CHAPTER 2

Fuzzy Cosets and Homomorphisms of N-groups

In this chapter we consider the fuzzy ideals of N-group G where N is a zero-symmetric right near-ring.

In section-1 of this chapter we proved several new results on fuzzy ideals and fuzzy cosets of N-groups, which are used in the sequel.

In section-2 of this chapter, we introduce fuzzy ideal θ_μ of the quotient N-group G/μ with respect to a fuzzy ideal μ of G. If μ is a fuzzy ideal of G and θ a fuzzy ideal of G/μ such that $\theta_\mu \subseteq \theta$ and $\theta_\mu(0) = \theta(0)$, then the corresponding σ_θ: $G \to [0, 1]$ is defined and proved that it is a fuzzy ideal of G such that $\mu \subseteq \sigma_\theta$ and $\mu(0) = \sigma_\theta(0)$.

In section-3 of this chapter, we prove some results on homomorphisms and fuzzy ideals of N-groups. The image and preimage of fuzzy ideal μ under N-group homomorphism were studied. Finally it is obtained that if f: $G \to G^1$ is an epimorphism of N-groups, then there is an order preserving bijection between the fuzzy ideals of G^1 and the fuzzy ideals of G that are constant on ker f. Some examples related to these concepts were illustrated.

Section-2.1: Fuzzy Ideals

In this section we provide some preliminary results on fuzzy ideals of N-groups. We start this section by recollecting the definition: fuzzy ideal of N-group G.

2.1.1. Definition: (Saikia H.K & Bharthakur L.K [19]) Let μ: G $\rightarrow$ [0, 1] be a mapping. μ is said to be a *fuzzy ideal of* G if the following conditions hold:

(i) $\mu(g + g^1) \geq \min\{\mu(g), \mu(g^1)\}$

(ii) $\mu(y + x - y) = \mu(x)$

(iii) $\mu(-x) = \mu(x)$

(iv) $\mu(n(g + x)-ng) \geq \mu(x)$ for all x, y, g, $g^1 \in G$, $n \in N$.

If μ: G $\rightarrow$ [0, 1] is a constant function, then μ is a fuzzy ideal of G.

Now we present a non-trivial example for fuzzy ideal of an N-group with detailed verification.

2.1.2 Example: Write N = Z, the set of integers under the usual operations '+' and '.', (N, + , .) form a near-ring.

Write G = Z.

Then (G, +) where + is usual addition of integers is a group.

Now consider G as N-group.

Define $\mu: G \rightarrow [0, 1]$ by $\mu(x) = \begin{cases} 1 & \text{if } x = 4n \text{ for some } n \in Z \\ 0.5 & \text{if } x = 2n \text{ for some } n \in Z \text{ and } x \neq 4m \text{ for any } m \in Z \\ 0 & \text{if } x \text{ is odd} \end{cases}$

Since $\mu: G \rightarrow [0, 1]$, we have that μ is a fuzzy set.

Now we verify that μ is a fuzzy ideal of G.

Step1: First we verify the condition: $\mu(g + g^1) \geq \min\{\mu(g), \mu(g^1)\}$.

Let $g, g^1 \in G$.

Case (i): Suppose $g = 4n$ and $g^1 = 4m$ for some $m, n \in G$.

Then $\mu(g + g^1) = \mu(4n + 4m)$

$= \mu(4(n + m))$

$= 1$ (by def of μ).

≥ 1

$= \min\{1, 1\} = \min\{\mu(4n), \mu(4m)\}$

$= \min\{\mu(g), \mu(g^1)\}$.

Case(ii): Suppose that one of the g, g^1 is of the form 4n and other is not in 4m form but 2m form.

Without loss of generality, we assume that $g = 4n$, $g^1 = 2m$ but $g^1 \neq 4k$ for any $k \in Z$.

Now it is clear that $g + g^1 \neq 4k$ for any $k \in Z$ and

$$g + g^1 = 4n + 2m = 2(2n + m).$$

So $\mu(g + g^1) = ½$.

$\mu(g) = 1$ and $\mu(g^1) = ½$.

