CHAPTER 3

Prime Fuzzy Ideals in N-Groups

In this chapter we consider the fuzzy ideals of N-group G where N is a near-ring. We introduce the concept: fuzzy prime ideal; and it is obtained that a non-constant fuzzy ideal μ of G is a prime fuzzy ideal if and only if for any $t \in [0, \mu(0)]$, either $\mu_t = G$ or μ_t is a prime fuzzy ideal of G. If μ is a prime fuzzy ideal, then the N-group G/μ is a prime ideal and converse holds if $|\text{Im}\ \mu| = 2$. Some examples related to these concepts were illustrated.

This chapter is divided into three sections.

In section-1, we introduced the prime fuzzy ideal of an N-group. We obtained the equivalent conditions of prime fuzzy ideal in terms of generalized characteristic function. Also we characterized the prime fuzzy ideal in terms of level ideals.

In section-2, we observed the image and inverse image of fuzzy ideals of N-groups under homomorphism. We provided some related examples.

In section-3, we considered the quotient N-group G/μ of a given fuzzy ideal μ. We proved that if μ is a prime fuzzy ideal of G, then G/μ is a prime N-group, conversely, if μ is a fuzzy ideal of G such that $|\text{Im}\ \mu| = 2$ and G/μ is a prime N-group, then μ is a prime fuzzy ideal of G.

Section-3.1: Prime Fuzzy Ideals

Now we start this section with the definition 'prime ideal' of an N-group.

3.1.1 Definition: A proper ideal I of an N-group G is said to be a *prime ideal* if for any $0 \neq n \in N$ and $g \notin I$, we have $ng \notin I$. G is said to be a *prime N-group* if the ideal (0) is a prime ideal of G.

3.1.2 Theorem: If I is an ideal of G, then G/I is a non-zero prime N-group $\Leftrightarrow$ I is a prime ideal of G.

Proof: Suppose G/I is a non-zero prime N-group.

Then the zero ideal $\{0 + I\}$ is a prime ideal of G/I.

Since G/I is non-zero N-group, it follows that I is a proper ideal of G.

Let $0 \neq n \in N$, $g \in G$ such that $ng \in I$.

Since $\{0 + I\}$ is a prime ideal of G and $n(g + I) = ng + I = 0 + I \in \{0 + I\}$, it follows that $g + I \in \{0 + I\}$ which implies $g \in I$.

Hence I is a prime ideal of G.

Conversely, suppose that I is a prime ideal of G.

Since $I \neq G$, it follows that G/I is a non-zero N-group.

Clearly $\{0 + I\}$ is a proper ideal of G/I.

Let $0 \neq n \in N$ and $g + I \in G/I$ such that $n(g + I) \in \{0 + I\}$.

Then $ng + I = 0 + I$ and so $ng \in I \Rightarrow g \in I$ (since I is a prime ideal of G and $n \neq 0$) $\Rightarrow g + I \in \{0 + I\}$.

Therefore the zero ideal $\{0 + I\}$ is a prime ideal of G/I and so G/I is a non-zero prime N-group.

We now introduce the concept "prime fuzzy ideal" of an N-group.

3.1.3 Definition: A non-constant fuzzy ideal $\mu: G \to [0, 1]$ is said to be a *prime fuzzy ideal* of G if $\mu(ng) \leq \mu(g)$ for all $0 \neq n \in N$, $g \in G$.

3.1.4 Note: Let μ be a fuzzy ideal of G. Then μ is a prime fuzzy ideal of G $\Leftrightarrow$ $\mu(ng) = \mu(g)$ for all $0 \neq n \in N$ and $g \in G$. This fact follows directly from the definitions of "fuzzy ideal" and "prime fuzzy ideal" (Please refer Note 2.1.7).

3.1.5 Example: If R is a division ring, then every non-constant fuzzy ideal of G is a prime fuzzy ideal

[**Verification**: Since every non-zero element of a division ring possess inverse, it follows from Proposition 2.1.3(i)].

3.1.6 Remark: Let G be an N-group and define $\mu: G \to [0, 1]$ by $\mu(x) = 0.5$ for all $x \in G$. Then μ is a constant fuzzy ideal. Hence μ is not a prime fuzzy ideal of G.

3.1.7 Theorem: For an ideal I of G the following three conditions are equivalent:

(i) I is a prime ideal of G;

(ii) The generalized characteristic function μ of I defined by

$\mu(x) = \begin{cases} \beta & \text{if } x \in I \\ \alpha & \text{otherwise} \end{cases}$ is a prime fuzzy ideal; and

(iii) The characteristic function χ_I of I is a prime fuzzy ideal of G.

