Chapter 9

Special Kinds of Colorable Complement Graph of a Graph

This chapter aims at the study of the $k$-colorable complement graph $G^C_k$ and the $k(i)$-colorable complement graph $G^{C}_{k(i)}$ of a graph $G$ and its relationship with other graph theoretic parameters are explored.

9.1 Introduction

In 1998, Sampathkumar et al., [90] and [91] was initiate the concept of generalized complement of a graph as follows. Let $G$ be a graph and $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V$. Then $k$-complement $G^P_k$ and $k(i)$-complement $G^{P}_{k(i)}$ (with respect to $P$) are defined as follows: For all $V_l$ and $V_j$, $l \neq j$, remove the edges between $V_l$ and $V_j$, and add the edges which are not in $G$ between $V_l$ and $V_j$. The graph $G^P_k$ thus obtained is called the $k$-complement of a graph $G$ with respect to $P$. Similarly, the $k(i)$-complement of $G^{P}_{k(i)}$ of a graph $G$ is obtained by removing the edges in $\langle V_l \rangle$ and $\langle V_j \rangle$ and adding the missing edges in them for $l \neq j$. The graph $G^c$ obtained is the complement graph of a graph $G$ and the graph $G$ is self-complementary if $G \cong G^c$.

Several paper written on the subject of $k$-switchings of a graphs, which is analogous
to the generalized complement of a graph. For more detail on these concepts we refer
[1], [18], [44], [92] and [101].

A graph is said to be $k$-vertex colorable (or $k$-colorable) if it is possible to
assign one color from a set of $k$-colors to each vertex such that no two adjacent
vertices have the same color. The set of all vertices with any one color is independent
and is called a color class. The $k$-coloring of a graph $G$ uses $k$-colors: it there
by partitions $V$ into $k$-color classes. The chromatic number $\chi(G)$ is defined as the
minimum $k$ for which $G$ has an $k$-coloring. Hence, graph $G$ is a $k$-colorable if and
only if $\chi(G) \leq k$, see [62] and [85].

Analogously, we define the theory of colorings in generalized complement graphs
in terms of the $k$-colorable complement graph $G_k^C$ and the $k(i)$-colorable comple-
ment graph $G_{k(i)}^C$ of a graph $G$.

### 9.2 $k$-Colorable Complement

Let $G = (V, E)$ be a graph. If there exists a $k$-coloring of a graph $G$ if and only
if $V(G)$ can be partitioned into $k$ subsets $C_1, C_2, \ldots, C_k$ such that no two vertices in
color classes of $C_t$, $t = 1, 2, \ldots, k$, are adjacent. Then, we have the following:

The $k$-colorable complement graph $G_k^C$ (with respect to $C$) of a graph $G$ is
obtained by for every $C_i$ and $C_j$, $i \neq j$, remove the edges between $C_i$ and $C_j$ in $G$,
and add the edges which are not in a graph $G$.

The graph $G$ is $k$-self colorable complement graph, if $G \cong G_k^C$.

The graph $G$ is $k$-co-$\infty$ self colorable complement graph, if $G^C \cong G_k^C$. 
Lemma 9.2.1. Let $G$ be a $k$-colorable graph. Then in any $k$-coloring of $G$, the subgraph induced by the union of any two color classes is connected.

Proof. If possible, let $C_1$ and $C_2$ be two color classes of vertex set $V(G)$ such that the subgraph induced by $C_1 \cup C_2$ is disconnected. Let $G_1$ be a component of the subgraph induced by $C_1 \cup C_2$. Obviously, no vertex of $C_1$ is adjacent to a vertex in $V(G) - V(G_1)$, which assigns the color either $C_1$ or $C_2$. Thus interchanging the colors of the vertices in $G_1$ and retaining the original colors for all other vertices, we get a different $k$-coloring of a graph $G$, which is a contradiction. \(\square\)

Theorem 9.2.2. Let $G$ be a $(n, m)$-graph. If for every $C_l$ and $C_j$, $l \neq j$, and each vertex of $C_l$ is adjacent to each vertex of $C_j$, then $m(G_k^C) = \emptyset$.

