Chapter 1

Prologue

This chapter provides an introduction to the essential factors of graph theoretic definitions and terminologies which we use throughout this thesis. Also we give the preliminary results and elementary concepts of the theory of neighborhood and coloring in succeeding chapters.

1.1 Basic Definitions and Terminologies

The term “graph” was introduced in 1878 by J. J. Sylvester in one of his paper published in Nature [106], where he drawn an analogy between “quantic invariants” and “co - variants” of algebra and molecular diagrams.

A graph $G$ consists of a pair $G = \{V(G), E(G)\}$ where $V(G)$ is a nonempty finite set, whose elements are called points or vertices and $E(G)$ are called lines or edges such that each edge is identified with an unordered pair of vertices. Specifically, let $G$ be a graph with vertex set $V(G) = V$ of order $n = |V(G)|$ and edge set $E(G) = E$ of size $m = |E(G)|$ and a graph $G$ with $n$ - vertices and $m$ - edges is called an $(n, m)$ - graph. A graph $G$ is said to be trivial if $n = 1$, and nontrivial otherwise. If the
edge $e = \{u, v\}$ is an edge of a graph $G$, then $u$ and $v$ are adjacent vertices and $e$ and $u$ are incident with each other.

The degree of $v$ in $G$, denoted by $\text{deg}(v)$, is the number of edges of a graph $G$ incident with $v$. Thus, if $G$ is a simple graph, then $\text{deg}(v) = |N(v)|$. A vertex $v$ of a graph $G$ is said to be an isolated vertex if $\text{deg}(v) = 0$ and is called as an endvertex or pendant vertex or hanging vertex if $\text{deg}(v) = 1$. Assume that $V(G) = \{v_1, v_2, \ldots, v_n\}$, then the sequence $\{\text{deg}(v_1), \text{deg}(v_2), \ldots, \text{deg}(v_n)\}$ is called the degree sequence of $G$. Usually, the minimum degree $\delta(G) = \text{Min}\{\text{deg}(v) : v \in V(G)\}$ and the maximum degree $\Delta(G) = \text{Max}\{\text{deg}(v) : v \in V(G)\}$. A graph $G$ is said to be regular if $\Delta(G) = \delta(G)$. More precisely, the graph $G$ is $r$-regular if $\Delta(G) = r = \delta(G)$; that is $\text{deg}(v) = r$ for each $v \in V(G)$.

Theorem 1.1.1. [49] For any nontrivial graph $G$,

$$\sum_{x_i \in V} \text{deg}(x_i) = 2m.$$  

Further, the number of vertices of odd degree in a graph is always even.

Theorem 1.1.2. [14] For any nontrivial graph $G$,

$$\delta(G) \leq \frac{2m}{n} \leq \Delta(G).$$

A graph $H$ is called a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and a subgraph is a proper subgraph of a graph $G$ if $H \neq G$. A subgraph $H$ of $G$ is said to be spanning subgraph if $V(H) = V(G)$. A nonempty subset $S \subseteq V(G)$, the subgraph of $G$ induced by $S$, denoted by $\langle S \rangle$, is the graph having the vertex set $S$ and its edge
set consisting of all the edges of $G$ joining the vertices in $S$. We call $\langle S \rangle$ the induced subgraph of $S$ in $G$.

Let $N = \{1, 2, 3, \ldots\}$ be the set of positive integers, and let $n \in N$. A graph is an empty graph if it contains no edges. The empty graph of order $n$ is denoted by $O_n$. In contrast with empty graphs, a graph is said to be complete if any two of its vertices are adjacent. The complete graph of order $n$ is denoted by $K_n$. Clearly, $m(K_n) = \binom{n}{2}$. Let $M \subseteq V(G)$, then $M$ is called clique in graph $G$ if every two vertices in $M$ are adjacent. On the other hand, $M$ is called an independent set in a graph $G$ if no two vertices in $M$ are adjacent in $G$. The clique number of $G$, denoted by $\omega(G)$ and is defined as $\omega(G) = \max \{|M| : M \text{ is a clique in } G\}$.

