CHAPTER-2

Classification of liquids, fundamental equations and boundary conditions

2.1 Introduction

This chapter provides the classification of fluids, Basic laws, concepts and review of non-Newtonian fluids. Beside that some definition required for subsequent chapters and magnetic field, inclined channel also included. The chapter also discusses the relevant basic equations related to the dynamics of Newtonian and fractional second grade model due to peristaltic transport. The various velocity and temperature boundary conditions made use in the problems are furnished. The dimensionless parameters arising in the problems under investigation are also put in a general manner.

2.1.1 Classification of fluid

Based on the application of stress in the liquid and its response to the same, we can classify fluids as Newtonian and non-Newtonian types.

2.1.2 Newtonian Fluids

According to Newton’s law viscosity is the constant of proportionality between the shear stress and the velocity gradient, as described by

\[
\tau = \mu \dot{\gamma}
\]

(2.1)

where the \( \tau \) is the shear stress exerted by fluid (“drag”) \( \mu \) is the fluid velocity, \( \dot{\gamma} \) is the velocity gradient perpendicular to the direction of shear. It is well known that
Newtonian fluids possess a property called viscosity and obey a relation through Newton’s law between a stress and strain. Fluids such as water, air, ethanol, and benzene are Newtonian. This means that a plot of shear stress versus shear rate at a given temperature is a straight line with a constant slope that is independent of the shear rate. We call this slope the viscosity of the fluid. Also, the plot passes through the origin, that is, the shear rate is zero when the shear stress is zero. All low molecular weight liquids, and solutions of low molecular weight substances in liquids are usually Newtonian. Some examples are aqueous solutions of sugar or salt.

2.1.3 Non-Newtonian Fluids

Any fluid that does not obey the Newtonian relationship between the shear stress and shear rate is called non-Newtonian. The subject of “Rheology” is devoted to the study of the behavior of such fluids. High molecular weight liquids, which include polymer melts and solutions of polymers, as well as liquids in which fine particles are suspended (slurries and pastes), are usually non-Newtonian. In this case, the slope of the shear stress versus shear rate curve will not be constant as we change the shear rate.

2.1.4 Classification of non-Newtonian fluid

According to the relationships between shear stress and shear rate, non-Newtonian fluids are commonly grouped in three general classes (1) time independent non-Newtonian fluids; (2) time-dependent non-Newtonian fluids; and (3) viscoelastic non-Newtonian fluids.
1. Time-independent fluids are those for which the rate of shear $\dot{\gamma}$, or the velocity gradient, is a unique but non-linear function of the instantaneous shear stress $\tau$ at that point. For the time-independent fluid, the relationship is

$$\dot{\gamma} = f(\tau)$$  

(2.2)

The time-independent non-Newtonian fluids can be characterized by the flow curves of $z$ versus $y$, as shown in Figure 2.1. These are: (a) Bingham plastics curve A; (b) pseudo plastic fluids (shear thinning), curve B; and (c) dilatant fluids (shear thickening), curve C.

2. Time-dependent fluids have more complex shear stress and shear rate relationships. In these fluids, the shear rate depends not only on the shear stress, but also on shearing time, or on the previous shear stress rate history of the fluid. These materials are usually classified into two groups, thixotropic fluids and rheopectic fluids, depending upon whether the shear stress decreases or increases in time at a given shear rate and under constant temperature.

3. A viscoelastic material exhibits both elastic and viscous properties, and shows partial recovery upon the removal of the deformable shear stress. The rheological properties of such a substance at any instant will be a function of the recent history of the material and can not be described by relationships between shear stress and shear rate alone, but will require inclusion of the time derivative of both quantities.
2.2 Basic equations

The investigation of any fluid motion involves solving a set of non-linear partial differential equations called the fundamental equations of fluid dynamics. The fundamental equations governing any flow phenomena are stated below:

2.2.1. Continuity equation:

In fluid dynamics, the continuity equation is a mathematical statement that, in any steady state process, the rate at which mass enters a system is equal to the rate at which mass leaves the system.