Proof. If for every $C_l$ and $C_j$, $l \neq j$ in a $(n, m)$-graph with $(C_k)$ is totally disconnected, where $C_k$ is the partition of color classes of vertex set $V(G)$, then by the definition of $k$-colorable complement, $m(G_k^C) = \emptyset$ follows. Conversely, suppose the given condition is not satisfied, then there exist at least two vertices $u$ and $v$ such that $u \in C_l$ is not adjacent to vertex $u \in C_j$ with $l \neq j$. Thus by above lemma, this implies that $m(G_k^C) \geq 1$, which is a contradiction. \(\square\)

A graph that can be decomposed into two partite sets but not is fewer, bipartite; three sets but not fewer, tripartite; $k$-sets but not fewer, $k$-partite; and an unknown number of sets, multipartite. A 1-partite graph is same as an independent set, or an empty graph. A 2-partite graph is same as a bipartite graph. A graph that can be decomposed into $k$-partite sets is also said to be $k$-colorable. That is $\chi(K_n) = n$. 
but the chromatic number of complete $k$-partite graph $\chi(K_{r_1,r_2,r_3,\ldots,r_k}) = k < n$ for $r_t > 2$, where $t = 1, 2, \ldots, k$. By virtue of the facts, we have following corollaries.

**Corollary 9.2.3.** Let $G$ be a complete graph $K_n$; $n \geq 1$ vertices and $m = \binom{n}{2}$ edges with $\chi(K_n) = n$. Then $m(G^c_n) = \emptyset$.

**Corollary 9.2.4.** Let $G$ be a complete bipartite graph $K_{r_1,r_2}$; $1 \leq r_1 \leq r_2$, with $\chi(K_{r_1,r_2}) = 2$ for $n = (r_1 + r_2)$ vertices and $m = (r_1r_2)$ edges. Then $m(G^c_2) = \emptyset$.

**Theorem 9.2.5.** Let $G$ be a path $P_n$ with $\chi(P_n) = 2$; $n \geq 2$ vertices. Then

$$m(G^c_2) = \begin{cases} \frac{1}{4}(n-2)^2 & \text{if } n \text{ is even} \\ \frac{1}{4}(n-1)(n-3) & \text{if } n \text{ is odd}. \end{cases}$$

**Proof.** Let $G$ be a path $P_n$ with $\chi(P_n) = 2$; $n \geq 2$ vertices, and $C = \{C_1, C_2\}$ be a partition of colorable class of vertex set of $P_n$. We have the following cases:

**Case 1.** If $\{u_1, u_2, \ldots, u_{t-1}, u_t\} \in C_1$ and $\{v_1, v_2, \ldots, v_{t-1}, v_t\} \in C_2$ with $v_1 - v_t$ is path of even length. Then $u_1, u_2, \ldots, u_{t-1}$ are adjacent $(t-2)$-vertices, that is $deg(u_i) = (t-2)$ if $1 \leq i \leq t-1$. Similarly, $v_1, v_2, \ldots, v_t$ are adjacent to $(t-2)$-vertices that is $deg(u_i) = (t-2)$ if $2 \leq i \leq t-1$, and $v_1$ and $u_t$ are adjacent to $(t-1)$-vertices in $G^c_2$. Thus, $2(t-1) + (n-2)(t-2) = 2m(G^c_2)$. By Theorem (1.1.1), with the fact that $n = 2t$ and $m(G) = n - 1$. Hence $m(G^c_2) = \frac{1}{4}(n-2)^2$.

**Case 2.** If $\{u_1, u_2, \ldots, u_{t-1}, u_t\} \in C_1$ and $\{v_1, v_2, \ldots, v_t, v_{t+1}\} \in C_2$ with $v_1 - v_{t+1}$ is path of even length. Then $u_1, u_2, \ldots, u_t$ are adjacent $(t-1)$-vertices, $v_2, v_3, \ldots, v_t$ are adjacent to $(t-2)$-vertices and, $v_1$ and $u_{t-1}$ are adjacent to $(t-1)$-vertices in
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$G_2^C$. Thus, $t(t - 1) + (t - 1)(t - 2) + 2(t - 1) = 2m(G_2^C)$. Theorem (1.1.1), with the fact that $n = 2t + 1$ and $m(G) = n - 1$. Hence $m(G_2^C) = \frac{1}{4}(n - 1)(n - 3)$.

\[\Box\]

**Theorem 9.2.6.** Let $G$ be a cycle $C_n$; $n \geq 3$ vertices. Then

(i) $m(G_2^C) = \frac{(n - 4)n}{4}$, if $\chi(C_n) = 2$ and $n$ is even.

