\partial}{\partial t} \left(\frac{m}{\kappa} \left(\frac{\partial}{\partial t}\right) + 1\right) + Nm \frac{\partial}{\partial t}\right] \mathbf{u} &= \left(1 + \lambda \frac{\partial}{\partial t}\right) \left(\frac{m}{\kappa} \left(\frac{\partial}{\partial t}\right) + 1\right) [-\nabla \delta p + g \delta \rho + \frac{\mu_\pi}{4\pi} ((\nabla \times h) \times H - (\nabla \times H) \times h) + \left(1 + \lambda_0 \frac{\partial}{\partial t}\right) \mu \nabla^2 \mathbf{u}.
\end{align*}
\]

(4.1.15)

Equations (4.1.9) - (4.1.12) and (4.1.14) in the Cartesian form are

\[
\begin{align*}
\left(1 + \lambda \frac{\partial}{\partial t}\right) \left[\rho \frac{\partial}{\partial t} \left(\frac{m}{\kappa} \left(\frac{\partial}{\partial t}\right) + 1\right) + Nm \frac{\partial}{\partial t}\right] \mathbf{u} &= \left(1 + \lambda \frac{\partial}{\partial t}\right) \left(\frac{m}{\kappa} \left(\frac{\partial}{\partial t}\right) + 1\right) [-\frac{\partial}{\partial x} \delta p + \frac{\mu_\pi}{4\pi} (h_z \frac{\partial H_0}{\partial z}) + \\
\left(1 + \lambda_0 \frac{\partial}{\partial t}\right) \left(\frac{m}{\kappa} \left(\frac{\partial}{\partial t}\right) + 1\right) \mu \nabla^2 \mathbf{u}.
\end{align*}
\]

(4.1.16)
We ascribe all the quantities describing the perturbation dependence on \(x, y\) and \(t\) of the form

\[ f(z) \exp \left( ik_x x + ik_y y + nt \right), \tag{4.1.25} \]

where \(f(z)\) is the same function of \(z\) only; \(k_x, k_y\) are the wavenumbers in the \(x\) and \(y\)-directions, \(k = (k_x^2 + k_y^2)^{1/2}\) is the resultant wavenumber and \(n\) is a constant which can be complex, in general. The solution of the stability problem requires the specifications of the state for each \(k\).

Equations (4.1.16)-(4.1.24), using the expression (4.1.25), become

\[
(1 + \lambda n) \left[ \rho n \left( \frac{m}{k'} + 1 \right) + Nmn \right] u = (1 + \lambda n) \left( \frac{m}{k'} + 1 \right) \left[ -ik_x \delta \rho + \frac{\mu_e H_0}{4\pi} (ik_x h_x - ik_y h_y) \right] + \\
(1 + \lambda_0 n) \left( \frac{m}{k'} + 1 \right) \mu (D^2 - k^2) u, \tag{4.1.26}
\]

\[
(1 + \lambda n) \left[ \rho n \left( \frac{m}{k'} + 1 \right) + Nmn \right] v = (1 + \lambda n) \left( \frac{m}{k'} + 1 \right) \left[ -ik_y \delta \rho + \frac{\mu_e H_0}{4\pi} (ik_x h_y - ik_y h_x) \right] + \\
(1 + \lambda_0 n) \left( \frac{m}{k'} + 1 \right) \mu (D^2 - k^2) v, \tag{4.1.27}
\]

\[
(1 + \lambda n) \left[ \rho n \left( \frac{m}{k'} + 1 \right) + Nmn \right] w = (1 + \lambda n) \left( \frac{m}{k'} + 1 \right) \left[ -D \delta \rho - g \delta \rho + \frac{\mu_e H_0}{4\pi} (ik_x h_z - Dh_x - \frac{h_x}{H_0} D H_0) \right] + (1 + \lambda_0 n) \left( \frac{m}{k'} + 1 \right) \mu (D^2 - k^2) w, \tag{4.1.28}
\]

\[ ik_x u + ik_y v + Dw = 0, \tag{4.1.29} \]

\[ \delta \rho = -w \frac{D \rho}{n}, \tag{4.1.30} \]

\[ ik_x h_x + ik_y h_y + Dh_z = 0, \tag{4.1.31} \]
\[ h_x = \frac{1}{n} \left( H_0 ik_x u - wDH_0 \right), \]  \hspace{1cm} (4.1.32) \\
\[ h_y = \frac{1}{n} \left( H_0 ik_x v \right), \]  \hspace{1cm} (4.1.33) \\
\[ h_z = \frac{1}{n} \left( H_0 ik_x w \right), \]  \hspace{1cm} (4.1.34) \\

where D stands for \( \frac{d}{dz} \).

Now substituting the values of \( h_x, h_y, h_z \) from equations (4.1.32) - (4.1.34) in equations (4.1.26)-(4.1.28), we get

\[ (1 + \lambda n) \left[ \rho n \left( \frac{m}{K} + 1 \right) + Nmn \right] u = (1 + \lambda n) \left( \frac{m}{K} + 1 \right) \left[ -ik_x \delta p + \frac{\mu e}{4\pi n} (H_0 DH_0) ik_x w \right] + \\
(1 + \lambda_0 n) \left( \frac{m}{K} + 1 \right) \mu (D^2 - k^2) u, \]  \hspace{1cm} (4.1.35) \\

\[ (1 + \lambda n) \left[ \rho n \left( \frac{m}{K} + 1 \right) + Nmn \right] v = (1 + \lambda n) \left( \frac{m}{K} + 1 \right) \left[ -ik_y \delta p + \frac{\mu e H_0}{4\pi n} \left( -k_x^2 H_0 v \right) - \right. \\
\left. ik_y (ik_x H_0 u - wDH_0) \right] + (1 + \lambda_0 n) \mu (D^2 - k^2) \left( \frac{m}{K} + 1 \right) v, \]  \hspace{1cm} (4.1.36) \\

\[ (1 + \lambda n) \left[ \rho n \left( \frac{m}{K} + 1 \right) + Nmn \right] w = (1 + \lambda n) \left( \frac{m}{K} + 1 \right) \left[ -D \delta p - g \delta \rho + \frac{\mu e H_0}{4\pi n} \left[ ic_x (H_0 ik_x w) - \\
D (H_0 ik_x u - wDH_0) \right] \right] + (1 + \lambda_0 n) \left( \frac{m}{K} + 1 \right) \mu (D^2 - k^2) w. \]  \hspace{1cm} (4.1.37) \\

Multiplying equations (4.1.35) and (4.1.36) by \( -ik_y \) and \( ik_x \), respectively and then adding, we get

\[ \zeta = 0. \]  \hspace{1cm} (4.1.38) \\

where \( \zeta = ik_x v - ik_y u \) is the z-component of vorticity. \hspace{1cm} (4.1.39) \\

Substituting the value of \( \zeta \) in equation (4.1.36), it reduces to

\[ (1 + \lambda n) \left[ \rho n \left( \frac{m}{K} + 1 \right) + Nmn \right] v = (1 + \lambda n) \left( \frac{m}{K} + 1 \right) \left[ -ik_y \delta p + \frac{\mu e}{4\pi n} ik_y wDH_0 \right] + \\
\left( \frac{m}{K} + 1 \right) (1 + \lambda_0 n) \mu (D^2 - k^2) v. \]  \hspace{1cm} (4.1.40) \\

On solving equations (4.1.29) and (4.1.39), we get

\[ u = \frac{ik_x Dw}{k^2}. \]  \hspace{1cm} (4.1.41) \\

Multiplying equations (4.1.35) and (4.1.40) by \( -ik_x \) and \( -ik_y \), respectively and then adding the resulting equations, we get

\[ (1 + \lambda n) \left[ \rho n \left( \frac{m}{K} + 1 \right) + Nmn \right] Dw = (1 + \lambda n) \left( \frac{m}{K} + 1 \right) \left[ -k^2 \delta p + \frac{\mu e k^2}{4\pi n} H_0 (DH_0) w \right] + \\
\left( \frac{m}{K} + 1 \right) (1 + \lambda_0 n) \mu (D^2 - k^2) w. \]  \hspace{1cm} (4.1.42) \\

Eliminating \( \delta p \) from equations (4.1.42) and (4.1.37) and using the value of \( u \) from (4.1.41), we get
\[(1 + \lambda n)nv(D^2 - k^2)^2 w - (1 + \lambda n)(D^2 - k^2) \left[ \frac{\mu_k e k_x^2 H_0^2}{4\pi \rho} + n^2 \left( 1 + \frac{Nm/\rho}{m \,(k+1)} \right) \right] w - \left\{ n^2 (1 + \lambda n) \frac{D\mu}{\rho} + (1 + \lambda n) \frac{(D^2 - k^2) n}{\rho} - (1 + \lambda n) \frac{\mu_k e k_x^2}{4\pi \rho} D(H_0)^2 \right\} Dw - \]

\[(1 + \lambda n)gk^2 \frac{D\rho}{\rho} w = 0. \quad (4.1.43)\]

where $V_A^2 = \frac{\mu_k e H_0^2}{4\pi \rho}$ is the square of the Alfvén velocity.