The differential form of the continuity equation is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$  \hspace{1cm} (2.3)

Here, $\rho$ is the density and $\mathbf{u}$ is the velocity of the fluid.
2.2.2. Equation of motion:

The total force acting on a fluid mass enclosed in an arbitrary volume fixed in space is equal to the time rate of change of linear momentum (Law of conservation of momentum). The general form of Nervier –Stokes equation is

\[ \rho \frac{Du}{Dt} = \nabla P = \rho f \]  \hspace{1cm} (2.4)

where \( u \) is the velocity, \( \rho \) is density of fluid, \( \frac{D}{Dt} \) is substantive derivative, \( f \) is the body force vector, and \( p \) is a tensor that represents the surface forces applied on a fluid particle.

2.2.3. Equation of energy

Physical principle: The energy added to a closed system increases the internal energy per unit mass of the fluid (Law of conservation of energy). In general energy equation can be defined as

\[ \rho \frac{de}{dt} = \tau \cdot L \cdot \text{div} \bar{Q} + \rho r_i \]  \hspace{1cm} (2.5)

Here \( e = \rho C_p \) is the specific internal energy, \( r_i \) is the radiant heating. According to Fourier law \( \tilde{Q} = -k \text{grad} \tilde{T} \), where \( k \) is the constant of thermal conductivity and \( \tilde{T} \) is the temperature.
2.2.4. Caputo’s definition

Caputo’s definition (2011) of the fractional order derivative is defined as

\[
D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_b^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+n}} d\tau \quad (n-1 < \text{Re}(\alpha) \leq n, \, n \in \mathbb{N}),
\]  

(2.6)

Where, \(\alpha\) is the order of derivative and is allowed to be real or even complex, \(b\) is the initial value of function \(f\). For the Caputo’s derivative we have

\[
D^\alpha t^\beta = \begin{cases} 
0 & (\beta \leq \alpha - 1) \\
\Gamma(\beta + 1) t^{\beta - \alpha} & (\beta > \alpha - 1)
\end{cases}
\]  

(2.7)

2.2.5. Fractional second grade model

The constitutive equation for viscoelastic fluid with fractional second grade model is given by

\[
s = \mu \left( 1 + \lambda_1^a \frac{\partial^\alpha}{\partial t^\alpha} \right) \dot{\gamma}
\]  

(2.8)

Where \(t, s, \dot{\gamma}\) and \(\lambda_1\) is the time, shear stress, rate of shear strain and material constants respectively, \(\mu\) is viscosity, and \(\alpha\) is the fractional time derivative parameters such that \(0 < \alpha \leq 1\). This model reduces to second grade model with \(\alpha = 1\), and Classical Navier Stokes model is obtained by substituting \(\lambda_1 = 0\).
2.4 Flow in porous medium

A porous medium is a matter which contains a number of small holes distributed throughout the matter. Flows through porous medium occur in filtration of fluids and seepage of water in river beds. Movement of underground, water and oils are some important examples of flows through porous medium. An oil reservoir mostly contains of sedimentary formation such as limestone and sandstone in which oil is entrapped. Another example of flow through porous medium is the seepage under a dam which is very important. There are examples of natural porous medium such as beach sand, rye bread, wood, filter, loaf of bread, human lung, gall bladder and bile with stones, in petroleum production engineering and in many other processes as well [Fig.(2.1)].

Darcy’s law:

Henri Darcy’s experiments on steady-state unidirectional flow in a uniform medium revealed proportionality between flow rate and the applied pressure difference. In modern notation this is expressed by

\[ u = -\frac{k_0}{\mu} \frac{\partial p}{\partial x} \]  

(2.9)

Here \( \frac{\partial p}{\partial x} \) is the pressure gradient in the flow direction and is dynamic viscosity of fluid. The coefficient \( k_0 \) is independent of the nature of the fluid but it depends on the geometry of the medium. It has dimensions (length) and is called the specific permeability or intrinsic permeability of the medium. In three dimensions, equation generalizes to:
\[ q = -\frac{1}{\mu} k \nabla p \]  \hspace{1cm} (2.10)

a) Extension of Darcy’s law

Many early authors on convection in porous media an extension of equation of the form:

\[ \rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = -\nabla p - \frac{\mu}{k} u \]  \hspace{1cm} (2.11)

b) Brinkman’s equations:

An alternative to Darcy’s equation is what is commonly known as Brinkman’s equation. With inertial terms omitted this takes the form:

\[ \nabla p = \frac{\mu}{k} u + \tilde{\mu} \nabla^2 u \]  \hspace{1cm} (2.12)

In equation (2.13), the first term is usual Darcy term and the second is analogous to the Laplacian term that appears in the Navier-Stokes equation. The coefficient \( \tilde{\mu} \) is an effective viscosity.