(ii) $m(G_3^C) = \frac{(n + 1)(n - 3)}{4}$, if $\chi(C_n) = 3$ and exactly one vertex is contain in any one colorable class of a vertex partition set of an odd cycle $C_n$.

*Proof.* By Theorem (9.2.5), with an even cycle of $C_n$, and an exactly one vertex is contain in any one colorable class of a vertex partition set of an odd cycle $C_n$. Thus the results (i) and (ii) follows. \[\Box\]

**Theorem 9.2.7.** Let $G$ be a Wheel graph $W_n$; $n \geq 4$ vertices and $m = 2(n - 1)$ edges. Then

(i) $m(G_4^C) = \frac{(n - 4)n}{4}$, if $\chi(C_n) = 4$ and $n$ is even.

(ii) $m(G_5^C) = \frac{(n + 1)(n - 3)}{4}$, if $\chi(W_n) = 3$ and exactly one vertex is contain in any one colorable class of a vertex partition set of an odd cycle $C_{n-1}$ of $W_n$.

*Proof.* By Theorem (9.2.6) and $m(K_1) = 0$ due to the fact of $W_n = K_1 + C_{n-1}$, the result follows. \[\Box\]

**Theorem 9.2.8.** Let $T$ be a nontrivial tree with $\chi(T) = 2$. Then

$m(G_2^C) = (r_1, r_2) - n(T) + 1$. 
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Proof. Let $C = \{C_1, C_2\}$ be a partition of colorable class of a tree $T$ with $n \geq 2$ vertices and $m(T) = n(T) - 1$. If every vertex in $C_1$ is adjacent to every vertex in $C_2$, that is $K_{r_1,r_2}$ with $m(K_{r_1,r_2}) = r_1 \cdot r_2$. By definition of $G^C_k$ with $\chi(T) = 2$, we have $m(G^C_2) = m(K_{r_1,r_2}) - m(T)$. Thus the results follows. □

**Theorem 9.2.9.** For any nontrivial graph $G$ is $k$-self colorable complement if and only if $G \cong P_7$ or $2K_2$.

Proof. By definition of $k$-self colorable complement. It is clear that both $G$ and $G^C_2$ are isomorphic to $P_7$ or $2K_2$ with $\chi(P_7) = \chi(2K_2) = 2$. On the other hand, suppose $G$ is $k$-self colorable complement, when $G$ is not isomorphic with $P_7$ or $2K_2$. Then there exist at least two adjacent vertices $u$ and $v$ in $G$ such that $u \in C_1$ and $v \in C_2$ are in disjoint color classes of $C = \{C_1, C_2\}$ with $\chi(P_7) = \chi(2K_2) = 2$. This implies that, $u$ and $v$ are not adjacent in $G^C_2$ or they are in one color classes in $G^C_1$, that is totally disconnected graph. Thus the graph $G$ and its colorable complements $G^C_k$ are not isomorphic to each other, which is a contradiction. Hence the results follows. □

**Theorem 9.2.10.** Let $G$ be a $k$-self colorable complement graph. Then $G$ has a vertex of degree at least $\frac{n(\chi(G) - 1)}{2\chi(G)}$.

Proof. Let $G$ be a $(n,m)$-graph with $G \cong G^C_k$ and $C = \{C_1, C_2, \ldots, C_k\}$ be a partition of color classes of a vertex set $V(G)$. Suppose, if $\chi(G) = k$ and $V(G)$ is partitioned into $k$ independent sets $C_1, C_2, \ldots, C_k$. Thus, $n = |V(G)| = |C_1, C_2, \ldots, C_k| = \sum_{i=1}^{k}|V(G)| \leq k\beta(G)$, where $\beta(G)$ is the independence number of a graph $G$. Therefore $\chi(G) = k = \frac{n}{\beta(G)}$. Also, suppose $v \in C_i$, where $C_i$ is a colorable set in $C$
with at most $\frac{n}{\chi(G)}$. Then the sum of the degree of $v$ in $G$ and $G^C_k$ is greater than $\frac{n(\chi(G) - 1)}{\chi(G)}$. This implies that the degree of $v$ is at least $\frac{1}{2} \left( n - \frac{n}{\chi(G)} \right)$. Hence the result follows.