The second last term in equation (4.1.43), represents the effect of the heterogeneity of the fluid on the inertia is neglected in comparison with the effect on potential energy.

Thus equation (4.1.43) reduces to

\[(1 + \lambda n)nv(D^2 - k^2)^2 w - (1 + \lambda n)(D^2 - k^2) \left[ V_A^2 k_x^2 + n^2 \left( 1 + \frac{Nm/\rho}{m \,(k+1)} \right) \right] w - \]

\[(1 + \lambda n)gk^2 \frac{D\rho}{\rho} w = 0. \quad (4.1.44)\]

Equation (4.1.44) constitutes a characteristic equation, the solutions of which need to be sought.

### 4.1.3 The case of exponentially varying stratifications

In order to obtain the solution of the stability problem under consideration, we assume that the stratifications in suspended particle number density, viscosity, density and magnetic field exponentially along the vertical of the form

\[N = N_0 e^{\beta_1 z}, \quad \mu = \mu_0 e^{\beta_1 z}, \quad \rho = \rho_0 e^{\beta_1 z}, \quad H_0^2 = H_1^2 e^{\beta_1 z}, \quad (4.1.45)\]

where $\rho_0$, $\mu_0$, $N_0$, $H_1$ and $\beta_1$ are constants and so the kinematic viscosity, $\nu_0 = \left( \frac{\mu_0}{\rho_0} \right)$, the kinematic viscoelasticity $\nu_0' = \left( \frac{\mu_0}{\rho_0} \right)$ and the Alfvén velocity,

\[V_A = \left( \frac{\mu_k e H_0^2}{4\pi \rho_0} \right)^{1/2} = \left( \frac{\mu_k e H_1^2}{4\pi \rho_0} \right)^{1/2} \]

are constant from layer to layer.

Here the case of two free boundaries is considered and it is assumed that $\beta_1 d \ll 1$, i.e. the variations of density at two neighbouring points in the velocity field which is much less than the average density has a negligible effect on the inertia of the fluid.

The appropriate boundary conditions for the case of two free surfaces are

\[w = D^2 w = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = d. \quad (4.1.46)\]
Using these boundary conditions, equation (4.1.44) implies that $D^4w = 0$ at $z = 0$ and $z = d$. By differentiating equation (4.1.44) twice, we get $D^6w = 0$ at $z = 0$ and $z = d$.

This process can be continued and it can be shown that all the even order derivatives of $w$ must vanish for $z = 0$ and $z = d$ and hence, the proper solution of $w$ is given by

$$w = A \sin \left( \frac{m_1 \pi z}{d} \right).$$

(4.1.47)

where $m_1$ is an integer and $A$ is a constant.

Substituting the value of $w$ from equation (4.1.47) in equation (4.1.44) and also using expressions (4.1.45), we get

$$\lambda m \left[ \frac{m_1^2 \pi^2}{d^2} + k^2 \right] n^2 + \left( \frac{m_1^2 \pi^2}{d^2} + k^2 \right) \left[ m \lambda_0 \nu_0 \left( \frac{m_1^2 \pi^2}{d^2} + k^2 \right) + (m + K') \lambda + \frac{\lambda m \nu_0 k^2}{\rho_0} \right] n^3 +$$

$$\left[ (m + K') \lambda_0 \nu_0 \left( \frac{m_1^2 \pi^2}{d^2} + k^2 \right)^2 + \left( m \lambda \nu_0 V_A^2 \right) k_x^2 + K' \left( 1 + \frac{m \nu_0}{\rho_0} \right) \left( \frac{m_1^2 \pi^2}{d^2} + k^2 \right) \lambda m \nu_0 k^2 \beta_1 \right] n^2 +$$

$$\left[ K' \nu_0 \left( \frac{m_1^2 \pi^2}{d^2} + k^2 \right)^2 + (m + K') \lambda \left( \frac{m_1^2 \pi^2}{d^2} + k^2 \right) V_A^2 k_x^2 - g k^2 \beta_1 \right] n + K' \left[ V_A^2 k_x^2 \left( \frac{m_1^2 \pi^2}{d^2} + k^2 \right) - g k^2 \beta_1 \right] = 0.$$  

(4.1.48)

Equation (4.1.48) is bi-quadratic equation in $n$ and is the dispersion relation governing the effects of variable horizontal magnetic field, kinematic viscosity, suspended particles number density, stress-relaxation time and strain-retardation time on the stability of stratified Oldroyd-B fluid.

(a) Case of Stable Stratification ($i.e. \beta_1 < 0$). Equation (4.1.48) does not admit of any positive real root or complex root with positive real part using Routh–Hurwitz criterion; therefore, the system is always stable for disturbances of all wavenumbers.

(b) Case of Unstable Stratification ($i.e. \beta_1 > 0$). If $V_A^2 k_x^2 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) < g k^2 \beta_1$, equation (4.1.48) has at least one root with positive real part or complex root with positive real part using Routh–Hurwitz criterion; so the system is unstable for all wavenumbers satisfying the inequality

$$k^2 < \frac{g \beta_1 \sec^2 \theta}{V_A^2} - \frac{m_1^2 \pi^2}{d^2},$$

where $\theta$ is the angle between $k_x$ and $k$, i.e. $k_x = k \cos \theta$. 

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If \( k^2 > \frac{g \beta_1 \sec^2 \frac{k}{\rho_0}}{V_A^2} - \frac{m_1 n^2}{d^2} \), equation (4.1.48) does not admit of any positive real root or complex root with positive real part, therefore, the system is stable. The system is clearly unstable in the absence of variable magnetic field. However, the system can be completely stabilized by a large enough magnetic field.

Thus if \( \beta_1 > 0 \) and \( V_A^2 k_x^2 \left( k^2 + \frac{m_1 n^2}{d^2} \right) < g k^2 \beta_1 \), equation (4.1.48) has at least one positive root. Let \( n_0 \) denotes the positive root of equation (4.1.48). Therefore,

\[
\lambda m \left[ \frac{m_1 n^2}{d^2} + k^2 \right] n_0^4 + \left( \frac{m_1 n^2}{d^2} + k^2 \right) \left[ m \lambda_0 v_0 \left( \frac{m_1 n^2}{d^2} + k^2 \right) + (m + K' \lambda) + \frac{\lambda m n_0 k'}{\rho_0} \right] n_0^3 + \left[ (m + K' \lambda_0) v_0 \left( \frac{m_1 n^2}{d^2} + k^2 \right)^2 + \left( m \lambda V_A^2 k_x^2 + K' \left( 1 + \frac{m n_0}{\rho_0} \right) \right) \left( \frac{m_1 n^2}{d^2} + k^2 \right) \lambda m g k^2 \beta_1 \right] n_0^2 + \left[ K' v_0 \left( \frac{m_1 n^2}{d^2} + k^2 \right)^2 + (m + K' \lambda) \left( \left( \frac{m_1 n^2}{d^2} + k^2 \right) V_A^2 k_x^2 - g k^2 \beta_1 \right) \right] n_0 + K' \left[ V_A^2 k_x^2 \left( \frac{m_1 n^2}{d^2} + k^2 \right) \right] = 0, \tag{4.1.49}
\]

as \( v_0 = \frac{\mu_0}{\rho_0} = \frac{\mu}{\rho} = v \) and \( V_A^2 = \frac{\mu e H^2}{4 \pi \rho_0} = \frac{\mu e H^2}{4 \pi \rho} \).