Proof: By Corollary 2.1.17, we have that χ_I is a fuzzy ideal of G $\Leftrightarrow$ I is an ideal of G for any subset I of G.

(ii) $\Rightarrow$ (i): Suppose μ is a prime fuzzy ideal of G

Since μ is a non-constant fuzzy ideal of G, it follows that I is a proper ideal of G.

Let $0 \neq n \in N$ and $g \notin I$.

Since $g \notin I$ we have that $\mu(g) = \alpha$.

Since μ is a prime fuzzy ideal, we have $\mu(ng) = \mu(g) = \alpha$, which implies $ng \notin I$.

This shows that I is a prime ideal of G.

(i) $\Rightarrow$ (ii): Suppose that I is a prime ideal of G.

Then I is a proper ideal of G and so μ is a non-constant fuzzy ideal of G.

Let $0 \neq n \in N$ and $g \in G$.

Suppose $g \notin I$. Then $\mu(g) = \alpha$.

Since I is a prime ideal of G, we have $ng \notin I$ and so $\mu(ng) = \alpha$.

So we get that $\mu(ng) = \mu(g)$.

Suppose $g \in I$. Then $\mu(g) = \beta$.

So $\mu(ng) \geq \mu(g) = \beta$.

This shows that $\mu(ng) = \beta$.

Therefore $\mu(ng) = \mu(g)$.

This shows that μ is a prime fuzzy ideal of G.

In the above proof, by taking $\beta = 1$ and $\alpha = 0$ we get (i) $\Leftrightarrow$ (iii)

3.1.8 Theorem: Let μ be a non-constant fuzzy ideal of G. Then μ is a prime fuzzy ideal of G $\Leftrightarrow$ for any $t \in [0, \mu(0)]$, either $\mu_t = G$ or μ_t is a prime ideal of G.

Proof: Suppose μ is a prime fuzzy ideal of G.

Let $t \in [0, \mu(0)]$.

If $\mu_t = G$, then there is nothing to prove.

Suppose $\mu_t \neq G$.

Since μ is a fuzzy ideal of G, by Remark 2.1.10 we have μ_t is a ideal of G.

Since $\mu_t \neq G$, we have that μ_t is a proper ideal of G.

Let $g \notin \mu_t$ and $0 \neq n \in N$.

Then $\mu(g) < t$.

Since μ is a prime fuzzy ideal, it follows that $\mu(ng) = \mu(g) < t$.

This implies $ng \notin \mu_t$.

Thus we proved that if $g \notin \mu_t$, then $ng \notin \mu_t$ for all $0 \neq n \in N$.

This shows that μ_t is a prime ideal of G.

Conversely, suppose that for any $t \in [0, \mu(0)]$, either $\mu_t = G$ or μ_t is a prime ideal of G.

Let $0 \neq n \in N$ and $g \in G$

Suppose that $\mu(ng) = t$.

Then $ng \in \mu_t$.

If $\mu_t = G$, then $g \in \mu_t$.

If $\mu_t \neq G$, then μ_t is a prime ideal of G.

So $g \in \mu_t$.

Therefore in any case, $g \in \mu_t$, which implies $\mu(g) \geq t = \mu(ng)$.

So $\mu(ng) = \mu(g)$ for all $0 \neq n \in N$, $g \in G$.

This shows that μ is a prime fuzzy ideal of G.

3.1.9 Note: Let us observe the statement of the Theorem 3.1.8. The condition "μ is non-constant" in the statement is essential. Define μ: G → [0, 1] by $\mu(x) = 0.5$ for all $x \in G$. Then $\mu_t = G$ for all $t \in [0, \mu(0)]$. So the statement "for any $t \in [0, \mu(0)]$, either $\mu_t = G$ or μ_t is a prime ideal of G" is true. But this fuzzy ideal μ cannot be a prime fuzzy ideal.