**Theorem 9.2.11.** Let $G$ be a $k$ - self colorable complement graph. Then

$$\frac{(k-1)(2n-k)}{4} \leq m(G) \leq \frac{2n(n-k)+k(k-1)}{4}. $$

**Proof.** Let $G$ be a $k$ - self colorable complement graph and $C = \{C_1, C_2, \ldots, C_k\}$ be a partition of color classes of a vertex set $V(G)$. If $|C_t| = n_t$ for $1 \leq t \leq k$, then the total number of edges between $C_l$ and $C_j$ in $C$, $l \neq j$, in both the graph $G$ and its colorable complement graph $G^C_k$ is $\sum_{l \neq j} n_t n_j$. Since the graph $G$ is $k$ - self colorable complement graph $G^C_k$, half of these edges are not there in $G$. Hence $m(G) \leq \binom{n}{2} - \sum_{l \neq j} n_t n_j$. Clearly, $\sum_{l \neq j} n_t n_j$ is minimum, when $n_t = 1$ for $k - 1$ of the indices. Thus, we have

$$m(G) \leq \binom{n}{2} - \frac{1}{2} \left[ \binom{k-1}{2} + (k-1)(n-k+1) \right].$$

Hence the upper bound follow. To establish the lower bound, the graph $G$ being $k$ - self colorable complement has at least $\sum_{l \neq j} n_t n_j$ - edges. So,

$$\frac{1}{2} \left[ \binom{k-1}{2} + (k-1)(n-k+1) \right] \leq m(G)$$

and the result follows.

**Theorem 9.2.12.** For any nontrivial graph $G$ is $k$ - co - self colorable complement if and only if $G \cong K_n$. 
Proof. On contrary, suppose given condition is not satisfied, then there exists at least three vertices $u, v$ and $w$ such that $v$ is adjacent to both $u$ and $w$, and $u$ is not adjacent to $w$. This implies that an edge $e = uvw \in G^c$ and induced subgraph $\langle u, v, w \rangle$ in $G^c_2$ is totally disconnected. Thus $E(G^c_2) \subset E(G^c)$, which is a contradiction to the fact of $G^c \cong G^c_n$ with $\chi(K_n) = n$. Converse is obvious. Thus the result proves. □

9.3 \( k(i) \) - Colorable Complement

Let $G = (V, E)$ be a graph and $C = \{C_1, C_2, \ldots, C_k\}$ be a partition of color classes of a vertex set $V(G)$. Then, we have the following:

The $k(i)$ - colorable complement graph $G^c_{k(i)}$ (with respect to $C$) of a graph $G$ is obtained by removing the edges in $\langle C_l \rangle$ and $\langle C_j \rangle$ and adding the missing edges in them for $l \neq j$.

The graph $G$ is $k(i)$ - self colorable complement graph, if $G \cong G^c_{k(i)}$.

The graph $G$ is $k(i)$ - co - self colorable complement graph, if $G^c \cong G^c_{k(i)}$.

Theorem 9.3.1. For any graph $G$, $m(G^c_{k(i)}) = \binom{n}{2}$. if and only if the graph $G$ is isomorphic with complete $n$ - partite graph $K_{r_1, r_2, r_3, \ldots, r_4}$ or $(K_n)^c$.

Proof. To prove the necessity, we use the mathematical induction. Let $G$ be a graph with $n = 1$ vertex. Then $\chi(G) = 1$ and $m(G^c_{k(i)}) = \emptyset$. Hence the result follows.

Suppose the graph $G$ with $n > 1$ vertices. Then the following cases arises:
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**Case 1.** If $G$ is totally disconnected, that is $(K_n)^c$, complement of a complete graph $K_n$, then $G$ has only one color class $C_1$ with $\chi((K_n)^c) = 1$. By the definition of $G_{1(i)}^C$, the induced subgraph of $\langle C_1 \rangle$ is complete, which form a $\binom{n}{2}$ - edges.