To find the role of viscosity, magnetic field, suspended particle number density, Stoke's drag coefficent, strain-retardation time parameter, and stress-relaxation time parameter on the growth rate of the unstable modes; we examine the nature of \( \frac{d n_0}{d v_0}, \frac{d n_0}{d V_A^2}, \frac{d n_0}{d n_0}, \frac{d n_0}{d K', \frac{d n_0}{d \lambda_0}, \text{and } \frac{d n_0}{d K} \text{ analytically.}}

Equation (4.1.49), yield that

\[
\frac{d n_0}{d v_0} = - \frac{(m \lambda_0 n^2 + K' \lambda_0 n^2 + K'n)}{(3 m n_0 v_0 + 2 K' \lambda_0 v_0 + K' v_0)}, \tag{4.1.50}
\]

\[
\frac{d n_0}{d V_A^2} = - \frac{2 m (m + K') + K'}{(m + K') V_A^2}, \tag{4.1.51}
\]

\[
\frac{d n_0}{d K'} = - \frac{n \left( K' + \frac{\lambda m n_0}{\rho_0} \right) + n \left( \lambda \right)}{(3 n + 2 L, \lambda) n_0}, \tag{4.1.52}
\]

\[
\frac{d n_0}{d L} = - \frac{n \left( L^2 + \frac{\lambda m n_0}{\rho_0} \right) + n \left( \lambda L \right)}{K \left( 3 n^2 \left( 1 + \frac{\lambda m n_0}{\rho_0} \right) + 2 L \lambda n_0 v_0 L^2 + 2 n L \left( 1 + \frac{\lambda m n_0}{\rho_0} \right) + V_A^2 \lambda L \right)}, \tag{4.1.53}
\]

\[
\frac{d n_0}{d \lambda_0} = - \frac{n (m n + K)}{(3 m n + 2 K')}, \tag{4.1.54}
\]
It is clear from equation (4.1.50)- (4.1.54) that the growth rate decreases with the increase in kinematic viscosity, magnetic field, suspended particle number density, Stoke's drag coefficient, strain retardation time parameter whereas, it is evident from equation (4.1.55) that \( \frac{dn_0}{d\lambda} \) is negative if

\[ L_1V_A^2k^2 > gk^2 \beta_1, \]

where \( L_1 = \frac{m_1^2\pi^2}{d^2} + k^2 \) and \( \beta_1 > 0 \),

implying thereby the stabilizing effect of stress-relaxation time parameter.

4.1.4 NUMERICAL RESULTS AND DISCUSSION

The dispersion relation (4.1.48) has been computed numerically for \( \beta_1 > 0 \) for different values of the physical parameters \( \lambda, \lambda_0, \nu_0 \) and the square of Alfvén velocity \( V_A^2 \), using the Software Mathematica version 5.2.

Figure 1 illustrates the variation of growth rate \( n_r \) (positive real part of \( n \)) with respect to the wavenumber \( k \) satisfying equation (4.1.48) for fixed permissible values of \( \beta_1 = 0.2, \lambda_0 = 0.3, m_1 = 1, \lambda = 0.8, d = 4, g = 980 \text{ cm/s}^2, \nu_0 = 4, V_A^2 = 10, k_x = k/\sqrt{2}, K = 0.001, \rho_0 = 1.02, m = 0.0001 \) for three values of number of molecules of suspended particles \( N_0 = 500000, 700000, 900000 \), respectively.

It is observed from this figure that there is a mode of maximum instability where the growth rate \( n_r \) increases with \( k \) increasing through the range \( 0 < k < k_{max} \) (at \( k_{max} = 1.5 \), growth rates arrive to the maximum instability), whereas the growth rate \( n_r \) starts to decrease as \( k \) increases and then goes to the complete stability at \( k_c = 6.5 \) (\( k_c \) is the critical value for stability, at this point \( n_r \) approaches to zero). This means that the kinematic viscosity has a critical capability to suppress the instability of viscoelastic fluids completely.
Figure 1: Variation of $n_r$ (positive real part of $n$) with wavenumber $k$ for three values of $N_0 = 500000, 700000, 900000$.

Figure 2 shows the variation of growth rate $n_r$ (positive real part of $n$) with respect to the wavenumber $k$ satisfying equation (4.1.48) for fixed permissible values of $\beta_1 = 0.2, \lambda_0 = 0.3, m_1 = 1, \lambda = 0.8, d = 4, g = 980 \text{ cm/s}^2, \nu_0 = 4, V_A^2 = 10, k_x = k/\sqrt{2}, N_0 = 500000, \rho_0 = 1.02, m = 0.0001$ for three values of Stoke's drag coefficient $K' = 0.001, 0.005, 1$ respectively.
Figure 2: Variation of $n_r$ (positive real part of $n$) with wavenumber $k$ for three values of $K' = 0.001$, 0.050, 1.

Again, it is observed from the graphs that there is a mode of maximum instability where the growth rate $n_r$ increases with $k$ increasing through the range $0 < k < k_{\text{max}}$ (at $k_{\text{max}} = 1.5$, growth rates arrive to the maximum instability), whereas the growth rate $n_r$ starts to decrease as $k$ increases and then goes to the complete stability at $k_c = 6.5$ ($k_c$ is the critical value for stability, at this point $n_r$ approaches to zero). This means that the Stoke's drag coefficient has a critical capability to suppress the instability of viscoelastic fluids completely.
Figure 3: Variation of $n_r$ (positive real part of $n$) with wavenumber $k$ for three values of $\lambda = 0.2, 0.8, 1.5$.

Figure 3 shows the variation of growth rate $n_r$ (positive real part of $n$) with respect to the wavenumber $k$ satisfying equation (4.1.48) for fixed permissible values of $\beta_1 = 0.2, \lambda_0 = 0.3, m_1 = 1, N_0 = 500000, d = 4, g = 980\text{ cm/s}^2, v_0 = 4, V_A^2 = 10, k_x = k/\sqrt{2}, K = 0.001, \rho_0 = 1.02, m = 0.0001$ for three values of the stress-relaxation parameter $\lambda = 0.2, 0.8, 1.5$, respectively. The graph shows that there is a mode of maximum instability where the growth rate $n_r$ increases with $k$ increasing through the range $0 < k < k_{\text{max}}$ (at $k_{\text{max}} = 1.5$, growth rates arrive to the maximum instability), whereas the growth rate $n_r$ starts to decrease as $k$ increases and then goes to the complete stability at $k_c = 6.25$ ($k_c$ is the critical value for stability, at this point $n_r$ approaches to zero). This means that the stress-relaxation parameter has a critical capability to suppress the instability of viscoelastic fluids completely.
Figure 4 shows the variation of growth rate $n_r$ (positive real part of $n$) with respect to the wavenumber $k$ satisfying equation (4.1.48) for fixed permissible values of $\beta_1 = 0.2$, $N_0 = 500000$, $m_1 = 1$, $\lambda = 0.8$, $d = 4$, $g = 980$ cm/s$^2$, $\nu_0 = 4$, $V_A^2 = 10$, $k_x = k/\sqrt{2}$, $K' = 0.001$, $\rho_0 = 1.02$, $m = 0.0001$ for three values of the strain retardation parameter $\lambda_0 = 0.3$, 0.9, 1.5, respectively.

Figure 4: Variation of $n_r$ (positive real part of $n$) with wavenumber $k$ for three values of $\lambda_0 = 0.2$, 0.3, 0.5.

It is clear from the figure 4 that there is again a mode of maximum instability where the growth rate $n_r$ increases with $k$ increasing through the range $0 < k < k_{\text{max}}$ (at $k_{\text{max}} = 1.5$), whereas the growth rate $n_r$ starts to decrease as $k$ increases and then goes to the complete stability at $k_c = 6.25$, which imply that the strain retardation parameter $\lambda_0$ has a critical capability to suppress the instability completely.

Figure 5 shows the variation of growth rate $n_r$ (positive real part of $n$) with respect to the wavenumber $k$ satisfying equation (4.1.48) for fixed permissible
values of $\beta_1 = 0.2$, $\lambda_0 = 0.3$, $m_1 = 1$, $\lambda = 0.8$, $d = 4$, $g = 980 \text{ cm/s}^2$, $\nu_0 = 4$, $N_0 = 500000$, $k_x = k/\sqrt{2}$, $K' = 0.001$, $\rho_0 = 1.02$, $m = 0.0001$ for three values of the square of the Alfvén velocity $V_A^2 = 10$, 11, 12, respectively. The graphs show that there is a mode of maximum instability where the growth rate $n_r$ increases with $k$ increasing through the range $0 < k < k_{\text{max}}$ (at $k_{\text{max}} = 1.5$), whereas the growth rate $n_r$ starts to decrease as $k$ increases and the critical wavenumber also goes on decreasing with the increase in the square of the Alfvén velocity i.e. $k_c = 6.2$, 5.9, 5.6, depicting thereby that horizontal magnetic field has critical strength to suppress the instability completely at very small values of normalized wavenumber and in this stage the system capitulates to the horizontal magnetic field effect.