Hence $\mu(g + g^1) = ½ \geq \min\{1, ½\} = \min\{\mu(g), \mu(g^1)\}$.

Case (iii): Suppose one of g, g^1 is of the form 4n and the other is not of the form 2k with $k \in Z$.

Without loss of generality assume that $g = 4n$ and g^1 is odd.

Now $g + g^1 \neq 2k$ for any $k \in Z$.

So $\mu(g + g^1) = 0$, $\mu(g) = 1$, $\mu(g^1) = 0$.

Hence $\mu(g + g^1) = 0 \geq \min\{1, 0\} = \min\{\mu(g), \mu(g^1)\}$.

Case-(iv): Suppose both g, g^1 are of the form 2n but no one of g, g^1 can be written in the form 4k for some $k \in Z$.

Then $\mu(g) = \mu(g^1) = ½$.

$\mu(g + g^1) = 1$ or ½.

So $\mu(g + g^1) \geq ½ = \min\{1/2, 1/2\} = \min\{\mu(g), \mu(g^1)\}$.

Case-(v): Suppose that one of the g, g^1 are of the form 2n and the other is an odd integer.

Without loss of generality, we may assume that $g = 2n$ and g^1 is odd.

Now $\mu(g) = 1/2$ and $\mu(g^1) = 0$.

Since $g + g^1 \neq 2k$ for any k, it follows that $\mu(g + g^1) = 0$.

So $\mu(g + g^1) = 0 \geq \min\{1/2, 0\} = \min\{\mu(g), \mu(g^1)\}$.

Case-(vi): Suppose g, g^1 both odd numbers.

Then $\mu(g + g^1) \geq 0$, $\mu(g) = 0 = \mu(g^1)$.

So $\mu(g + g^1) \geq \min\{0, 0\} = \min\{\mu(g), \mu(g^1)\}$.

From the above six cases, we conclude that

$\mu(g + g^1) \geq \min\{\mu(g), \mu(g^1)\}$ for all $g, g^1 \in Z$.

Step-2: Since $(G, +)$ is abelian it is clear that $\mu(g + x - g) = \mu(g - g + x) = \mu(x)$.

Step-3: By the definition of μ, we get that $\mu(-g) = \mu(g)$ for all $g \in G$; and $\mu(n(g + x) - ng) \geq \mu(x)$ for all $x, g \in G$ and $n \in N$.

Step-4: Now we verify that $\mu(kg) \geq \mu(g)$ for $k \in N$ and $g \in G$.

Let $k \in N$ and $g \in G$.

Case-(i): Suppose $k = 4n$ and $g = 4m$ for some $m, n \in G$.

Then $\mu(kg) = \mu(4n.4m) = \mu(16mn)) = \mu(4(4mn)) = 1$ (by def of μ).

$$\geq 1 = \mu(g)$$

Case-(ii): Suppose that one of the k, g is of the form 4n and other is not in 4m form but 2m form. Without loss of generality, we assume that $k = 4n$, $g = 2m$ but $g \neq 4s$ for any $s \in Z$. Now it is clear that $kg = 8mn = 4(2t)$ for some $t \in Z$.

So $\mu(kg) = 1 \geq \mu(g)$.

Case-(iii): Suppose one of k, g is of the form 4n and the other is not of the form 2s with $s \in Z$. Without loss of generality we assume that $k = 4n$ and g is odd.

Now $kg = 4t$ for some $t \in Z$.

So $\mu(kg) = 1 \geq \mu(g)$.

Case-(iv): Suppose both k, g are of the form 2n but no one of k, g can be written in the form 4s for some $s \in Z$.

Then $\mu(k) = \mu(g) = ½$.

$\mu(kg) = 1 \geq \mu(g)$.

Case-(v): Suppose one of the k, g are of the form 2n and the other is an odd integer. Without loss of generality, we may assume that $k = 2n$ and g is odd.

Now $\mu(k) = 1/2$ and $\mu(g) = 0$.

Now $kg = 2s$ for some $s \in Z$.

So $\mu(kg) = 0.5 \geq \mu(g)$

Case-(vi): Suppose k, g both odd numbers.