Section-3.2: Prime Fuzzy Ideals and N-group Homomorphism

3.2.1 Definition: Let S be a set, μ a fuzzy subset of S and f: S → S is a function. Then μ is said to be *f-invariant* if it satisfies the condition:

$$x, y \in S,\ f(x) = f(y) \Rightarrow \mu(x) = \mu(y).$$

3.2.2 Example: Define f: Z → Z by $f(x) = \begin{cases} 1 & \text{if } x \in 2Z \\ 0.5 & \text{if } x \notin 2Z \text{ and } x > 0 \\ 0 & \text{if } x \notin 2Z \text{ and } x < 0 \end{cases}$

Define μ: Z → [0, 1] by $\mu(x) = \begin{cases} 0.2 & \text{if } x \in 2Z \\ 0.4 & \text{if } x \notin 2Z \end{cases}$. Then μ is f-invariant.

3.2.3 Example: Write G = Z and $G^1 = Z_6$. Then G and G^1 are N-groups over Z, the ring of integers.

Let f: G → G^1 be the canonical N-group epimorphism.

Define μ: G → [0, 1] by $\mu(x) = \chi_{2Z}(x)$ for all $x \in G$.

Note that χ_{2Z} is the characteristic function of 2Z.

By Corollary 2.1.17, μ is a fuzzy ideal of G.

Now we verify that μ is f-invariant.

Let x, y ∈ Z be such that f(x) = f(y).

Then x - y is divisible by 6 and so (x - y) is divisible by 2.

If x is even, then y is even and so μ(x) = μ(y) = 1.

If x is odd, then y is odd, and so μ(x) = μ(y) = 0.

Therefore f(x) = f(y) ⇒ μ(x) = μ(y).

This shows that μ is f-invariant.

3.2.4 Note: There exists a non-constant fuzzy ideal μ, and a non-constant N-group epimorphism f, such that f(μ) is a constant fuzzy ideal. For this, we observe the following example.

3.2.5 Example: Write Z = the set of integers and Z_6 = {0, 1, 2, 3, 4, 5}, the set of integers modulo '6'.

Then G = Z and $G^1 = Z_6$ are N-groups over the near-ring N = Z of integers.

Let f: Z → Z_6 be the natural epimorphism of N-group.

It is clear that f is a non-constant N-group homomorphism.

Let α, β ∈ [0, 1] be such that β > α.

Define μ: Z → [0, 1] by $\mu(x) = \begin{cases} \beta & \text{if } x \in 5Z \\ \alpha & \text{otherwise} \end{cases}$.

Then μ is a generalized characteristic function and it is clear that μ is a non-constant fuzzy subset of G.

By Corollary 2.1.17, μ is a fuzzy ideal of G.

Now we verify that $f(\mu)$ is a constant fuzzy ideal of $G^{\cdot} = Z_6$.

$f(\mu)(0) = \sup\{\mu(x) / x \in f^{-1}(0)\} \geq \mu(0)$ (since $0 \in f^{-1}(0)$) $= \beta$ (since $\mu(0) = \beta$).

Therefore $f(\mu)(0) = \beta$.

Now $f(\mu)(1) = \sup\{\mu(x) / x \in f^{-1}(1)\} \geq \mu(25)$ (since $25 \in f^{-1}(1)$) $= \beta$ (since $\mu(25) = \beta$).

Therefore $f(\mu)(1) = \beta$.

In a similar way, we can verify that $f(\mu)(x) = \beta$ for all x.

3.2.6 Theorem: Suppose G and G^1 are two N-groups, $f: G \rightarrow G^1$ is a non-constant N-group epimorphism and $\mu: G \rightarrow [0, 1]$ is an f-invariant fuzzy ideal of G. If μ is a prime fuzzy ideal of G, then $f(\mu)$ is either constant or a prime fuzzy ideal of G^1.

Proof: Suppose μ is a prime fuzzy ideal of G.

By Proposition 2.3.3, $f(\mu)$ is a fuzzy ideal of G^1.

If $f(\mu)$ is constant, then there is nothing to prove.

Suppose $f(\mu)$ is non-constant.

Now we show that $f(\mu)$ is a prime fuzzy ideal.

Let $0 \neq n \in N$ and $g^1 \in G^1$.

Since f is an epimorphism, there exits $g \in G$ such that $f(g) = g^1$.

Since $f(ng) = nf(g) = ng^1$, we have that $ng \in f^{-1}(ng^1)$.

If $x_1, x_2 \in f^{-1}(ng^1)$, then $f(x_1) = f(x_2)$.

So $\mu(x_1) = \mu(x_2)$ (since μ is f-invariant).

This shows that $\{\mu(x) / x \in f^{-1}(ng^1)\} = \{\mu(ng)\}$ ....(i).

Since $f(g) = g^1$ we have $g \in f^{-1}(g^1)$.