**Figure 5:** Variation of $n_r$ (positive real part of $n$) with wavenumber $k$ for three values of square of Alfvén velocity $V_A^2 = 10$, 11, 12.
4.1.5 CONCLUSIONS

We thus conclude the whole analysis with the following results. The principal results drawn are as follows:

1. The criterion determining stability are found to be independent of the effects of suspended particles and viscoelastic parameters. However, the effects of kinematic viscosity, Stoke’s drag coefficient, stress-relaxation parameter, strain-retardation parameter and square of the Alfvén velocity on the growth rate of the most unstable mode has been investigated numerically.

2. It is found that the critical wavenumber $k_c$ of the most unstable growth rate remains unchanged with the increase in effects of kinematic viscosity, stress-relaxation parameter, strain-retardation parameter, Stokes’ drag coefficient and suspended particle number density on the stability of stratified viscoelastic Oldroyd-B fluid, however the growth rate decreases with the increase in these parameters for a fixed wavenumber, showing thereby the stabilizing effects.

3. The results indicate that the parameters play a major role in securing a complete stability at $k_c = 6.50$ and 6.25 for the system at hand

4. It is also found that the critical wavenumber $k_c$ goes on decreasing from 6.2 to 5.6 with the increase in square of the Alfvén velocity, depicting thereby the large enough stabilizing effect of square of the Alfvén velocity on the system due to the increase in dissipation.
NUMERICAL INVESTIGATIONS OF STABILITY OF STRATIFIED VISCOELASTIC OLDROYD-B FLUID IN THE PRESENCE OF SUSPENDED PARTICLES AND VARIABLE MAGNETIC FIELD SATURATING POROUS MEDIA

4.2.1 INTRODUCTION

The flow through porous medium has been of considerable interest in recent years particularly among geophysical fluid dynamicists. The gross effect, as the fluid slowly percolates through the pores of the rock, is that the usual viscous term in the equations of fluid motion are replaced by the resistance term $-\frac{\mu}{k_1}q$, where $\mu$ is the viscosity of fluid, $k_1$ the permeability of the medium and $q$ is the filter (seepage) velocity of fluid. Rayleigh-Taylor instability of viscoelastic fluids with suspended particles in porous medium has been investigated by Sharma and Rajput (1992). The Rayleigh Taylor instability of a plane interface between viscous and viscoelastic fluid through a porous medium has been investigated by Sharma and Kumar (1993). Sharma and Kumar (1994) have studied the Rayleigh Taylor instability of Oldroydian viscoelastic fluids in a porous medium in the presence of variable magnetic field. Agrawal and Geol (1998) has studied the instability of viscoelastic fluid in a porous medium and their analysis reveals a stabilizing character of viscosity and viscoelasticity and a destabilizing character of medium permeability and the shear velocity. Bhatia and Mathur (2003) have studied the instability of viscoelastic fluids in a horizontal magnetic field through porous medium and have found that the magnetic field stabilizes the unstable configuration for wavenumber band $k > k_c$ where $k_c$ is the critical wavenumber for which the system is unstable in the absence of magnetic field. It is also found that the viscosity, viscoelasticity and medium porosity have stabilizing influence while, elasticity and medium permeability have destabilizing influence.

Keeping in mind the importance of viscoelastic Oldroydian fluids in modern technology, industries and owing to the importance to the magnetic fields, porous medium in chemical engineering and geophysics, a numerical study has, therefore, been made of the stability of stratified viscoelastic Oldroyd-B fluid in the presence...
of suspended particles and variable magnetic field saturating porous media. These aspects form the subject matter of the present section.

4.2.2 FORMULATION OF THE PROBLEM AND PERTURBATION EQUATIONS

The initial stationary state whose stability we wish to examine is that of an incompressible heterogeneous infinitely conducting Oldroyd-B fluid of variable density and kinematic viscosity arranged in horizontal strata through a homogeneous porous medium permeated with suspended particles. This fluid-particle layer is acted on by gravity force $g(0,0,-g)$, a variable horizontal magnetic field $H(H_0(z),0,0)$, $z$-axis being taken as vertical. This layer is confined between the planes $z = 0$ and $z = d$.

The equations of motion, continuity, incompressibility and Maxwell's equations for the viscoelastic fluid-particle layer and equations of motions and continuity of the suspended particles relevant to the problem saturating porous media are

\[
\frac{\partial}{\partial t}\left(1 + \lambda \frac{\partial}{\partial t}\right)\left(\frac{\partial q}{\partial t} + (q \cdot \nabla)q\right) = \left(1 + \lambda \frac{\partial}{\partial t}\right)\left(-\nabla p + \rho g + \frac{\mu_e}{4\pi} (\nabla \times H) \times H + \frac{K^N}{\epsilon} (q_d - q)\right) - \left(1 + \lambda_0 \frac{\partial}{\partial t}\right)\frac{\mu}{\epsilon} q, \tag{4.2.1} \right.
\]

\[
\nabla \cdot q = 0, \tag{4.2.2} \]

\[
\nabla \cdot H = 0, \tag{4.2.3} \]

\[
\frac{\varepsilon}{\partial t} \frac{\partial H}{\partial t} = \nabla \times (q \times H), \tag{4.2.4} \]

\[
\frac{\varepsilon}{\partial t} \frac{\partial \rho}{\partial t} + (q \cdot \nabla)\rho = 0, \tag{4.2.5} \]

\[
m_0 \left(\frac{\partial q_d}{\partial t} + \frac{1}{\varepsilon} (q_d \cdot \nabla)q_d\right) = K^N(q - q_d). \tag{4.2.6} \]

\[
\frac{\varepsilon}{\partial t} \frac{\partial N}{\partial t} + \nabla \cdot (N \cdot q_d) = 0. \tag{4.2.7} \]

where $\rho$, $\mu$, $p$, $\mu_e$ and $q$ denote, respectively, the density, the viscosity, the pressure, the magnetic permeability and the filter velocity of the fluid (initially zero). $q_d(l,r,s)$ and $N(x,t)$ denote the velocity and number density of particles.
\[ K' = 6 \pi n \eta', \eta' \text{ being the particle radius, } K' \text{ is the Stoke's drag and } \bar{x} = (x, y, z) \text{ and } mN \text{ is the mass of particles per unit volume. Assuming uniform particle size, spherical shape and small relative velocities between the fluid and particles, the net effect of the particles on the fluid is equivalent to an extra body force per unit volume } K'N(q_d - q). \]

Since the equilibrium state under consideration is static and quiescent one, (i.e. there are no motions and no settling of particles) as follows:

\[
\begin{aligned}
q &= (0, 0, 0) \\
q_d &= (0, 0, 0) \\
\rho &= \rho(z) \\
p &= p(z) \\
H &= (H_0(z), 0, 0) \\
N &= N_0 (\text{a constant})
\end{aligned}
\]

The character of the equilibrium of this stationary state can be determined by disturbing the system slightly and then, following its further evolution.

Let \( \delta \rho, \delta p, q, q_d, N \) and \( h(h_x, h_y, h_z) \) denote, respectively, the perturbations in density \( \rho(z) \), pressure \( p(z) \), filter velocity \( q(0,0,0) \), particle velocity \( q_d(0,0,0) \), suspended particle number density and horizontal magnetic field \( H(H_0(z),0,0) \). Then using linear theory and the steady state solution (4.2.8), equations (4.2.1)-(4.2.7) in the linearized perturbation form are

\[
\frac{\rho}{\varepsilon} \left( 1 + \lambda \frac{\partial}{\partial t} \right) \left( \frac{\partial q}{\partial t} + (q \cdot \nabla) q \right) = \left( 1 + \lambda \frac{\partial}{\partial t} \right) \left( -\nabla p + \delta \rho g + \frac{\mu}{4 \pi} (\nabla \times H) \times H + K'N(q_d - q) \right) - \\
\frac{\mu}{k_1} \left( 1 + \lambda \rho \frac{\partial}{\partial t} \right) q.
\]