Let $u$ and $v$ be (not necessarily distinct) the vertices of a graph $G$. A $u - v$ walk of $G$ is finite and having alternating sequence $u = u_0, e_1, u_1, e_2, \ldots, e_n, u_n = v$ of vertices and edges beginning with vertex $u$ and ending with vertex $v$ such that $e_i = u_{i-1}u_i$, $i = 1, 2, \ldots, n$. The number $n$ is called the length of the walk. The walk is said to be open if $u$ and $v$ are distinct and it is closed otherwise. A walk is a trail if none of the edges are repeated. A trail in which all the vertices are distinct is called a path. A closed trail $(u_0, u_1, \ldots, u_n)$ in which $(u_0, u_1, u_2, \ldots, u_{n-1})$ are distinct is called a cycle. A path on $n$-vertices is denoted by $P_n$ and a cycle on $n$-vertices is denoted by $C_n$.

A graph is said to be unicyclic if it contains exactly one cycle as a subgraph. A forest is a graph containing no cycle, such a graph is also said to be acyclic. The length of a shortest cycle (if any) in $G$ is referred to as the girth of $G$, denoted by
g(G). A tree is a connected forest. A spanning subgraph \( H \) of a connected graph \( G \) such that \( H \) is a tree is called spanning tree of \( G \). A tree is called a caterpillar if a path results when all the leaves are removed.

**Theorem 1.1.3.** [14]

(i) A connected graph \( G \) is a tree \( T \) if and only if it has \((n - 1)\) edges.

(ii) A graph \( G \) is a tree \( T \) if and only if every two vertices of \( G \) are connected by a unique path.

(iii) Every connected graph contains a spanning tree.

A graph \( G \) is connected if any two vertices in \( G \) are joined by a path. It is disconnected otherwise. A component of \( G \) is a connected subgraph of \( G \) which is not a proper subgraph of any connected subgraph of \( G \).

A cut vertex in a graph \( G \) is a vertex whose removal increases the number of components of \( G \). Analogously, a bridge in \( G \) is an edge whose removal increases the number of components. A block of \( G \) is an induced subgraph without cut vertices of maximum cardinality.

A simple graph \( G \) is bipartite if its vertices can be partitioned into two sets (called partite set) in such a way that no edge joins two vertices in the same set. A complete bipartite graph is a simple bipartite graph in which each vertex in one partite set is adjacent to all the vertices in the other partite set. If the two partite sets have cardinalities \( r \) and \( s \), then the graph is denoted by \( K_{r,s} \). A graph is multipartite (or \( t \)-partite) if its vertices can be partitioned into \( t \)-sets (called partite sets) in
such a way that no edge joins two vertices in the same set. A complete multipartite (or complete $t$ - partite graph) is a simple $t$ - partite graph in which two vertices are adjacent if and only if they are in different partite sets, if the $t$ - partite sets have orders $r_1 \leq r_2 \leq \ldots \leq r_t$.

**Theorem 1.1.4.** [25] A graph $G$ is bipartite if and only if it contains no odd cycles.

The distance $d(x, y)$ of two vertices $x$ and $y$ of a graph $G$ is the length of a path of minimum length with end vertices $x$ and $y$. The eccentricity, $e(v)$ of a vertex $v$ in a connected graph is the maximum $d(u, v)$ for all $u$ in $G$. The minimum eccentricity is called the radius denoted by $\text{rad}(G)$ and maximum eccentricity is the diameter denoted by $\text{diam}(G)$ of the graph. A vertex $v$ is a central vertex if $e(v) = \text{rad}(G)$ and the center of a graph $G$ is the set of all central vertex.

**Theorem 1.1.5.** Every tree has a center consisting of either one vertex or two adjacent vertices.

**Theorem 1.1.6.** [49] For any nontrivial connected graph $G$,

(i) $m \leq \frac{n(n-2)}{2}$ if $\text{rad}(G) = 1$,

(ii) $m \leq \frac{n(n-2)}{2}$ if $\text{rad}(G) = 2$,

(iii) $m \leq \frac{1}{2} [n^2 - 4n \cdot \text{rad}(G) + 5n + 4 (\text{rad}(G))^2 - 6 \text{rad}(G)]$ if $\text{rad}(G) = 3$.