If $y_1, y_2 \in f^{-1}(g^1)$, then $f(y_1) = f(y_2)$ and so $\mu(y_1) = \mu(y_2)$ (since μ is f-invariant).

This shows that $\{\mu(y) / y \in f^{-1}(g^1)\} = \{\mu(g)\}$................(ii).

Now $(f(\mu))(ng^1) = \sup\{\mu(x) / x \in f^{-1}(ng^1)\} = \sup\{\mu(ng)\}$ (by (i))

$= \mu(ng) = \mu(g)$ (since μ is a prime fuzzy ideal)

$= \sup\{\mu(g)\} = \sup\{\mu(x) / x \in f^{-1}(g^1)\}$ (by (ii))

$= f(\mu)(g^1)$.

So $(f(\mu))(ng^1) = (f(\mu))(g^1)$ is true for all $0 \neq n \in N$ and $g^1 \in G^1$.

Hence $f(\mu)$ is a prime fuzzy ideal of G^1.

3.2.7 Note: There exists a non-constant fuzzy ideal μ^1 and a non-constant N-group homomorphism f, such that $f^{-1}(\mu^1)$ is a constant fuzzy ideal. For this, observe the following example.

3.2.8 Example: Write Z = the set of integers.

Then $G = Z$ and $G^1 = Z$ are N-groups over the near-ring $N = Z$ of integers.

Define $f: G \to G^1$ by $f(x) = 2x$.

Then f is a non-constant N-group homomorphism.

Let $\alpha, \beta \in [0, 1]$ be such that $\beta > \alpha$.

Define $\mu^1: G^1 \to [0, 1]$ by $\mu^1(x) = \begin{cases} \beta & \text{if } x \in 2Z \\ \alpha & \text{otherwise} \end{cases}$.

Then μ^1 is a generalized characteristic function and it is clear that μ^1 is a non-constant fuzzy subset of $G^1 = Z$.

By Corollary 2.1.17, μ^1 is a fuzzy ideal of G^1.

Now we verify that $f^{-1}(\mu^1)$ is a constant fuzzy ideal of $G = Z$.

For any $x \in G$, $f^{-1}(\mu^1)(x) = \mu^1(f(x)) = \mu^1(2x) = \beta$.

This shows that $f^{-1}(\mu^1)$ is constant.

Moreover, the only level set of $f^{-1}(\mu^1)$ is $f^{-1}(\mu^1)_\beta = 2Z$, an ideal of G.

So by Remark 2.1.10, $f^{-1}(\mu^1)$ is a fuzzy ideal of G.

Thus $f^{-1}(\mu^1)$ is a constant fuzzy ideal of G.

3.2.9 Theorem: Let G, G^1 be two N-groups, f: $G \rightarrow G^1$ a non-constant N-group homomorphism, and μ^1: $G^1 \rightarrow [0, 1]$ a fuzzy ideal of G^1. If μ^1 is a prime fuzzy ideal of G^1, then $f^{-1}(\mu^1)$: $G \rightarrow [0, 1]$ defined by $f^{-1}(\mu^1)(x) = \mu^1(f(x))$ is either constant or a prime fuzzy ideal of G.

Proof: Suppose μ^1 is a prime fuzzy ideal of G^1.

By Proposition 2.3.5, $f^{-1}(\mu^1)$ is a fuzzy ideal of G.

If $f^{-1}(\mu^1)$ is constant, then there is nothing to prove.

Suppose $f^{-1}(\mu^1)$ is non-constant.

Now we show that $f^{-1}(\mu^1)$ is a prime fuzzy ideal of G.

Let $g \in G$ and $0 \neq n \in N$.

Now $f^{-1}(\mu^1)(ng) = \mu^1(f(ng)) = \mu^1(nf(g))$ (since f is a N-group homomorphism)

$= \mu^1(f(g))$ (since μ^1 is a prime fuzzy ideal of G^1)

$= f^{-1}(\mu^1)(g)$.

This shows that $f^{-1}(\mu^1)$ is a prime fuzzy ideal of G.

Section-3.3: Fuzzy cosets of Prime Fuzzy Ideals

In chapter-2 we have discussed the fuzzy cosets of N-groups. In this chapter we consider the fuzzy cosets of prime fuzzy ideals.

3.3.1 Theorem: If μ is a prime fuzzy ideal of G, then G/μ is a prime N-group.

Proof: Suppose μ is a prime fuzzy ideal of G.