(4.2.9)

\[ \nabla \cdot q = 0, \] (4.2.10)

\[ \varepsilon \frac{\partial \rho}{\partial t} = -w \frac{dp}{dz}, \] (4.2.11)

\[ \nabla \cdot h = 0, \] (4.2.12)

\[ \varepsilon \frac{\partial h}{\partial t} = (H \cdot \nabla) u - (u \cdot \nabla) H, \] (4.2.13)

\[ \left( \frac{m}{K' \frac{\partial}{\partial t} + 1} \right) q_d = q. \] (4.2.14)
Operating equation (4.2.9) by \((m \frac{\partial}{K \partial t} + 1)\) to eliminate \(q_d\) between equations (4.2.10) and (4.2.14), we get
\[
\frac{1}{\varepsilon} \left(1 + \lambda \frac{\partial}{\partial t}\right) \left[\rho \frac{\partial}{\partial t} \left(\frac{m \frac{\partial}{\partial t}}{K \frac{\partial t}} + 1\right) + N m \frac{\partial}{\partial t}\right] q = \left(1 + \lambda \frac{\partial}{\partial t}\right) \left(\frac{m \frac{\partial}{\partial t}}{K \frac{\partial t}} + 1\right) \left[-\nabla \delta \rho + g \delta \rho + \frac{\mu_t}{4\pi} ((\nabla \times H) \times H - (\nabla \times H) \times h)\right]
\]
\[
- \mu \left(1 + \lambda \frac{\partial}{\partial t}\right) \left(\frac{m \frac{\partial}{\partial t}}{K \frac{\partial t}} + 1\right) q.
\]
(4.2.15)

Equations (4.2.9) - (4.2.13) in the Cartesian form are
\[
\frac{1}{\varepsilon} \left(1 + \lambda \frac{\partial}{\partial t}\right) \left[\rho \frac{\partial}{\partial t} \left(\frac{m \frac{\partial}{\partial t}}{K \frac{\partial t}} + 1\right) + N m \frac{\partial}{\partial t}\right] u = \left(1 + \lambda \frac{\partial}{\partial t}\right) \left(\frac{m \frac{\partial}{\partial t}}{K \frac{\partial t}} + 1\right) \left[-\frac{\partial}{\partial x} \delta \rho + \frac{\mu_t H_0}{4\pi} \left(\frac{\partial h_x}{\partial x} - \frac{\partial h_y}{\partial y}\right)\right] -
\]
\[
\frac{\mu}{k_1} \left(1 + \lambda \frac{\partial}{\partial t}\right) \left(\frac{m \frac{\partial}{\partial t}}{K \frac{\partial t}} + 1\right) u,
\]
(4.2.16)
\[
\frac{1}{\varepsilon} \left(1 + \lambda \frac{\partial}{\partial t}\right) \left[\rho \frac{\partial}{\partial t} \left(\frac{m \frac{\partial}{\partial t}}{K \frac{\partial t}} + 1\right) + N m \frac{\partial}{\partial t}\right] v = \left(1 + \lambda \frac{\partial}{\partial t}\right) \left(\frac{m \frac{\partial}{\partial t}}{K \frac{\partial t}} + 1\right) \left[-\frac{\partial}{\partial y} \delta \rho + \frac{\mu_t H_0}{4\pi} \left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right)\right]
\]
\[
- \frac{\mu}{k_1} \left(1 + \lambda \frac{\partial}{\partial t}\right) \left(\frac{m \frac{\partial}{\partial t}}{K \frac{\partial t}} + 1\right) v,
\]
(4.2.17)
\[
\frac{1}{\varepsilon} \left(1 + \lambda \frac{\partial}{\partial t}\right) \left[\rho \frac{\partial}{\partial t} \left(\frac{m \frac{\partial}{\partial t}}{K \frac{\partial t}} + 1\right) + N m \frac{\partial}{\partial t}\right] w = \left(1 + \lambda \frac{\partial}{\partial t}\right) \left(\frac{m \frac{\partial}{\partial t}}{K \frac{\partial t}} + 1\right) \left[-\frac{\partial}{\partial z} \delta \rho + g \delta \rho + \frac{\mu_t H_0}{4\pi} \left(\frac{\partial h_z}{\partial x} - \frac{\partial h_x}{\partial y}\right)\right]
\]
\[
- \frac{\partial h_x}{\partial z} - \frac{h_x}{H_0} \frac{\partial H_0}{\partial z} \right] - \frac{\mu}{k_1} \left(1 + \lambda \frac{\partial}{\partial t}\right) \left(\frac{m \frac{\partial}{\partial t}}{K \frac{\partial t}} + 1\right) w,
\]
(4.2.18)
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,
\]
(4.2.19)
\[
\varepsilon \frac{\partial}{\partial t} \delta \rho = -w \frac{\partial \rho}{\partial z},
\]
(4.2.20)
\[
\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z} = 0,
\]
(4.2.21)
\[
\varepsilon \frac{\partial h_x}{\partial t} = H_0 \frac{\partial u}{\partial x} - w \frac{\partial H_0}{\partial z},
\]
(4.2.22)
\[
\frac{\partial h_y}{\partial t} = H_0 \frac{\partial v}{\partial x},
\]
(4.2.23)
\[
\frac{\partial h_z}{\partial t} = H_0 \frac{\partial w}{\partial x}.
\]
(4.2.24)

Now ascribing all the quantities describing the perturbations dependence on \(x,y\) and \(t\) of the form
\[ f(z) \exp(ik_x x + ik_y y + nt). \]  \hfill (4.2.25)

Equations (4.2.16)-(4.2.24), using the expression (4.2.25), become

\[
\frac{1}{\varepsilon} (1 + \lambda n) \left[ \rho n \left( \frac{mn}{K} + 1 \right) + Nmn \right] u = (1 + \lambda n) \left( \frac{mn}{K} + 1 \right) \left[-ik_x \delta \rho + \frac{\mu_\varepsilon}{4\pi} (h_z DH_0) \right] - \mu \left(1 + \lambda_0 n \right) \left( \frac{mn}{K} + 1 \right) (D^2 - k^2) u, \tag{4.2.26}
\]

\[
\frac{1}{\varepsilon} (1 + \lambda n) \left[ \rho n \left( \frac{mn}{K} + 1 \right) + Nmn \right] v = (1 + \lambda n) \left( \frac{mn}{K} + 1 \right) \left[-ik_y \delta \rho + \frac{\mu_\varepsilon}{4\pi} (ik_z h_y - ik_y h_x) \right] - \mu \left(1 + \lambda_0 n \right) \left( \frac{mn}{K} + 1 \right) (D^2 - k^2) v, \tag{4.2.27}
\]

\[
\frac{1}{\varepsilon} (1 + \lambda n) \left[ \rho n \left( \frac{mn}{K} + 1 \right) + Nmn \right] w = (1 + \lambda n) \left( \frac{mn}{K} + 1 \right) \left[-D \delta \rho - g \delta \rho + \frac{\mu_\varepsilon H_0}{4\pi} (ik_x h_z - Dh_x - \frac{h_x}{H_0} DH_0) \right] - \frac{\mu}{k_1} \left(1 + \lambda_0 n \right) \left( \frac{mn}{K} + 1 \right) (D^2 - k^2) w, \tag{4.2.28}
\]

\[ ik_x u + ik_y v + Dw = 0, \]  \hfill (4.2.29)

\[ \delta \rho = -w \frac{\partial \rho}{\partial n}, \]  \hfill (4.2.30)

\[ ik_x h_x + ik_y h_y + Dh_z = 0, \]  \hfill (4.2.31)

\[ h_x = \frac{1}{\varepsilon n} (H_0 ik_x u - wD H_0). \]  \hfill (4.2.32)

\[ h_y = \frac{1}{\varepsilon n}(H_0 ik_x v), \]  \hfill (4.2.33)

\[ h_z = \frac{1}{\varepsilon n}(H_0 ik_x w). \]  \hfill (4.2.34)

where \( D \) stands for \( \frac{d}{dz} \).