The connectivity $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected graph or $K_1$, the trivial graph. The edge connectivity
\(\lambda(G)\) of a graph \(G\) is the minimum number of edges whose removal results in a disconnected graph.

**Theorem 1.1.7.** [14] For any nontrivial graph \(G\),

\[ \kappa(G) \leq \lambda(G) \leq \delta(G). \]

A vertex and an edge are said to cover each other if they are incident. A set of vertices which cover all the edges of \(G\) is called a cover for \(G\). The minimum number of \(n\) vertices which covers all the edges of \(G\) is called its covering number and is denoted by \(\alpha(G)\). A set of edges which cover all the vertices of \(G\) is called an edge cover of \(G\). The minimum number of edges in any edge cover of \(G\) is called edge covering number of \(G\) and is denoted by \(\alpha_1(G)\).

A subset \(S\) of the vertex set in a graph \(G\) is said to be independent if no two vertices in \(S\) are adjacent in \(G\). The maximum number of vertices in an independent set is called the independence number of \(G\) and is denoted by \(\beta(G)\). A set \(S\) of edges in a graph \(G\) is said to be independent in no two edges in \(S\) are adjacent in \(G\). \(S\) is called a maximal independent set provided it is not a proper subset of some other independent set. The maximum cardinality of an edge set of \(G\) is called the edge independence number of \(G\) and is denoted by \(\beta_1(G)\). A set of independent edges covering all the vertices of \(G\) is called a 1-factor or a perfect matching of \(G\).

**Theorem 1.1.8.** [49] For any nontrivial connected graph \(G\),

\[ \alpha(G) + \beta(G) = \alpha_1(G) + \beta_1(G) = n. \]
Theorem 1.1.9. [14] In a bipartite graph $G$, $\alpha_1(G) = \beta(G)$. Consequently, if a graph $G$ has no vertex of degree 0, then $\alpha(G) = \beta_1(G)$.

Theorem 1.1.10. [14] The minimum cardinality of a perfect matching in $G$ is equal to the minimum cardinality of a vertex cover of its edges.

The open neighborhood $N(v)$ of a vertex $v$ in a graph $G$ is the set of all vertices adjacent to $v$ in $G$. $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of $v$. The private neighborhood $PN(v, X)$ of $v \in X$ of a graph $G$ is defined by $PN(v, X) = N[v] - N[X - \{v\}]$.

A graph $G_1$ is isomorphic to a graph $G_2$ if there exists a bijection $\phi$ from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. If $G_1$ is isomorphic to $G_2$, we write $G_1 \cong G_2$.

The graph $G^c$ is called the complement of $G$ and is defined as the graph with vertex set $V(G^c) = V(G)$ and edge set $E(G^c)$ such that $e \in E(G^c)$ if and only if $e \notin E(G)$. Further, a graph is self-complementary if it is isomorphic to its complement, i.e., $G \cong G^c$.

The $G_2$-corona of $G_1$ is the graph $G_1 \circ G_2$ formed from one copy of $G_1$ and $n(G_1)$ copies of $G_2$ where the $i^{th}$ vertex of $G_1$ is adjacent to every vertex in the $i^{th}$ copy of $G_2$.

For a real number $x > 0$, let $\lceil x \rceil$ be the least integer not less than $x$ and $\lfloor x \rfloor$ be the greatest integer not greater than $x$. If $n$ is a positive integer and $r$ is a nonnegative integer with $r \leq n$, then $\binom{n}{r}$ is the binomial coefficient that stands for the number of ways of choosing $r$-objects out of $n$-objects.
In 1958, domination was formalized as a theoretical area in graph theory by [12]. He referred to the domination number as the coefficient of external stability. In 1962, Ore [82] was first to use the term “Domination” for undirected graphs. During the past 30 years the study of dominating sets in graphs has emerged as a significant area of research not only in graph theory but in combinatorial optimization and analysis of algorithms as well.