2.5 FRACTIONAL SECOND GRADE FLUID

There are many fluids in industry and technology whose behavior cannot be explained by the classical linearly viscous Newtonian model. The departure from the Newtonian behavior manifests itself in a variety of ways: non-Newtonian viscosity shear thinning or shear thickening, stress relaxation, nonlinear creeping, development of normal stress differences, and yield stress. The Navier-Stokes equations are inadequate to
predicted the behavior of such type of fluids; therefore, many constitutive relations of
non-Newtonian fluids are proposed. These constitutive relations give rise to the
differential equations, which, in general, are more complicated and higher order than the
Navier-Stokes equations. Therefore, it is difficult to obtain exact analytical solutions for
non-Newtonian fluids. Modeling of the equation governing the behaviors of non-
Newtonian fluids in different circumstance is important from many points of view. For
examples, plastics and polymers are extensively handled by the chemical industry,
whereas biological and biomedical devices like hemodialyser make use of the rheological
behavior. In general, the analysis of the behavior of the fluid motion of non-Newtonian
fluids tends to be much more complicated and subtle in comparison with that of the
Newtonian fluids.

The fractional calculus, almost as old as the standard differential and integral one,
is increasingly seen as an efficient tool and subtle frame work within which useful
generalization is quite long and arguments almost yearly. It includes fractal media,
fractional wave diffusion, fractional Hamiltonian dynamics, and biopolymer dynamics as
well as many other topics in physics. Fractional calculus is useful in the field of bio
rheology and bioengineering, in part, because many tissue-like materials polymers, gels,
emulsions, composites, and suspensions exhibit power-law responses to an applied stress
or strain. An example of such power-law behavior in elastic tissue was observed recently
for viscoelastic measurements of the aorta, both in vivo and in vitro, and the analysis of
these data was most conveniently performed using fractional order viscoelastic models.

The starting point of the fractional derivative model of non-Newtonian model is
usually a classical differential equation which is modified by replacing the time
derivative of an integer order by the so-called Riemann-Liouville/Caputo fractional calculus operators. This generalization allows one to define precisely non-integer order integrals or derivatives. In general, fractional model of viscoelastic fluids is derived from well-known ordinary model by replacing the ordinary time derivatives, to fractional order time derivatives and this plays an important role to study the valuable tool of viscoelastic properties. We include here some investigation in which the fractional calculus approach has been adopted for the flows of non-Newtonian fluids. Furthermore, the one-dimensional fractional derivative Maxwell model has been found very useful in modeling the linear viscoelastic response of some polymers in the glass transition and the glass state. In other cases, it has been shown that the governing equations employing fractional derivatives are also linked to molecular theories. The use of fractional derivatives within the context of viscoelasticity was firstly proposed by Germain. Later, Bagley and Torvik demonstrated that the theory of viscoelasticity of coiling polymers predicts constitutive relations with fractional derivatives, and Makris et al. achieved a very good fit of the experimental data when the fractional derivative Maxwell model has been used instead of the Maxwell model for the silicon gel fluid. Furthermore, it is worth pointing out that Palade et al. developed a fully objective constitutive equation for an incompressible fluid reducible to the linear fractional derivative Maxwell model under small deformations hypothesis.