Now substituting the values of \( h_x, h_y, h_z \) from equations (4.2.32) - (4.2.34) in equations (4.2.26)-(4.2.28), we get

\[
(1 + \lambda n) \left[ \rho n \left( \frac{mn}{K} + 1 \right) + \frac{1}{\varepsilon} Nmn \right] u = (1 + \lambda n) \left( \frac{mn}{K} + 1 \right) \left[-ik_x \delta \rho + \frac{\mu_\varepsilon}{4\pi n} H_0 DH_0 ik_x w \right] - \frac{\mu}{k_1} \left(1 + \lambda_0 n \right) \left( \frac{mn}{K} + 1 \right) u, \tag{4.2.35}
\]

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(1 + \lambda n) \left[ \rho_n \left( \frac{mn}{\kappa'} + 1 \right) + \frac{1}{\epsilon} Nmn \right] v = (1 + \lambda n) \left( \frac{mn}{\kappa'} + 1 \right) \left[ -ik_y \delta p + \frac{\mu\varepsilon H_0}{4\pi n} \{ik_x (ik_x H_0 v) - ik_y (ik_x H_0 u - w D H_0) \} \right] - \frac{\mu}{k_1} (1 + \lambda_0 n) \left( \frac{mn}{\kappa'} + 1 \right) v. \quad (4.2.36)

(1 + \lambda n) \left[ \rho_n \left( \frac{mn}{\kappa'} + 1 \right) + \frac{1}{\epsilon} Nmn \right] w = (1 + \lambda n) \left( \frac{mn}{\kappa'} + 1 \right) \left[ -D \delta p - g \delta \rho + \frac{\mu\varepsilon H_0}{4\pi n} \{ik_x (H_0 ik_x w) - D (H_0 ik_x u - w D H_0) \} \right] - \frac{\mu}{k_1} (1 + \lambda_0 n) \left( \frac{mn}{\kappa'} + 1 \right) w. \quad (4.2.37)

Multiplying equations (35) and (36) by \(-ik_y\) and \(ik_x\), respectively and then adding the resulting equations

\[ \zeta = 0. \quad (4.2.38) \]

where, \( \zeta = ik_x v - ik_y u \) is the z-component of vorticity

So, \( ik_x v - ik_y u = 0 \).

Substituting the value of \( \zeta \) in equation (4.2.36), it reduces to

\[
(1 + \lambda n) \left[ \rho_n \left( \frac{mn}{\kappa'} + 1 \right) + Nmn \right] v = (1 + \lambda n) \varepsilon \left( \frac{mn}{\kappa'} + 1 \right) \left[ -ik_y \delta p + \frac{\mu\varepsilon H_0}{4\pi n} ik_x w D H_0 \right] - (1 + \lambda_0 n) \varepsilon \left( \frac{mn}{\kappa'} + 1 \right) \frac{\mu}{k_1} v. \quad (4.2.40)
\]

On solving equations (4.2.29) and (4.2.39), we get

\[ u = \frac{ik_x D w}{k^2}. \quad (4.2.41) \]

Multiplying equations (4.2.35) and (4.2.40) by \(-ik_x\) and \(-ik_y\), respectively and then adding us, we get

\[
(1 + \lambda n) \left( \rho n \left( \frac{mn}{\kappa'} + 1 \right) + Nmn \right) D w = (1 + \lambda n) \varepsilon \left( \frac{mn}{\kappa'} + 1 \right) \left( -k^2 \delta p + \frac{\mu k^2}{4\pi n} H_0 (D H_0) w \right) - (1 + \lambda_0 n) \varepsilon \left( \frac{mn}{\kappa'} + 1 \right) \frac{\mu}{k_1} w. \quad (4.2.42)
\]

Eliminating \( \delta p \) from equations (4.2.42) and (4.2.37) and using (4.2.41), we get
\( (1 + \lambda_0 n) \left( \frac{mn}{\kappa'} + 1 \right) \mu_n \mu_n' (D^2 - k^2) w + (1 + \lambda n) (D^2 - k^2) \left[ \frac{\mu_k e_k^2 H_0^2}{4\pi \rho} \left( \frac{mn}{\kappa'} + 1 \right) + n^2 \left( \frac{mn}{\kappa'} + 1 + Nm/\rho \right) \right] w + (1 + \lambda n) \left( \frac{mn}{\kappa'} + 1 \right) g k^2 \frac{Dp}{\rho} w + \left[ n^2 (1 + \lambda n) \left( \frac{mn}{\kappa'} + 1 \right) \frac{Dp}{\rho} \right] Dw + \left[ (1 + \lambda_0 n) (D\mu_0) \frac{en}{\rho k_1} + (1 + \lambda n) \left( \frac{mn}{\kappa'} + 1 \right) \frac{\mu_k e_k^2}{4\pi \rho} D(H_0)^2 \right] = 0, \quad (4.2.43) \)

where \( V_A^2 = \frac{\mu_k e_k^2 H_0^2}{4\pi \rho} \) is the square of the Alfvén velocity.

Second last term in equation (4.2.43), represents the effect of the heterogeneity of the fluid on the inertia is neglected in comparison with the effect on potential energy.

Thus equation (4.2.43) reduces to

\[
(1 + \lambda_0 n) \left( \frac{mn}{\kappa'} + 1 \right) \frac{\mu_n}{\rho k_1} (D^2 - k^2) w + (1 + \lambda n) (D^2 - k^2) \left[ V_A^2 \frac{e_k^2}{(\kappa')^2} + n^2 \left( \frac{mn}{\kappa'} + 1 + Nm/\rho \right) \right] w + (1 + \lambda n) \left( \frac{mn}{\kappa'} + 1 \right) g k^2 \frac{Dp}{\rho} w + \left( \frac{mn}{\kappa'} + 1 \right) \left[ (1 + \lambda_0 n) (D\mu) \frac{en}{\rho k_1} + (1 + \lambda n) \frac{\mu_k e_k^2}{4\pi \rho} D(H_0)^2 \right] = 0. \quad (4.2.44)\]

Equation (4.2.44) constitutes a characteristic equation, the solutions of which need to be sought satisfying certain boundary conditions.

### 4.2.3 THE CASE OF EXPONENTIALLY VARYING STRATIFICATIONS

In order to obtain the solutions of the stability problem under consideration, we assume that the stratifications in suspended particle number density, viscosity, medium porosity, density, medium permeability and magnetic field exponentially along the vertical of the form

\[
N = N_0 e^{\beta_1 z}, \quad \mu = \mu_0 e^{\beta_1 z}, \quad \epsilon = \epsilon_0 e^{\beta_1 z}, \quad \rho = \rho_0 e^{\beta_1 z}, \quad k_1 = k_{10} e^{\beta_1 z}, \quad H_0^2 = H_1^2 e^{\beta_1 z}, \quad (4.2.45)\]

where \( \rho_0, \mu_0, N_0, H_1, H_0, k_1, k_{10} \) and \( \beta_1 \) are constants and so the kinematic viscosity, \( \nu_0 = \left( \frac{\mu}{\rho} \right) = \left( \frac{\mu_0}{\rho_0} \right) \), the kinematic viscoelasticity \( \nu_0' = \left( \frac{\mu}{\rho} \right) = \left( \frac{\mu_0}{\rho_0} \right) \) and the Alfvén velocity, \( V_A = \left( \frac{\mu_k e_k^2 H_0^2}{4\pi \rho_0} \right)^{1/2} = \left( \frac{\mu_k e_k^2 H_1^2}{4\pi \rho_0} \right)^{1/2} \) are constant from layer to layer.
Inspite the nature of the bounding surfaces, the boundary conditions appropriate to the problem are

\[ w = 0 \text{ at } z = 0 \text{ and } z = d. \]  

(4.2.46)

Using stratifications from (4.2.45) in equation (4.2.44) becomes

\[
[(1 + \lambda_0 n) \left( \frac{m_n}{k} + 1 \right) \frac{\varepsilon_{en}}{k_1} + (1 + \lambda n) \left\{ V_A^2 k_x^2 \left( \frac{m_n}{k} + 1 \right) + n^2 \left( \frac{m_n}{k} + 1 + \frac{N m}{\rho_0} \right) \right\}] D^2 w + \\
\left[ (1 + \lambda_0 n) \beta_1 \frac{\varepsilon_{en}}{k_1} + (1 + \lambda n) V_A^2 k_x^2 \beta_1 \right] \left( \frac{m_n}{k} + 1 \right) D w - \left[ (1 + \lambda_0 n) \left( \frac{m_n}{k} + 1 \right) \frac{\varepsilon_{en}}{k_1} + \\
(1 + \lambda n) \left\{ V_A^2 k_x^2 - (1 + \lambda n) g \beta_1 \right\} \left( \frac{m_n}{k} + 1 \right) + n^2 \left( \frac{m_n}{k} + 1 + \frac{N m}{\rho_0} \right) \right] k^2 w. 
\]  

(4.2.47)

The general solution of equation (47) is given by

\[ w = A_1 e^{q_1 z} + A_2 e^{q_2 z}, \]  