**Case 2.** If the graph $G$ is complete $t$ - partite graph $K_{r_1,r_2,r_3,...,r_t}$, then for every two color classes $C_l$ and $C_j$ for $l \neq j$, and each vertex $C_l$ adjacent to each vertex of $C_j$ in complete $t$ - partite graph $K_{r_1,r_2,r_3,...,r_t}$ with $m(K_{r_1,r_2,r_3,...,r_t}) = r_1r_2r_3 \hdots r_t$. By the definition of $G_{n(i)}^C$ with $G = K_{r_1,r_2,r_3,...,r_t}$, we have

$$m(G_{n(i)}^C) = \binom{r_1}{2} + \binom{r_2}{2} + \hdots + \binom{r_t}{2} + r_1r_2r_3 \hdots r_t,$$

where $\binom{r_s}{2}$ is the maximum number edges of induced subgraph $\langle C_s \rangle$ if $s = 1, 2, \ldots, t$ which are complete. This form a $\binom{n}{2}$ - edges.

Conversely, suppose the graph $G$ is not isomorphic to complete $n$ - partite graph $K_{r_1,r_2,r_3,...,r_n}$ or $(K_n)^c$. Then there exists at least three vertices $\{x, y, z\}$ such that at least two adjacent vertices $x$ and $y$ are not adjacent to isolated vertex $z$. By the definition of $G_{k(i)}^C$ with $\chi(G) = k \geq 2$, which form a path $(x - y - z)$ or $(y - x - z)$ of length 2, which is not a complete, a contradiction. This proves the sufficiency.

\[\square\]

**Theorem 9.3.2.** Let $G$ be a path $P_n$ with $\chi(P_n) = 2$; $n \geq 2$ vertices. Then

$$m(G_{2(i)}^C) = \begin{cases} 
\frac{1}{4}[n^2 + 2n - 4]^2 & \text{if } n \text{ is even} \\
\frac{1}{4}(n-1)(n+3) & \text{if } n \text{ is odd}
\end{cases}$$
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Proof. Let $G$ be a path $P_n$ with $\chi(P_n) = 2$; $n \geq 2$ vertices, and $C = \{C_1, C_2\}$ be a partition of colorable class of vertex set of $P_n$. We have the following cases:

Case 1. If $\{u_1, u_2, \ldots, u_{t-1}, u_t\} \in C_1$ and $\{v_1, v_2, \ldots, v_{t-1}, v_t\} \in C_2$ with $v_1 - u_t$ is path of even length. Then $\langle C_1 \rangle$ and $\langle C_2 \rangle$ are complete in $G_{2(i)}^C$ and also $v_1 - u_t$ path have $(n - 1)$ - edges in both the graph $G$ and its $k(i)$ - colorable complement graph $G_{2(i)}^C$. Thus, $m(G) + t(t - 1) = (n - 1) + \frac{n(n - 2)}{4} = m(C_{2(i)}^C)$ and this implies $m(C_{2(i)}^C) = \frac{1}{4}[n^2 + 2n - 4]^2$.

Case 2. If $\{u_1, u_2, \ldots, u_{t-1}, u_t\} \in C_1$ and $\{v_1, v_2, \ldots, v_{t-1}, v_t\} \in C_2$ with $v_1 - u_{t+1}$ is a path of odd length. Then $\langle C_1 \rangle$ and $\langle C_2 \rangle$ are complete in $G_{2(i)}^C$ and also $v_1 - u_{t+1}$ path have $(n - 1)$ - edges in both the graph $G$ and its $2(i)$ - colorable complement graph $G_{2(i)}^C$. Thus,

$$m(G) + \frac{t(t - 1)}{2} + \frac{t(t + 1)}{2} = (n - 1) \left[ 1 + \frac{n - 3}{8} + \frac{n + 1}{8} \right] = m(C_{2(i)}^C)$$

and this implies $m(C_{2(i)}^C) = \frac{1}{4}(n - 1)(n + 3)$. \qed

Theorem 9.3.3. Let $G$ be a cycle $C_n$; $n \geq 3$ vertices. Then

(i) $m(C_{2(i)}^C) = \frac{1}{4}[n(n + 2)]$, if $\chi(C_n) = 2$ and $n$ is even,

(ii) $m(C_{3(i)}^C) = \frac{1}{4}(n^2 + 3)$, if $\chi(C_n) = 3$ and exactly one vertex contains in any one colorable class of a vertex partition set of an odd cycle $C_n$.