The objective of this paper is twofold. Firstly, is to give few more exact analytical solutions for viscoelastic fluids with fractional derivative approach, which is more natural and appropriate tool to describe the complex behavior of such fluids. Secondly, is to study the slip effects on viscoelastic fluid flows, which is important due to their practical
applications. More precisely, our aim is to find the velocity field and the shear stress corresponding to the motion of a Maxwell fluid due to a sudden moved plate, where no-slip assumption is no longer valid. However, for completeness, we will determine exact solutions for a larger class of such fluids. Consequently, motivated by the above remarks, we solve our problem for Maxwell fluids with fractional derivatives. The general solutions are obtained using the discrete Laplace transforms. They are presented in series form in terms of the Wright generalized hyper geometric functions $\Psi^p_q$ and presented as sum of the slip contribution and the corresponding no-slip contributions. The similar solutions for ordinary Maxwell fluid can easily be obtained as limiting cases of general solutions. The Newtonian solutions are also obtained as special cases of fractional and ordinary Maxwell fluids. Furthermore, the solutions for fractional and ordinary Maxwell fluid for no-slip condition also obtained as special cases, and they are similar with previously known results in the literature. Finally, the influence of the material, slip and fractional parameters on the motion of fractional and ordinary Maxwell fluids is underlined by graphical illustrations. The difference among fractional Maxwell, ordinary Maxwell, and Newtonian fluid models is also highlighted.

The governing equation that describes the flow of a Newtonian fluid is the Navier–Stoke equation. However, some materials such as clay coatings, drilling muds, suspensions, certain oils and greases, polymer melts, elastomers, many emulsions have been treated as non-Newtonian fluids and they cannot be described by the Navier–Stokes equation. For this reason, many non-Newtonian models or constitutive equations have been proposed and most of them are empirical or semi-empirical. One of the most popular models for non-Newtonian fluids is the model that is called the second grade
fluid or fluid of second grade. It is reasonable to use the second grade fluid model to do numerical calculations.

This is particularly so due to the fact that the calculations will generally be simpler. The constitutive equation of a second grade fluid is a linear relation between the stress and the square of the first Rivlin–Ericksen tensor and the second Rivlin–Ericksen tensor (1962).

This constitutive equation has three coefficients. There are some restrictions on these coefficients due to the Clausius–Duhem inequality and also due to the assumption that Helmholtz free energy is minimum in equilibrium. A comprehensive discussion on there restrictions for these coefficients has been given by Dunn and Fosdick (1974) and Dunnand Rajagopal (1995). The restrictions on the two coefficients have not been confirmed by experiments and the sign of the moduli is the subject of much controversy.

During the last years, the fractional calculus has achieved a great success in the description of the complex dynamics. In particular it has been found to be quite flexible in describing the viscoelastic behavior (1998, 2000). A very good fit of the experimental data was achieved when the Maxwell model was used with its first-order derivatives replaced by fractional-order derivatives (1993). Especially, the rheological constitutive equations with fractional derivatives play an important role in the description of the behavior of polymer solutions and melts. In other cases, it has been shown that the constitutive equations employing fractional derivatives are linked to molecular theories (1991). The list of their applications is quite long, it including fractal media, fractional wave diffusion, fractional Hamiltonian dynamics as well as many other topics in physics (2008). In the last time, a lot of papers regarding these fluids have been published but we
remember here only a part of those concerning generalized second grade fluids have been studied Khan, Feecau e.al (2002, 2004, 2005, 2006, 2009, 2010).

2.6 Assumptions

(i) Long wavelength: It may be noted that the radius of bolus is very small when compared to the wave length. i.e. the wave number is \((2 \pi a / \lambda)\) is small.

(ii) Reynolds number: The inertia forces are small. This means that the Reynolds number is low.

(iii) Geometry: The phenomenon of peristalsis can be described in an axisymmetric geometry; we assume that the wall form is an infinite solitary wave. The wall itself moves in a purely transverse direction (extensible wall).

(iv) Amplitude ratio: In comparison to the other parameters considered, the amplitude has the maximum influence on the flow rate. Therefore, more deliberation is required in choosing an optimal value for \(\phi\) (amplitude ratio).

(v) Frame of reference: There are two types of frame of reference in which to study transport phenomena. First is the laboratory frame of reference, where a material point on the wall is up and down, and net transport of the interior fluid goes in the direction of wave propagation. Second is the wave frame of reference, which moves with constant speed \(c\) of traveling wave. Channel walls are stationary in the wave frame of reference.
2.7 BOUNDARY CONDITIONS

(a) Velocity Boundary Conditions

In case of viscous fluid, the fluid adheres to the rigid boundary because of friction. This implies that the velocity of viscous fluid at the rigid boundary will be that of boundary. This condition is known as no-slip condition. At the solid stationary boundaries, the no-slip conditions are:

\[ U = r = 0 \]

2.8 DIMENSIONLESS PARAMETERS

Dimensionless analysis results in sound, orderly arrangement of the various quantities involved in problem. The magnitude of individual quantities encountered in a physical problem can be assembled into dimensionless groups, using dimensional analysis and the differential equations governing the fluid flow can be recast into dimensionless forms using this method. The dimensionless parameters, which appear in the resulting equations, are the parameters of solutions and are the key factors in determining the qualitative and quantitative nature of the flow phenomenon.