(4.2.48)

where \( A_1, A_2 \) are arbitrary constants and \( q_1, q_2 \) are the roots of the equation given below

\[
\left[ (1 + \lambda_0 n) \left( \frac{m_n}{k} + 1 \right) \frac{\varepsilon_{en}}{k_1} + (1 + \lambda n) \left\{ V_A^2 k_x^2 \left( \frac{m_n}{k} + 1 \right) + n^2 \left( \frac{m_n}{k} + 1 + \frac{N m}{\rho_0} \right) \right\} \right] q^2 + \\
\left[ (1 + \lambda_0 n) \beta_1 \frac{\varepsilon_{en}}{k_1} + (1 + \lambda n) V_A^2 k_x^2 \beta_1 \right] \left( \frac{m_n}{k} + 1 \right) q - \left[ (1 + \lambda_0 n) \left( \frac{m_n}{k} + 1 \right) \frac{\varepsilon_{en}}{k_1} + (1 + \lambda n) \left\{ V_A^2 k_x^2 \left( \frac{m_n}{k} + 1 \right) - (1 + \lambda n) \left( \frac{m_n}{k} + 1 \right) g \beta_1 + n^2 \left( \frac{m_n}{k} + 1 + \frac{N m}{\rho_0} \right) \right\} \right] k^2 . 
\]  

(4.2.49)

Using both the boundary conditions given by (4.2.46) in (4.2.48), we get

\[ A = -B \] which yields that

\[ w = A(e^{q_1 z} - e^{q_2 z}). \]  

(4.2.50)

\[ w = 0 \text{ at } z = d \] implies that

\[ \exp(q_1 - q_2) d = 1, \text{ or } (q_1 - q_2) d = 2 im_1 \pi, \]  

(4.2.51)

where \( m_1 \) is an integer.

The roots of quadratic equation (4.2.49) are given by:
Now, substituting the values of $q_1$ and $q_2$ from (4.2.52) in equation (4.2.51) and after a little algebra, we get

$$A_8 n^8 + A_7 n^7 + A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0 = 0,$$  
(4.2.53)

where,

$$A_8 = 4 \lambda^2 m^2 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right),$$

$$A_7 = 8m \lambda^2 K \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) + 8N \lambda^2 m^2 K \rho_0 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) + 8 \lambda m^2 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) + \frac{8 \nu_0 \lambda \epsilon \lambda m^2}{k_1} \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right),$$

$$A_6 = \frac{4 \nu_0^2 \lambda_0 \rho_0^2 m^2}{k_1^2} \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) + 4 m^2 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) + 4 \lambda^2 K^2 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) +$$

$$4N^2 \lambda^2 m^2 K \rho_0^2 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) + 8N \lambda^2 m^2 \rho_0 K \frac{m_1^2 \pi^2}{d^2} \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) + 16 \lambda m^2 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) +$$

$$16N \lambda m^2 K \rho_0 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) + \frac{8 \nu_0 \lambda \epsilon \lambda m^2}{k_1} \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) + \frac{8 \nu_0 \lambda \epsilon \lambda m^2}{k_1} \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) +$$

$$4 \lambda^2 \nu_0^2 \lambda \epsilon \lambda m^2 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) + \frac{16 \nu_0 \lambda \epsilon \lambda K \rho_0}{k_1} \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) + \frac{16 \nu_0 \lambda \epsilon \lambda K \rho_0}{k_1} \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) -$$

$$4 \lambda^2 m^2 g \lambda \beta_1.$$
\[
A_4 = \frac{4\varepsilon v_0 m^2}{k_1^2} (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 4V_A^4 k_x^4 \lambda^2 m^2 (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \frac{8\varepsilon v_0 V_A^4 k_x^2 \lambda m^2}{k_1} (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \\
8\varepsilon v_0 V_A^2 k_x^2 \lambda m^2 (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \frac{4\varepsilon^2 v_0^2 \beta_1^2 \lambda_0 m^2}{k_1^2} (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 16\varepsilon v_0^2 \lambda_0^2 K m (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \\
16\varepsilon v_0 V_A^2 k_x^2 \lambda_0 K m (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 4K^2 (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 4N^2 m^2 \rho_0 K' (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \\
8N m \rho_0 K'^2 (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 8V_A^2 k_x^2 m^2 (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 8K v_0 e m (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \\
4\lambda V_A^2 k_x^2 m K' (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 16V_A^2 k_x^2 \lambda K' m (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 8\varepsilon v_0 N m^2 \rho_0 K (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \\
8\varepsilon v_0 V_A^2 k_x^2 \lambda m m^2 (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 8\varepsilon v_0 \lambda K N m \rho_0 K (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \\
\frac{4\varepsilon v_0 k_x^2 \beta_1^2 \lambda_0 m^2}{k_1} (k^2 + \frac{m_1^2 \pi^2}{d^2}) - 4N^2 \rho_0 k^2 g_\beta_1 \lambda^2 - 4\varepsilon v_0 k^2 \beta_1 \lambda_0 K' \lambda m - 4k^2 g_\beta_1 \lambda m K' - \\
4K^2 \lambda^2 g_\beta_1 \beta_1,
\]

\[
A_3 = 8V_A^4 k_x^4 \lambda m^2 (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \frac{8\varepsilon v_0 V_A^2 k_x^2 m^2}{k_1} (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \frac{8\varepsilon v_0^2 \lambda K}{k_1^2} (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \\
8\varepsilon v_0 V_A^2 k_x^2 \lambda K^2 (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 8\varepsilon v_0 \lambda m (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 8V_A^4 k_x^4 \lambda^2 K' m (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \\
16\varepsilon v_0 V_A^2 k_x^2 \lambda K m (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 16\varepsilon v_0 V_A^2 k_x^2 \lambda_0 K' m (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 8V_A^2 k_x^2 \lambda K m (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \\
8V_A^2 k_x^2 \lambda K m (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 8\varepsilon v_0 V_A^2 k_x^2 \lambda K m (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 12V_A^2 k_x^2 \lambda K^2 (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \\
8\varepsilon v_0 N m \rho_0 K^2 (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 4\lambda \rho_0 V_A^2 k_x^2 \lambda K^2 m (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 8\varepsilon v_0 N m \rho_0 K^2 (k^2 + \frac{m_1^2 \pi^2}{d^2}) + \\
4\lambda^2 \rho_0 V_A^2 k_x^2 \lambda K^2 m (k^2 + \frac{m_1^2 \pi^2}{d^2}) + 2V_A^4 k_x^4 \lambda m^2 \beta_1^2 + \frac{2\varepsilon v_0 V_A^2 k_x^2 \beta_1^2}{k_1^2} + \frac{2\varepsilon v_0 \lambda K \beta_1^2}{k_1^2} +
\]
Equation (4.2.53) is of degree eight in $n$ and is the dispersion relation governing the effects of variable horizontal magnetic field, kinematic viscosity, stress relaxation time, strain-retardation time, suspended particle number density, medium permeability and medium porosity on the stability of stratified Oldroyd-B fluid in porous medium.

**(a) Case of stable stratification (i.e. $\beta_1 < 0$)**

If $\beta_1 < 0$ (stable stratification), equation (4.2.53) does not admit of any positive real root or complex root with positive real part, using Routh-Hurwitz criterion; therefore, the system is always stable for disturbances of all wavenumbers.
(b) **Case of unstable stratification** (i.e. $\beta_1 > 0$)

If $\beta_1 > 0$ (unstable stratification), equation (4.2.53) has at least one root with positive real part or complex root with positive real part, using Routh-Hurwitz criterion; so the system is stable or unstable according as

$$V_A^2 k_x^2 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) > g k^2 \beta_1 \quad \text{or} \quad V_A^2 k_x^2 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right) < g k^2 \beta_1.$$ 

The system is clearly unstable in the absence of magnetic field and for the case of non-viscoelastic fluid. However the system can be completely stabilized by large enough magnetic field if

$$V_A^2 > \frac{g k^2 \beta_1}{k_x^2 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right)}.$$

Thus if $\beta_1 > 0$, $V_A^2 < \frac{g k^2 \beta_1}{k_x^2 \left( k^2 + \frac{m_1^2 \pi^2}{d^2} \right)}$, equation (4.2.53) has at least one positive root and so the system is unstable for all wavenumbers satisfying the inequality

$$k^2 < \frac{g \beta_1 \sec^2 \theta}{V_A^2} - \frac{m^2 \pi^2}{d^2}, \quad \text{where} \ \theta \ \text{is the angle between} \ k_x \ \text{and} \ k \ (i.e. k_x = k \cos \theta).$$

Equation (4.2.53) does not admit of any positive real root or complex root with positive real part, therefore, the system is stable. However equation (4.2.53) is not very useful for determining the critical wavenumber $k_c$ for the assigned values of the parameters $\beta_1$, $m$, $d$, $g$, $\lambda_0$, $\lambda$, $\varepsilon$, $\nu_0$, $V_A^2$, $k_x = k \cos 45^0$. It is more convenient to evaluate $\eta$ directly as a function of $k$ (in accordance with equation (4.2.53) and locate the minimum numerically).