Proof. By Theorem (9.3.2), with even cycle of $C_n$ and exactly one vertex contains in any one colorable class of a vertex partition set of an odd cycle $C_n$. Thus the result (i) and (ii) follows. \qed
Theorem 9.3.4. Let $T$ be a nontrivial tree with $\chi(T) = 2$. If $C = \{C_1, C_2\}$ be a partition of colorable class of a tree $T$, then

$$m(G^C_{2(i)}) = \frac{1}{2}[r^2 + s^2 + n - 2],$$

where $|C_1| = r$ and $|C_2| = s$.

Proof. Let $C = \{C_1, C_2\}$ be a partition of colorable class of a tree $T$ with $\chi(T) = 2$ and $m(T) = n(T) - 1 = r + s + 1$. Then by definition of $G^C_{k(i)}$, we have $\langle C_1 \rangle$ and $\langle C_2 \rangle$ are complete. Therefore, $m(C_1) = \binom{r}{2}$ and $m(C_2) = \binom{s}{2}$. Thus, we have

$$m(G^C_{2(i)}) = \binom{r}{2} + \binom{s}{2} + m(T) = \frac{1}{2}[r(r + 1) + s(s + 1) - 2].$$

Hence the result follows.

Theorem 9.3.5. For any nontrivial graph $G$ is $k(i)$ - self colorable complement if and only if $G$ is isomorphic with $K_n$.

Proof. Let $G = K_n$ be a complete graph with $\chi(G) = n$. Then by the definition of $G^C_{k(i)}$, the induced subgraph $\langle C_t \rangle$ for $t = 1, 2, \ldots, n$ are connected and $|C_t| = 1$ for $t = 1, 2, \ldots, n$. Thus $G^C_{n(i)} \cong K_n$ and the result follows. Conversely, suppose given condition is not satisfied, then there exists at least two nonadjacent vertices $u$ and $v$ in $G$ such that $\chi(G) = 1$ and $m(G) = \emptyset$. By the definition of $G^C_{k(i)}$, we have $\chi(G^C_{1(i)}) = 2$ with an induced subgraph $\langle u, v \rangle$ in $G^C_{1(i)}$ is connected. Thus $m(G) < m(G^C_{1(i)})$, which is a contradiction to the fact of $G$ is isomorphic to $G^C_{k(i)}$. \qed
Here, we show the $G, G^p_k, G^p_{k(i)}$ - Realizability for some graph theoretic parameter.

Let $G$ be a graph. Then $S \subseteq V(G)$ is a separating set if $G - S$ has more than one component. The connectivity $\kappa(G)$ of $G$ is the minimum size of $S \subseteq V(G)$ such that $G - S$ is disconnected or has a single vertex. For any $k \leq \kappa(G)$, we say that $G$ is $k$ - connected. Then, we have

**Observation 9.3.6.** Let $G$ be a graph with $C = \{C_1, C_2\}$ be a partition of colorable class of a vertex set $V$. If $(C_1)$ and $(C_2)$ are $(t-1)$ - colorable with $\text{Max.} \{\chi(G^C_k), \chi(G^C_{k_i})\}$ \[ \geq t, \] then $\text{Min.} \{k(G), k(G^C_k), k(G^C_{k_i})\}$ has at least $(t - 1)$ - edges.

**Theorem 9.3.7.** Let $G$ be a $(n, m)$ - graph. Then

(i) $\chi(G^C_k) = 1$ if and only if the graph $G \cong K_n$ or $(K_n)^c$ or $K_{r_1, r_2, r_3, \ldots, r_k}$,

(ii) $\chi(G^C_{k(i)}) = n$ if and only if the graph $G \cong K_n$ or $(K_n)^c$ or $K_{r_1, r_2, r_3, \ldots, r_k}$.

**Proof.** By the definition of $G^C_k$ and Theorem (9.2.2), (i) follows. Also by the definition of $G^C_{k(i)}$ and Theorem (9.3.1), (ii) follows. \qed

By the definition of independence number $\beta(G)$ and clique number $\omega(G)$ with due to the fact of $\beta(G) = k$ if and only if $\omega(G^c) = k$. Then we have the following results without proof, which are straightforward.

**Theorem 9.3.8.** Let $G$ be a nontrivial graph. Then

(i) $\beta(G^C_{k(i)}) \leq \beta(G) \leq \beta(G^C_k)$,

(ii) $\omega(G^C_k) \leq \omega(G) \leq \omega(G^C_{k_i})$. 