The following dimensionless parameters are appeared in the thesis. The general definitions of various dimensionless numbers are given here. This actual expression of each parameter is given in the particular chapter.
Hartmann number- $M$

Hartmann number is defined as the ratio of electromagnetic force to the viscous force and is given by.

$$M = \sqrt[\sigma]{B_0 R_o \mu}$$

(2.14)

Where, $\sigma$ is the electrical conductivity, $\mu$ is the viscosity, $B_0$ is the magnetic field.

Reynolds number- $R$

Reynolds number is defined as the ratio of inertial force to viscous force and is given by.

$$R = \frac{\rho c a}{\nu}.$$ 

Grashof number- $Gr$

The thermal Grashof number is defined as

$$Gr = \frac{\rho g \alpha a^3 T_0}{\mu},$$

Where $T_0$ is the reference temperature, $g$ is the acceleration due to gravity. Physically, the Grashof number represents the measure of buoyancy force to the dissipation force of viscous and thermal dissipation.

Prandtl number- $Pr$

$$Pr = \frac{\mu c_p}{k}$$
Prandtl number is a property of the fluid not only of a particular flow, and plays a significant role especially under constrained system and system with through flow. Physically, it represents the ratio of viscous diffusion rate to the thermal diffusion rate, and is a measure of the relative importance of viscosity and heat conduction in the flow field. Temperature and velocity profile are identical when this parameter is unity.

**Eckert number- \( Ec \).**

Eckert number is defined as

\[
Ec = \frac{c^2}{c_p (T_1 - T_0)}.
\]

Eckert number is a key parameter in determining the relative importance in a heat transfer situation of the kinetic energy of a flow. It is the ratio of the kinetic energy to the enthalpy (or the dynamic temperature to the temperature) driving force for heat transfer.

**2.6 Assumptions**

Like any mathematical model of the real world, fluid mechanics makes some basic assumptions about the materials being studied. These assumptions are turned into equations that must be satisfied if the assumptions are to be held true. For example, consider an incompressible fluid in three dimensions. The assumption that mass is conserved means that for any fixed closed surface (such as a sphere) the rate of mass passing from outside to inside the surface must be the same as rate of mass passing the other way. (Alternatively, the mass inside remains constant, as does the mass outside). This can be turned into an integral equation over the surface.
Fluid mechanics assumes that every fluid obeys the following:

- Conservation of mass
- Conservation of momentum
- The continuum hypothesis, detailed below.

Further, it is often useful (and realistic) to assume a fluid is incompressible – that is, the density of the fluid does not change. Liquids can often be modelled as incompressible fluids, whereas gases cannot.

Similarly, it can sometimes be assumed that the viscosity of the fluid is zero (the fluid is in viscid). Gases can often be assumed to be in viscid. If a fluid is viscous, and its flow contained in some way (e.g. in a pipe), then the flow at the boundary must have zero velocity. For a viscous fluid, if the boundary is not porous, the shear forces between the fluid and the boundary results also in a zero velocity for the fluid at the boundary. This is called the no-slip condition. For a porous media otherwise, in the frontier of the containing vessel, the slip condition is not zero velocity, and the fluid has a discontinuous velocity field between the free fluid and the fluid in the porous media (this is related to the Beavers and Joseph condition).

**The continuum hypothesis**

Fluids are composed of molecules that collide with one another and solid objects. The continuum assumption, however, considers fluids to be continuous. That is, properties such as density, pressure, temperature, and velocity are taken to be well-
defined at "infinitely" small points, defining a REV (Reference Element of Volume), at
the geometric order of the distance between two adjacent molecules of fluid. Properties
are assumed to vary continuously from one point to another, and are averaged values in
the REV. The fact that the fluid is made up of discrete molecules is ignored.