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is said to be a dominating set of a graph $G$, if every vertex in $V - D$ is adjacent to some vertex in $D$. The minimum cardinality of a dominating set in $G$ is the domination number $\gamma(G)$. A dominating set $D$ with minimum cardinality is called $\gamma$ - set of a graph $G$. Similarly other sets can be expected.

The domatic number $d(G)$ of a graph $G$ is the maximum positive integer $k$ such that $V$ can be partitioned into $k$ pairwise disjoint dominating sets $D_1, D_2, \ldots, D_k$. A partition of $V$ into pairwise disjoint dominating sets is called a domatic partition. The concept of a domatic number was introduced in [32].

The word ‘domatic’ was created from the words ‘dominating’ and ‘chromatic’ in the same way the word ‘smog’ was created from the words ‘smoke’ and ‘fog’. In a certain sense a domatic number is analogous to the chromatic number of a graph, which is the minimum positive integer $k$ such that the vertex set can be partitioned into $k$ pairwise disjoint stable sets. For the complete review of the theory of domination and its related parameters, we refer [28], [50], [51], [52], [54], [68], [72] and [108].
1.2 Theory of Neighborhood

In 1985, E. Sampathkumar and P. S. Neeralagi [86], introduced an innovative idea of domination between the vertices and the edges, and vice versa. They introduced a new parameter called the neighborhood number of a graph, as follows. A set $S \subseteq V$ is a neighborhood set of $G$, if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of $G$ induced by $v$ and all vertices adjacent to $v$. The neighborhood number $\eta(G)$ is the minimum cardinality of a neighborhood set of $G$. A neighborhood set $S \subseteq V$ is a minimal neighborhood set of a graph $G$, if $S - v$ for all $v \in S$ is not a neighborhood set. For more information on neighborhood number we refer [19], [60] and [75].

**Theorem 1.2.1.** [86]

(i) For any nontrivial graph $G$, $\eta(G) = 1$ if and only if the graph $G$ has a vertex of degree $n - 1$. Thus $\eta(G)$ of each of the following graph is same as complete graph $K_n$ with $n \geq 1$ vertices or star $K_{1,n-1}$ with $n \geq 2$ vertices or wheel graph $W_n$ with $n \geq 1$ vertices.

(ii) For any bipartite graph $G$ with the bipartition $\{V_1, V_2\}$ of $V(G)$,

$$\eta(G) = \text{Min.}\{V_1, V_2\},$$

(iii) For any graph $G$ with no isolates and triangles,

$$\eta(G) = \alpha(G),$$

(iv) For any bipartite graph $G$ without isolated vertices,

$$\eta(G) = \alpha(G) = \beta_1(G).$$
Theorem 1.2.2. [86] Let $G$ be any graph and $S$ be any subset of $V(G)$. Then $S$ is an $\eta$-set of a graph $G$ if and only if every edge in $\langle V - S \rangle$ belongs to $\langle N[u] \rangle$ for some $u$ belongs to $S$.

Different graph theorists have defined different neighborhood parameters by imposing extra conditions on the neighborhood set $S$ of a graph $G$, because the neighborhood number is closely related to the domination number.

In [86], Sampathkumar and Neeralagi introduced the concepts of independent, perfect and connected neighborhood number as follows: A neighborhood set $S \subseteq V$ is a independent neighborhood set of $G$, if $\langle S \rangle$ is an independent and neighborhood set of $G$. The minimum cardinality of an independent neighborhood set is called \textit{independent neighborhood number} of $G$ and is denoted by $\eta_i(G)$. This concepts was also studied by Kulli and Soner [76].

A neighborhood set $S \subseteq V$ is a connected neighborhood set of a graph $G$, if $\langle V - S \rangle$ is connected. The minimum cardinality of a connected neighborhood set is called a \textit{connected neighborhood number} $\eta_c(G)$ of a graph $G$. For more detail on similar concept we refer [17].

A neighborhood set $S \subseteq V$ is a perfect neighborhood set of $G$, if vertex in $V - S$ is adjacent to exactly one vertex of $S$. The perfect neighborhood number $\eta_p(G)$ of $G$ is the minimum cardinality of a perfect neighborhood set of $G$.