### 4.2.4 NUMERICAL RESULTS AND DISCUSSION

The dispersion relation (4.2.53) has been computed numerically for the most unstable mode of perturbation for different values of the physical parameters $\lambda$, $\lambda_0$, $N_0$, $\varepsilon$, $k_1$ and square of Alfvén velocity $V_A^2$, using the Software Mathematica version 5.2.
Figure 1 shows the Variation of growth rate $n_r$ (positive real part of $n$) with respect to the wavenumber $k$ satisfying equation (4.2.53) for fixed permissible values of $\beta_1 = 0.2, \lambda_0 = 0.3, m_1 = 1, \lambda = 0.8, d = 4, g = 980 \text{ cm/sec}^2, \nu_0 = 4, V_A^2 = 10, \ k_x = k/\sqrt{2}, \ K' = 0.001, \ \rho_0 = 1.02, \ m = 0.0001, \ \varepsilon = 0.9, \ k_1 = 15$ for three value of suspended particles density $N_0 = 500000, 700000, 900000$, respectively. The graph shows that there is a mode of maximum instability where the growth rate $n_r$ increases with $k$ increasing through the range $0 < k < k_{\text{max}}$ (at $k_{\text{max}}$; growth rates arrives to the maximum instability), whereas the growth rate $n_r$ starts to decrease as $k$ increases and then goes to the complete stable at $k_c = 18$ ($k_c$ is the critical value for stability, at this point $n_r$ goes to zero). This means than the suspended particles number density has a critical capability to suppress the instability.

![Figure 1: Variation of $n_r$ (positive real part of $n$) with wavenumber $k$ for three values of $N_0 = 500000, 700000, 900000$.](image-url)
Figure 2 shows the variation of growth rate $n_r$ (positive real part of $n$) with respect to the wavenumber $k$ satisfying equation (4.2.53) for fixed permissible values of $\beta_1 = 0.2$, $\varepsilon = 0.9$, $\lambda_0 = 0.3$, $m_1 = 1$, $\lambda = 0.8$, $d = 4$, $g = 980 \text{ cm/s sec}^2$, $\nu_0 = 4$, $V_A^2 = 10$, $k_x = k/\sqrt{2}$, $N_0 = 500000$, $\rho_0 = 1.02$, $m = 0.0001$, $K' = 0.001$ for three different values of the kinematic permeability $k_1 = 4, 6, 8$ respectively. The graph shows that there is a mode of maximum instability where the growth rate $n_r$ increases with $k$ increasing through the range $0 < k < k_{max}$ (at $k_{max}$, growth rates arrives to the maximum instability), whereas the growth rate $n_r$ starts to decrease as $k$ increases and then goes to the complete stable at $k_c = 18$ ($k_c$ is the critical value for stability, at this point $n_r$ goes to zero). This means that the permeability has a critical capability to suppress the instability. However, the growth rate of the most unstable mode remains uninfluenced with the increases in medium permeability.

**Figure 2**: Variation of $n_r$ (positive real part of $n$) with wavenumber $k$ for three values of $k_1 = 4, 6, 8$. 
Figure 3 shows the Variation of growth rate $n_r$ (positive real part of $n$) with respect to the wavenumber $k$ satisfying equation (4.2.53) for fixed permissible values of $\beta = 0.2, \lambda_0 = 0.3, m_1 = 1, N_0 = 500000, d = 4, g = 980 \text{ cm/sec}^2, \nu_0 = 4, V_A^2 = 10, k_x = k' / \sqrt{2}, K' = 0.001, \rho_0 = 1.02, m = 0.0001, \lambda = 0.8, k_1 = 15$ for three value of number of molecules $\varepsilon = 0.5, 0.7, 0.9$ respectively. The graph shows that there is a mode of maximum instability where the growth rate $n_r$ increases with $k$ increasing through the range $0 < k < k_{max}$ (at $k_{max}$; growth rates arrives to the maximum instability), whereas the growth rate $n_r$ starts to decrease as $k$ increases and then goes to the complete stable at $k_c = 18$ ($k_c$ is the critical value for stability, at this point $n_r$ goes to zero). This means than the kinematic viscosity has a critical capability to suppress the instability.

**Figure 3:** Variation of $n_r$ (positive real part of $n$) with wavenumber $k$ for three values of $\varepsilon = 0.5, 0.7, 0.9$. 

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Figure 4 shows the Variation of growth rate $n_r$ (positive real part of $n$) with respect to the wavenumber $k$ satisfying equation (4.2.53) for fixed permissible values of $\beta_1 = 0.2$, $N_0 = 500000$, $m_1 = 1$, $\lambda_0 = 0.3$, $d = 4$, $g = 980 \text{ cm/sec}^2$, $v_0 = 4$, $V_A^2 = 10$, $k_x = k/\sqrt{2}$, $K' = 0.001$, $\rho_0 = 1.02$, $m = 0.0001$, $k_1 = 15$, $\varepsilon = 0.9$ for three values of stress-relaxation time $\lambda = 0.9$, 1.2, 1.5 respectively. The graph shows that there is a mode of maximum instability where the growth rate $n_r$ increases with $k$ increasing through the range $0 < k < k_{\text{max}}$ (at $k_{\text{max}} = 10$; growth rate arrives to the maximum instability), whereas the growth rate $n_r$ starts to decrease as $k$ increases and then goes to the complete stable at $k_c = 18$ ($k_c$ is the critical value for stability, at this point $n_r$ goes to zero). However, the growth rates slightly increases with the increase in the stress-relaxation time for a fixed wavenumber.

**Figure 4**: Variation of $n_r$ (positive real part of $n$) with wavenumber $k$ for three values of $\lambda = 0.9$, 1.2, 1.5.
Figure 5 shows the variation of growth rate $n_r$ (positive real part of $n$) with respect to the wavenumber $k$ satisfying equation (4.2.50) for fixed permissible values of $\beta_1 = 0.2$, $\varepsilon = 0.9$, $m_1 = 1$, $\lambda = 0.8$, $d = 4$, $g = 980 \text{ cm/sec}^2$, $v_0 = 4$, $N_0 = 500000$, $k_x = k/\sqrt{2}$, $K' = 0.001$, $\rho_0 = 1.02$, $m = 0.0001$, $V_A^2 = 10$, $k_1 = 15$ for three values of strain-retardation time $\lambda_0 = 0.3$, 0.4, 0.5 respectively. The graph shows that there is a mode of maximum instability where the growth rate $n_r$ increases with $k$ increasing through the range $0 < k < k_{\text{max}}$ (at $k_{\text{max}} = 10$; growth rates arrives to the maximum instability), whereas the growth rate $n_r$ starts to decrease as $k$ increases and then goes to the complete stable at $k_c = 18$ ($k_c$ is the critical value for stability, at this point $n_r$ goes to zero). However, the growth rate of the most unstable mode of perturbation is not influenced by the increase in $\lambda_0$.

![Figure 5: Variation of $n_r$ (positive real part of $n$) with wavenumber $k$ for three values of $\lambda_0 = 0.3$, 0.4, 0.5.](image)
4.2.5 CONCLUSIONS

The principle conclusions drawn are as follows:

(i) The system is stable for stable configuration and unstable for unstable configuration.

(ii) For the most unstable mode, the growth rate remains uninfluenced by the presence of medium permeability and very slight effect of the medium porosity.

(iii) The growth rates of the most unstable mode decrease with the increase in the suspended particles number density implying thereby the stabilizing effect of suspended particles.

(iv) The growth rate of the most unstable mode decreases slightly with the increase in stress-relaxation time implying thereby its stabilizing effect on the system.

(v) The growth rate of the most unstable mode is uninfluenced by the increase in the strain-retardation time.

However, the critical wavenumber remains the same i.e. $k_c = 18$ with the increase in the medium permeability and the suspended particles, number density, the stress-relaxation time and $k_c = 18$ with the increase in strain-retardation time.