The continuum hypothesis is basically an approximation, in the same way planets
are approximated by point particles when dealing with celestial mechanics, and therefore
results in approximate solutions. Consequently, assumption of the continuum hypothesis
can lead to results which are not of desired accuracy. That said, under the right
circumstances, the continuum hypothesis produces extremely accurate results.

Those problems for which the continuum hypothesis does not allow solutions of
desired accuracy are solved using statistical mechanics. To determine whether or not to
use conventional fluid dynamics or statistical mechanics, the Knudsen number is
evaluated for the problem. The Knudsen number is defined as the ratio of the molecular
mean free path length to a certain representative physical length scale. This length scale
could be, for example, the radius of a body in a fluid. (More simply, the Knudsen number
is how many times its own diameter a particle will travel on average before hitting
another particle). Problems with Knudsen numbers at or above unity are best evaluated
using statistical mechanics for reliable solutions.

2.9 Navier–Stokes equations

The Navier–Stokes equations (named after Claude-Louis Navier and George
Gabriel Stokes) are the set of equations that describe the motion of fluid substances such
as liquids and gases. These equations state that changes in momentum (force) of fluid particles depend only on the external pressure and internal viscous forces (similar to friction) acting on the fluid. Thus, the Navier–Stokes equations describe the balance of forces acting at any given region of the fluid.

The Navier–Stokes equations are differential equations which describe the motion of a fluid. Such equations establish relations among the rates of change the variables of interest. For example, the Navier–Stokes equations for an ideal fluid with zero viscosity state that acceleration (the rate of change of velocity) is proportional to the derivative of internal pressure.

This means that solutions of the Navier–Stokes equations for a given physical problem must be sought with the help of calculus. In practical terms only the simplest cases can be solved exactly in this way. These cases generally involve non-turbulent, steady flow (flow does not change with time) in which the Reynolds number is small.

For more complex situations, such as global weather systems like El Nino or lift in a wing, solutions of the Navier–Stokes equations can currently only are found with the help of computers. This is a field of sciences by its own called computational fluid dynamics.

**General form of the equation**

The general form of the Navier–Stokes equations for the conservation of momentum is:
\[
\rho \frac{Dv}{Dt} = \nabla P + \rho f
\]

Where

- \(\rho\) is the fluid density,
- \(\frac{D}{Dt}\) is the substantive derivative (also called the material derivative),
- \(v\) is the velocity vector,
- \(f\) is the body force vector, and
- \(\mathbb{P}\) is a tensor that represents the surface forces applied on a fluid particle (the commoving stress tensor).

Unless the fluid is made up of spinning degrees of freedom like vortices, \(\mathbb{P}\) is a symmetric tensor. In general, (in three dimensions) \(\mathbb{P}\) has the form:

\[
\mathbb{P} = \begin{pmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_{zz}
\end{pmatrix}
\]

Where

- \(\sigma\) are normal stresses,
- \(\tau\) are tangential stresses (shear stresses).
The above is actually a set of three equations, one per dimension. By themselves, these aren't sufficient to produce a solution. However, adding conservation of mass and appropriate boundary conditions to the system of equations produces a solvable set of equations.

**Newtonian versus non-Newtonian fluids**

A **Newtonian fluid** (named after Isaac Newton) is defined to be a fluid whose shear stress is linearly proportional to the velocity gradient in the direction perpendicular to the plane of shear. This definition means regardless of the forces acting on a fluid, it continues to flow. For example, water is a Newtonian fluid, because it continues to display fluid properties no matter how much it is stirred or mixed. A slightly less rigorous definition is that the drag of a small object being moved slowly through the fluid is proportional to the force applied to the object. (Compare friction). Important fluids, like water as well as most gases, behave to good approximation as a Newtonian fluid under normal conditions on Earth.