Since μ is non constant, it follows that $\{0 + \mu\}$ is a proper ideal of G/μ.

Let $0 \neq n \in N$, $g + \mu \in G/\mu$ be such that $n(g + \mu) = 0 + \mu$.

Now $ng + \mu = 0 + \mu$.

This implies $(ng + \mu)(0) = (0 + \mu)(0)$

$\Rightarrow \mu(ng - 0) = \mu(0)$

$\Rightarrow \mu(0) = \mu(ng) = \mu(g)$ (since μ is a prime fuzzy ideal)

$\Rightarrow g + \mu = 0 + \mu$ (by Proposition 2.1.23).

This shows that $\{0 + \mu\}$ is a prime ideal of G/μ.

3.3.2 Note: Suppose μ is a fuzzy ideal of G such that $|\text{Im } \mu| = 2$.

(i) Then μ is the generalized characteristic function of G_μ.

[**Verification**: Suppose Im $\mu = \{\alpha, \beta\}$ such that $\alpha > \beta$. Since $\mu(0) \geq \mu(x)$ for all $x \in G$, it follows that $\mu(0) = \alpha$].

(ii) Thus $\mu(x) = \begin{cases} \alpha & \text{if } x \in G_\mu \\ \beta & \text{otherwise} \end{cases}$. Hence μ is the generalized characteristic function.

(iii) If G_μ is a prime ideal of G, then by Theorem 3.1.7, μ is a prime fuzzy ideal of G.

3.3.3 Theorem: If μ is a fuzzy ideal of G such that $|\text{Im } \mu| = 2$ and G/μ is a prime N-group, then μ is a prime fuzzy ideal of G.

Proof: Let μ be a fuzzy ideal of G such that $|\text{Im } \mu| = 2$.

Suppose that G/μ is a prime N-group.

By Proposition 2.3.1, $G/\mu \cong G/G_\mu$ and so G/G_μ is a prime N-group

$\Rightarrow G_\mu$ is a prime ideal of G (by Theorem 3.1.2) $\Rightarrow \mu$ is a prime fuzzy ideal of G (by Theorem 3.1.7).

3.3.4 Theorem: Let μ be a prime fuzzy ideal of G. Then the fuzzy set $\theta_\mu: G/\mu \to [0, 1]$ (defined in Definition 2.2.1, as $\theta_\mu(x + \mu) = \mu(x)$) is a prime fuzzy ideal of G/μ.

Proof: Suppose μ is a prime fuzzy ideal of G.

By Theorem 2.2.2, θ_μ is a fuzzy ideal of G/μ.

Since μ is a prime fuzzy ideal, it is a non-constant fuzzy ideal of G, and so we have that θ_μ is a non-constant fuzzy ideal of G/μ.

Let $0 \neq n \in N$ and $g + \mu \in G/\mu$.

Now $\theta_\mu(n(g + \mu)) = \theta_\mu(ng + \mu)$

$= \mu(ng)$

$= \mu(g)$ (since μ is a prime fuzzy ideal)

$= \theta_\mu(g + \mu)$.

This shows that θ_μ is a prime fuzzy ideal of G/μ.

3.3.5 Theorem: Let μ and σ be two fuzzy ideals of G such that $\mu \subseteq \sigma$ with $\sigma(0) = \mu(0)$ and σ a prime fuzzy ideal of G. Then the fuzzy set h_σ: $G/\mu \rightarrow [0, 1]$ defined by $h_\sigma(x + \mu) = \sigma(x)$ for all $x + \mu \in G/\mu$, is a prime fuzzy ideal of G/μ. Moreover, $\theta_\mu \subseteq h_\sigma$ and $\theta_\mu(0) = h_\sigma(0)$.

Proof: Suppose σ is a prime fuzzy ideal of G.

By Corollary 2.2.3, h_σ is a fuzzy ideal of G/μ such that $\theta_\mu \subseteq h_\sigma$ and $\theta_\mu(0) = h_\sigma(0)$.

Since σ is a non-constant fuzzy ideal of G, we have h_σ is a non-constant fuzzy ideal of G/μ.

Let $x + \mu \in G/\mu$, $0 \neq n \in N$.

Now $h_\sigma(n(x + \mu)) = h_\sigma(nx + \mu)$

$= \sigma(nx)$ (by definition of h_σ)

$= \sigma(x)$ (since σ is a prime fuzzy ideal)

$= h_\sigma(x + \mu)$.