Then $\mu(kg) \geq 0$, $\mu(k) = 0 = \mu(g)$.

So $\mu(kg) \geq \mu(g)$.

Step 5: Take $k \in N$, $g, x \in G$.

Consider $\mu(k(g + x) - kg) = \mu((kg + kx) - kg)$

$= \mu(kg - kg + kx)$ (since $(Z , +)$ is abelian)

$= \mu(kx)$

$\geq \mu(x)$ (by Step 4).

Hence μ is a fuzzy ideal of G.

The proof is complete.

2.1.3 Proposition: Let G be N-group, where N is a near-ring with unity and $\mu: G \to [0, 1]$ is a fuzzy set with $\mu(ng) \geq \mu(g)$ for all $g \in G$, $n \in N$. Then the following two conditions are true.

(i). For all $0 \neq n \in N$, $\mu(ng) = \mu(g)$ if n is left invertible; and

(ii). $\mu(-g) = \mu(g)$.

Proof: (i) Let n^1 be a left inverse of n.

Then $n^1n = 1$.

Given that $\mu(ng) \geq \mu(g)$.

Now $\mu(g) = \mu(1.g) = \mu(n^1ng) \geq \mu(ng)$ (by the given condition)

$\Rightarrow \mu(g) \geq \mu(ng)$.

Hence $\mu(ng) = \mu(g)$ for all $g \in G$, and for all left invertible elements $0 \neq n \in N$.

(ii) Follows from (i) by taking $n = -1$.

2.1.4 Corollary: If μ is a fuzzy ideal of G and $g, g^1 \in G$, then

$$\mu(g - g^1) \geq \min\{\mu(g), \mu(g^1)\}.$$

Proof: Given μ is a fuzzy ideal of G.

Now $\mu(g - g^1) = \mu(g + (-g^1)) \geq \min\{\mu(g), \mu(-g^1)\}$ (since μ is a fuzzy ideal)

$$\geq \min\{\mu(g), \mu(g^1)\} \text{ (by the Proposition 2.1.3).}$$

Therefore $\mu(g - g^1) \geq \min\{\mu(g), \mu(g^1)\}$ for all $g, g^1 \in G$.

2.1.5 Proposition: If μ is a fuzzy ideal of G, and $g, g^1 \in G$ with $\mu(g) > \mu(g^1)$, then $\mu(g + g^1) = \mu(g^1)$. In other words, if $\mu(g) \neq \mu(g^1)$, then $\mu(g + g^1) = \min\{\mu(g), \mu(g^1)\}$.

Proof: Suppose $\mu(g) > \mu(g^1)$.

By definition, $\mu(g + g^1) \geq \min\{\mu(g), \mu(g^1)\}$

$$= \mu(g^1) \text{ (since } \mu(g) > \mu(g^1)) \text{ .... (i)}$$

Take $\mu(g^1) = \mu(g^1 + g - g) \geq \min\{\mu(g^1 + g), \mu(-g)\}$

$$= \min\{\mu(g^1 + g), \mu(g)\} \text{ .... (ii)}$$

If $\mu(g + g^1) \geq \mu(g)$, then $\mu(g^1) \geq \mu(g)$ by (ii)

$$> \mu(g^1) \text{ (by given condition)}$$

and so $\mu(g^1) > \mu(g^1)$, a contradiction.

Therefore $\mu(g + g^1) < \mu(g)$ .... (iii)

Now $\mu(g^1) \geq \min\{\mu(g^1 + g), \mu(g)\}$ (by (ii))

$$= \mu(g^1 + g) \text{ (by (iii)) .... (iv)}$$

From (i) and (iv), we can conclude that

$$\mu(g + g^1) = \mu(g^1).$$

The rest follows by using similar steps.

The proof is complete.

2.1.6 Corollary: If μ: G $\rightarrow$ [0, 1] is a mapping satisfies the condition $\mu(ng) \geq \mu(g)$ for all g $\in$ G and n $\in$ N, then the following two conditions are equivalent:

(i) $\mu(g - g^1) \geq \min\{\mu(g), \mu(g^1)\}$; and

(ii) $\mu(g + g^1) \geq \min\{\mu(g), \mu(g^1)\}$.