By contrast, stirring a non-Newtonian fluid can leave a "hole" behind. This will gradually fill up over time – this behaviour is seen in materials such as pudding, oobleck, or sand (although sand isn't strictly a fluid). Alternatively, stirring a non-Newtonian fluid can cause the viscosity to decrease, so the fluid appears "thinner" (this is seen in non-drip paints). There are many types of non-Newtonian fluids, as they are defined to be something that fails to obey a particular property for example, most fluids with long molecular chains can react in a non-Newtonian manner.
Equations for a Newtonian fluid

The constant of proportionality between the shear stress and the velocity gradient is known as the viscosity. A simple equation to describe Newtonian fluid behaviour is

\[ \tau = -\mu \frac{dv}{dx} \]

Where

- \( \tau \) is the shear stress exerted by the fluid ("drag")
- \( \mu \) is the fluid viscosity – a constant of proportionality
- \( \frac{dv}{dx} \) is the velocity gradient perpendicular to the direction of shear

For a Newtonian fluid, the viscosity, by definition, depends only on temperature and pressure, not on the forces acting upon it. If the fluid is incompressible and viscosity is constant across the fluid, the equation governing the shear stress (in Cartesian coordinates) is

\[ \tau_{ij} = \mu \left( \frac{dv_i}{dx_j} + \frac{dv_j}{dx_i} \right) \]

Where \( \tau_{ij} \) is the shear stress on the \( i^{th} \) face of a fluid element in the \( j^{th} \) direction.

- \( v_i \) is the velocity in the \( i^{th} \) direction.
- \( x_j \) is the \( j^{th} \) direction coordinate.
If a fluid does not obey this relation, it is termed a non-Newtonian fluid, of which there are several types.

Among fluids, two rough broad divisions can be made: ideal and non-ideal fluids. An ideal fluid really does not exist, but in some calculations, the assumption is justifiable. An Ideal fluid is non viscous- offers no resistance whatsoever to a shearing force.

One can group real fluids into Newtonian and non-Newtonian. Newtonian fluids agree with Newton's law of viscosity. Non-Newtonian fluids can be either plastic, bingham plastic, pseudo plastic, dilatant, thixotropic, rheopectic, viscoelatic.

"Using a whole body of mathematical methods (not only those inherited from the antique theory of ratios and infinitesimal techniques, but also the methods of the contemporary algebra and fine calculation techniques), Arabic scientists raised statics to a new, higher level. The classical results of Archimedes in the theory of the centre of gravity were generalized and applied to three-dimensional bodies, the theory of ponderable lever was founded and the 'science of gravity' was created and later further developed in medieval Europe. The phenomena of statics were studied by using the dynamic approach so that two trends – statics and dynamics – turned out to be interrelated within a single science, mechanics. The combination of the dynamic approach with Archimedean hydrostatics gave birth to a direction in science which may be called medieval hydrodynamics. Numerous fine experimental methods were developed for determining the specific weight, which were based, in particular, on the theory of balances and weighing. The classical works of al-Biruni and al-Khazini can by right be considered as the beginning of the application of experimental methods in medieval science."
2.10 Herschel–Bulkley fluid

The Herschel–Bulkley fluid is a generalized model of a non-Newtonian fluid, in which the stress experienced by the fluid is related to the strain in a complicated, non-linear way. Three parameters characterize this relationship: the consistency $k$, the flow index $n$, and the yield shear stress $\tau_0$. The consistency is a simple constant of proportionality, while the flow index measures the degree to which the fluid is shear-thinning or shear-thickening. Ordinary paint is one example of a shear-thinning fluid, while oobleck provides one realization of a shear-thickening fluid. Finally, the yield stress quantifies the amount of stress that the fluid may experience before it yields and begins to flow.

This non-Newtonian fluid model was introduced by Herschel and Bulkley in 1926.

**Definition**

The viscous stress tensor is given, in the usual way, as a viscosity, multiplied by the rate-of-strain tensor:

$$\tau_{ij} = 2\mu E_{ij} = \mu \left( \frac{dv_i}{dx_j} + \frac{dv_j}{dx_i} \right)$$

where in contrast to the Newtonian fluid, the viscosity is itself a function of the strain tensor. This is constituted through the formula

$$\mu = \begin{cases} \mu_0, & \pi \leq \pi_0 \\ k\pi_o^{n-1} + \tau_0\pi^{-1}, & \pi \leq \pi_0 \end{cases}$$

Where $\pi$ is the second invariant of the rate-of-strain tensor.

$$\pi = \sqrt{2E_{ij}E^{ij}}.$$

87
If \( n = 1 \) and \( \tau_0 = 0 \), this model reduces to the Newtonian fluid. If \( n < 1 \) the fluid is shear-thinning, while \( n > 1 \) produces a shear-thickening fluid. The limiting viscosity \( \mu_0 \) is chosen such that \( k\pi_0^{n-1} + \tau_0\pi^{-1} \). A large limiting viscosity means that the fluid will only flow in response to a large applied force. This feature captures the Bingham-type behaviour of the fluid.