Therefore h_σ is a prime fuzzy ideal of G/μ.

3.3.6 Theorem: Let μ be a fuzzy ideal of G and θ a prime fuzzy ideal of G/μ such that $\theta_\mu \subseteq \theta$ and $\theta_\mu(0) = \theta(0)$. Then σ_0: $G \rightarrow [0, 1]$ (defined in Definition 2.2.1 by $\sigma_\theta(x) = \theta(x + \mu)$ for all $x \in G$) is a prime fuzzy ideal of G such that $\mu \subseteq \sigma_\theta$ and $\mu(0) = \sigma_\theta(0)$.

Proof: By Theorem 2.2.4, σ_θ is a fuzzy ideal of G such that $\mu \subseteq \sigma_\theta$ and $\mu(0) = \sigma_\theta(0)$.

Since θ is non-constant, we have σ_θ is a non-constant fuzzy ideal of G.

Now we show that σ_θ is a prime fuzzy ideal of G.

Let $0 \neq n \in N$ and $g \in G$.

Now $\sigma_\theta(ng) = \theta(ng + \mu)$

$= \theta(n(g + \mu))$

$= \theta(g + \mu)$ (since θ is a prime fuzzy ideal)

$= \sigma_\theta(g)$.

Therefore σ_θ is a prime fuzzy ideal of G.

3.3.7 Notation: Let μ be a fuzzy ideal of G.

We write $P = \{\sigma \ / \ \sigma$ is a prime fuzzy ideal of G, $\mu \subseteq \sigma$, $\sigma(0) = \mu(0)\}$

$Q = \{\theta \ / \ \theta$ is a prime fuzzy ideal of G/μ, $\theta_\mu \subseteq \theta$ and $\theta(0) = \theta_\mu(0)\}$.

3.3.8 Theorem: Let μ be a fuzzy ideal of G. Then there exists an order preserving bijective mapping between the sets P and Q.

Proof: Define $\eta: P \to Q$ by $\eta(\sigma) = h_\sigma$ where $h_\sigma: G/\mu \to [0, 1]$ by $h_\sigma(x + \mu) = \sigma(x)$ for $\sigma \in P$.

By Theorem 3.3.5, $\eta(\sigma) = h_\sigma$ is a prime fuzzy ideal of G/μ such that $\theta_\mu \subseteq h_\sigma$ and $\theta_\mu(0) = h_\sigma(0)$.

This implies $\eta(\sigma) = h_\sigma \in Q$.

Now $\sigma_1 = \sigma_2 \Rightarrow h_{\sigma_1} = h_{\sigma_2} \Rightarrow \eta(\sigma_1) = \eta(\sigma_2)$ and so η is well defined.

Let $\sigma, \beta \in P$ be such that $\eta(\sigma) = \eta(\beta)$.

Then $h_\sigma = h_\beta$.

Now $\sigma(x) = h_\sigma(x + \mu) = h_\beta(x + \mu) = \beta(x)$.

This is true for every $x \in G$, and that implies $\sigma = \beta$.

This shows that η is one-one.

Let $\theta \in Q$.

Define $\sigma_\theta: G \rightarrow [0, 1]$ by $\sigma_\theta(x) = \theta(x + \mu)$ for all $x \in G$.

Then by Theorem 3.3.6, $\sigma_\theta \in P$.

Now we verify that $\eta(\sigma_\theta) = \theta$.

For any $x + \mu \in G/\mu$, $(\eta(\sigma_\theta))(x + \mu) = h_{(\sigma_\theta)}(x + \mu) = \sigma_\theta(x) = \theta(x + \mu)$.

Therefore $\eta(\sigma_\theta) = \theta$. So η is onto.

This shows that $\eta: P \rightarrow Q$ is a bijection.

Let $\sigma, \beta \in P$ be such that $\sigma \subseteq \beta$.

Now $(\eta(\sigma))(x + \mu) = h_\sigma(x + \mu)$ (by the definition of η)

$= \sigma(x)$ (by the definition of h_σ)

$\leq \beta(x)$ (since $\sigma \subseteq \beta$)

$= h_\beta(x + \mu)$ (by the definition of h_β)

$= (\eta(\beta))(x + \mu)$ (by the definition of η).

Since this is true for all $x + \mu \in G/\mu$, it follows that $\eta(\sigma) \subseteq \eta(\beta)$.

Thus $\eta: P \rightarrow Q$ is an order preserving bijection.