Proof: (i) $\Rightarrow$ (ii): Suppose (i).

Now $\mu(g + g^1) = \mu(g - (-g^1)) \geq \min\{\mu(g), \mu(-g^1)\}$ (by supposition)

$\geq \min\{\mu(g), \mu(g^1)\}$ (by the given condition with n = -1).

Therefore $\mu(g + g^1) \geq \min\{\mu(g), \mu(g^1)\}$.

(ii) $\Rightarrow$ (i): follows from Corollary 2.1.4

2.1.7 Note: Since N is a zero symmetric near-ring, by taking g = 0 $\in$ G, the condition (iv) of Definition 2.1.1 gives that $\mu(nx) = \mu(nx + 0) = \mu(n(0 + x) + n.0) \geq \mu(x)$.

Thus $\mu(nx) \geq \mu(x)$ for all n $\in$ N and x $\in$ G.

2.1.8 Remark: From the above Note 2.1.7 and the Corollary 2.1.6, we can conclude that the condition "$\mu(g + g^1) \geq \min\{\mu(g), \mu(g^1)\}$" in the Definition 2.1.1, may be replaced by "$\mu(g - g^1) \geq \min\{\mu(g), \mu(g^1)\}$".

2.1.9 Proposition: If μ: G $\rightarrow$ [0, 1] is a fuzzy ideal, then (i) $\mu(0) \geq \mu(g)$ for all g $\in$ G; and (ii) $\mu(0) = \sup_{g \in G} \mu(g)$.

Proof: (i) $\mu(0) = \mu(x - x) \geq \min\{\mu(x), \mu(-x)\} = \mu(x)$ for all x in G.

(ii) Follows from (i).

2.1.10 Remark [Lemma 1.7 of Saikia and Barthakur [20]: A fuzzy subset μ of G is a fuzzy ideal of G $\Leftrightarrow$ the level set μ_t is an ideal of G for all $t \in [0, \mu(0)]$.

2.1.11 Definition: Let μ be any fuzzy ideal of G. The ideals μ_t, $t \in [0, \mu(0)]$ where $\mu_t = \{x \in G / \mu(x) \geq t\}$ are called *level ideals* of μ. In particular, the level set denoted by $G_\mu = \{x \in G / \mu(x) = \mu(0)\} = \mu_s$ where $s = \mu(0)$ is an ideal of G.

2.1.12 Examples: (i) Consider N, G and μ as in Example 2.1.2.

In this example, $\mu: G \rightarrow [0, 1]$ is a fuzzy ideal defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 4n \text{ for some } n \in Z \\ 0.5 & \text{if } x = 2n \text{ for some } n \in Z \text{ and } x \neq 4m \text{ for any } m \in Z \\ 0 & \text{if } x \text{ is odd} \end{cases}$$

Some level ideals of μ are listed below

$\mu_0 = \{x \in G / \mu(x) \geq 0\} = G$

$\mu_{0\cdot 25} = \{x \in G / \mu(x) \geq 0.25\} = 2Z$

$\mu_{0.5} = \{x \in G / \mu(x) \geq 0.5\} = 2Z$

$\mu_{0\cdot 75} = \{x \in G / \mu(x) \geq 0.75\} = 4Z$

$\mu_1 = \{x \in G / \mu(x) \geq 1\} = 4Z$

(ii) Consider Z_8 = {0, 1, 2, 3, 4, 5, 6, 7}, the set of integers modulo 8. Then Z_8 is a N-group over Z (here N = Z), the near-ring of integers. Define μ: $Z_8 \rightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 3/4 & \text{if } x = 4 \\ 1/2 & \text{if } x \in \{2, 6\} \\ 1/4 & \text{if } x \in \{1, 3, 5, 7\} \end{cases}$$

. This μ is a fuzzy ideal of G.

The level ideals of μ are given below:

$\mu_1 = \{0\}$, $\mu_{3/4} = \{0, 4\}$, $\mu_{1/2} = \{0, 2, 4, 6\}$, $\mu_{1/4} = Z_8$.