**Channel flow**

A schematic diagram pressure-driven horizontal flow. The flow is uni-directional in the direction of the pressure gradient.

A frequently-encountered situation in experiments is pressure-driven channel flow (see diagram). This situation exhibits an equilibrium in which there is flow only in the horizontal direction (along the pressure-gradient direction), and the pressure gradient and viscous effects are in balance. Then, the Navier-Stokes equations, together with the rheological model, reduce to a single equation:
Velocity profile of the Herschel–Bulkley fluid for various flow indices $n$. In each case, the non-dimensional pressure is $\pi_0 = -10$. The continuous curve is for an ordinary Newtonian fluid (Poiseuille flow), the broken-line curve is for a shear-thickening fluid, while the dotted-line curve is for a shear-thinning fluid.

\[
\frac{\partial \dot{p}}{\partial x} = \frac{\partial}{\partial z} \left( \frac{\mu}{\frac{\partial u}{\partial z}} \right) = \begin{cases} \mu \frac{\partial^2 u}{\partial z^2}, & \left| \frac{\partial u}{\partial z} \right| < \gamma_0 \\ \frac{\partial}{\partial z} \left( k \left| \frac{\partial u}{\partial z} \right|^{n-1} + \tau_0 \left| \frac{\partial u}{\partial z} \right| \right), & \left| \frac{\partial u}{\partial z} \right| \geq \gamma_0 \end{cases}
\]

To solve this equation it is necessary to non-dimensionalize the quantities involved. The channel depth $H$ is chosen as a length scale, the mean velocity $V$ is taken as a velocity scale, and the pressure scale is taken to be $P_o = K \left( \frac{V}{H} \right)^n$. This analysis introduces the non-dimensional pressure gradient

\[
\pi_o = \frac{H \frac{\partial \dot{p}}{\partial x}}{P_o},
\]

which is negative for flow from left to right, and the Bingham number:

89
\[ B_n = \frac{\tau_0}{k} \left( \frac{H}{V} \right)^n. \]

Next, the domain of the solution is broken up into three parts, valid for a negative pressure gradient:

- A region close to the bottom wall where \( \partial u / \partial z > \gamma_0 \);
- A region in the fluid core where \( |\partial u / \partial z| < \gamma_0 \);
- A region close to the top wall where \( \partial u / \partial z < -\gamma_0 \);

**Herschel-Bulkley fluid**

A fluid described by a three-parameter rheological model. A Herschel-Bulkley fluid can be described mathematically as follows:

Where

\[ \tau = \tau_0 + k (\gamma)^n \]

\( \tau \) = Shear stress
\( \tau \) = Yield stress
\( k \) = Consistency
\( \gamma \) = Shear rate
\( n \) = Power law exponent.

The Herschel-Bulkley equation is preferred to power law or Bingham relationships because it results in more accurate models of rheological behavior when adequate experimental data are available. The yield stress is normally taken as the 3 rpm.

The Herschel-Bulkley equation is preferred to power law or Bingham relationships because it results in more accurate models of rheological behavior when adequate
experimental data are available. The yield stress is normally taken as the 3 rpm reading, with the n and K values then calculated from the 600 or 300 rpm values or graphically.

**X-Y Plot of Herschel-Bulkley fluid.**

Herschel-Bulkley fluid. Some drilling fluids conform to the Herschel-Bulkley fluid model, requiring a certain minimum stress to initiate flow, but less stress with increasing shear.
Herschel-Bulkley fluid. Fluids are described as Newtonian or non-Newtonian depending on their response to shearing. The shear stress of a Newtonian fluid (upper left) is proportional to the shear rate. Some drilling fluids conform to the Herschel-Bulkley fluid model (lower right), requiring a certain minimum stress to initiate flow, but decreasing stress with increasing shear. Most drilling muds are non-Newtonian fluids, with viscosity decreasing as shear rate increases.