A neighborhood set $S \subseteq V$ is a paired neighborhood set, if $\langle S \rangle$ contains at least one perfect matching of $G$. The minimum cardinality of a paired neighborhood set is
called \textit{paired neighborhood number} of \(G\) and is denoted by \(\eta_{pr}(G)\). This concept was introduced by Soner et al., [99] and also studied by [19] and [104].

\textbf{Theorem 1.2.3.} [99]. \textit{For any graph} \(G\) \textit{with no isolates, then}

\begin{enumerate}
\item \(\eta_{pr}(G) \geq \max \left\{ \left\lfloor \frac{n}{\Delta(G)} \right\rfloor, \left\lfloor \frac{2n}{\Delta(G) + 1} \right\rfloor \right\},\)
\item \(\eta_{pr}(G) \geq \frac{(4n - 2m)}{3},\)
\item \(\eta_{pr}(G) \geq \eta(G).\)
\end{enumerate}

The concept of maximal neighborhood number was introduced by Soner et al., [100] as follows. A neighborhood set of \(S \subseteq V\) is a maximal neighborhood set of \(G\), if \(V - S\) does not contain a neighborhood set of \(G\). The minimum cardinality of a maximal neighborhood set is called \textit{maximal neighborhood number} \(\eta_m(G)\) of a graph \(G\). This concept is also studied [102].

\textbf{Theorem 1.2.4.} [100] \textit{A} neighborhood set \(S\) \textit{of a graph} \(G\) \textit{is a maximal neighborhood set of} \(G\) \textit{if and only if there exist two adjacent vertices} \(u, v \in S\) \textit{such that every vertex} \(w \in V - S\) \textit{is adjacent to at most one of} \(u\) \textit{and} \(v\).

Inverse neighborhood number was introduced by Kulli et al., [73] and it is defined as a neighborhood set \(S \subseteq V\) is a inverse neighborhood set, if \(V - S\) contain a neighborhood set of \(G\). The minimum cardinality of an inverse neighborhood set is called \textit{inverse neighborhood number} \(\eta^{-1}(G)\) of a graph \(G\).

A neighborhood set \(S \subseteq V\) is a split (strong split) neighborhood set of a graph \(G\), if \(\langle V - S \rangle\) is disconnected (totally disconnected). The minimum cardinality of
a split (strong split) neighborhood set is called a *split (strong split) neighborhood number* of \( G \) and is denoted by \( \eta_s(G) \) (\( \eta_{ss}(G) \)). This concept was studied by [70].

A neighborhood set \( S \subseteq V \) is a nonsplit neighborhood set, if \( (V - S) \) is connected. The minimum cardinality of a nonsplit neighborhood set is called a *nonsplit neighborhood number* \( \eta_{ns}(G) \) of a graph \( G \) and is also known as co-connected neighborhood number, which is denoted by \( \eta_{cc}(G) \). This concept was introduced by Kulli and Janakiram [71] and also studied by [17], [57] and [105].

The *uniform neighborhood number* of a graph \( G \) is defined to be the least positive integer \( k \) such that any \( k \)-element set of \( V(G) \) is a neighborhood set in \( G \) and it is denoted by \( \eta_u(G) \). This concept was initiated by Soner and Chaluvaraju [98] and also studied by [17].

In 1987, Jayaram was first introduce the concept of nomatic number of a graph in the following sense. The *nomatic number* of \( G \) which is denoted by \( N(G) \) is the largest number of sets in a partition of \( V \) into disjoint minimal neighborhood sets of a graph \( G \). This concept is also studied by [46] and [59].

**Theorem 1.2.5.** [59] *For any graph \( G \),*

(i) \( N(G) \leq \delta(G) + 1 \),

(ii) \( \eta(G) + \eta(G^c) \leq n + 1 \), and equality holds iff \( G \cong K_n \) or \( K_n^c \), \( n \geq 1 \),

(iii) \( \eta(G) + N(G) \leq n + 1 \), and equality holds iff \( G \cong K_n \) or \( K_n^c \), \( n \geq 1 \),

(iv) \( N(G) = 1 \) iff \( G \cong K_n^c \) or \( C_{2r+1} \); \( r \geq 2 \), and \( N(G) = n \) iff \( G \cong K_n \).
A neighborhood set $S$ of a graph $G$ with $|S| = \eta(G)$ is called \( \eta \)-set. Similarly, the other sets can be expected. To date many papers have been published which study the existence of neighborhood sets in graphs. Perhaps most notable among these are the approximately 60 papers (too numerous to mention here) which study various parameters of neighborhood number. For a comprehensive survey on neighborhood number and its related invariants, we refer [10], [61], [76], [87] and [88].

1.3 Theory of Coloring

Chromatic graph theory goes back to a problem, posed some 150 years ago, relating to the coloring of maps, either real or imaginary. The condition postulated was that countries with a common border line (and not just a border point) should receive different colors. The question was, “How many colors are needed to cover all the different maps imaginable?”

Apart from being an exercise in abstract thinking, what practical application does this have? The coloring theory brings one immediate application to mind. If you want to make a timetable for an exam, one common condition is that you cannot have two papers written by students at the same time if one or more of the students has to write both papers. If you rephrase the problem correctly it turns out to be a simple coloring matter. The idea of using the minimum number of colors then translates to, “What is the minimum number of sessions you need to set up the timetable?”

The first known mention of coloring problems was in 1852, when August De Morgan, Professor of Mathematics at University College, London, wrote Sir William
Rowan Hamilton in Dublin about a problem posed to him by a former student, named Francis Guthrie. Guthrie noticed that it was possible to color the countries of England using four colors so that no two adjacent countries were assigned the same color. The question raised thereby was whether four colors would be sufficient for all possible decompositions of the plane into regions, which is well known Four Color Conjecture.

The vertex coloring of a graph $G$ is a function $f : V \rightarrow C$, where $C$ is an color assigned elements. A vertex coloring is proper if two adjacent vertices are always assigned different colors. We say that a graph $G$ is $k$ - colorable if it can be colored using (at most) $k$ - colors. The smallest number $k$ for which the graph $G$ is $k$ - colorable is called the chromatic number of $G$ and is denoted by $\chi(G)$. A graph is $k$ - chromatic if $\chi(G) = k$.

In a given coloring of a graph $G$, a set consisting of all those vertices assigned the same color is referred to as a color class. If $\chi(G) = k$ and every $k$ - coloring of $G$ induced by the same partition of $V$, then $G$ is called uniquely $k$ - colorable or simply uniquely colorable. For complete review on theory of coloring and its related parameters, we refer [15], [26], [62] and [92].

**Theorem 1.3.1.** [49] A graph $G$ has $\chi(G) = 2$ if and only if $G$ is nonempty bipartite graph.

**Theorem 1.3.2.** Let $G$ be a nontrivial connected graph. Then $\chi(G) = n$ if and only if the graph $G$ is isomorphic with $K_n$. 
Theorem 1.3.3. [25] For any \((n, m)\) - graph \(G\),

(i) \(\chi(G) \geq \omega(G)\),

(ii) \(\chi(G) \geq \frac{n}{\beta(G)}\),

(iii) \(\chi(G) \leq \Delta(G) + 1\),

(iv) \(\chi(G) \leq \frac{1}{2} + \sqrt{(2m + \frac{1}{4})}\),

(v) \(\frac{n}{n - \delta(G)} \leq \chi(G)\),

(vi) \(\frac{n^2}{n^2 - 2m} \leq \chi(G)\),

(vii) \(\chi(G) \leq 1 + Max. \{\delta(H)\}\), where the maximum is taken over all induced subgraphs \(H\) of \(G\).

Theorem 1.3.4. (Brooks [26]) Let \(G\) be a connected graph. If \(G\) is neither complete nor an odd cycle, then \(\chi(G) \leq \Delta(G)\).

Throughout this thesis, unless specified otherwise, by a graph, we mean, a finite, undirected graph without loops or multiple edges. However, any terminology or notions not described in this thesis may be found in [12], [24], [42], [49] and [62].