2.1.13 Definition: Let A and B be two sets such that $A \subseteq B$. Suppose that α and β are two numbers in [0, 1] such that α > β. Define a function μ from A to B by

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in A \\ \beta & \text{otherwise} \end{cases}.$$

Then the function μ is called as the *generalized characteristic function* of A.

2.1.14 Example: Every characteristic function is a generalized characteristic function.

2.1.15 Theorem: Let $I \subseteq G$, α and β are two numbers in [0, 1] such that α > β. Define

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in I \\ \beta & \text{otherwise} \end{cases}.$$

Then the following two conditions are equivalent:

(i) μ is a fuzzy ideal; and (ii) I is a ideal of G.

Proof: (i) ⇒ (ii): Let x, y ∈ I. Now μ(x) = μ(y) = α.

Now $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ (since μ is fuzzy ideal)

$$= \min\{\alpha, \alpha\} = \alpha \Rightarrow \mu(x - y) \geq \alpha \Rightarrow x - y \in I.$$

Let $x \in I$.

Since μ is fuzzy normal, it follows that

$\mu(y + x - y) = \mu(x) = \alpha$, for all $y \in G$.

Therefore $y + x - y \in I$ for all $y \in G$.

Take $x \in I$, $g \in G$ and $n \in N$.

Since μ is a fuzzy ideal of G, we have that

$\mu(n(g + x) - ng)) \geq \mu(x) = \alpha$ and so $n(g + x) - ng \in I$.

Hence I is an ideal of G.

(ii) $\Rightarrow$ (i): Let $x, y \in G$.

If $x, y \in I$, then $x - y \in I$ and so

$$\mu(x - y) = \alpha \geq \min\{\alpha, \alpha\}$$

$$= \min\{\mu(x), \mu(y)\}.$$

If $x \in I$ and $y \notin I$, then $x - y \notin I$ and

so $\mu(x - y) = \beta \geq \min\{\alpha, \beta\} = \min\{\mu(x), \mu(y)\}$.

If $x \notin I$, $y \notin I$, then $\mu(x - y) \geq \beta = \min\{\mu(x), \mu(y)\}$.

Take $x \in I$.

Since I is an ideal of G, we have that $y + x - y \in I$ and so $\mu(y + x - y) = \alpha = \mu(x)$.

If $\mu(y + x - y) = \beta$, then $y + x - y \notin I$ and so $x \notin I$.

This shows that $\mu(y + x - y) = \beta = \mu(x)$.

Take $x \in I$, $g \in G$ and $n \in N$.

Since I is an ideal G, we have $n(g + x) - ng \in I$.

Therefore $\mu(n(g + x) - ng) = \alpha = \mu(x)$.

If $x \notin I$, then $\mu(n(g + x) - ng) \geq \beta = \mu(x)$.

Thus (ii) $\Rightarrow$ (i).

The proof is complete.

2.1.16 Corollary (Saikia & Bharthakur [19]): Let $I \subseteq G$. Define

$\mu(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$. Then the following conditions are equivalent:

(i) μ is a fuzzy ideal; and

(ii) I is an ideal of G.

2.1.17 Corollary: Suppose $I \subseteq G$. Then the following three conditions are equivalent:

(i) I is a ideal of G;

(ii) The generalized characteristic function $\mu : M \to [0,1]$ by $\mu(x) = \begin{cases} \beta & \text{if } x \in N \\ \alpha & \text{otherwise} \end{cases}$ for

some $\beta > \alpha$, is a fuzzy ideal of G; and

(iii) The characteristic function χ_I is a fuzzy ideal of G.

Proof: Follows from Theorem 2.1.15 and Corollary 2.1.16.

In the statement of 2.1.15, the condition $\alpha > \beta$ is necessary. For this observe the following remarks.

2.1.18 Remark: If we replace the condition "$\alpha > \beta$" of the statement of 2.1.15 by "$\alpha = \beta$", then the new statement is not true.

If $\alpha = \beta$, then $\mu(x) = \alpha$ for all $x \in G$ and so μ is a constant fuzzy set. We know that every constant function $\mu: G \to [0, 1]$ is a fuzzy ideal. Thus μ is a fuzzy ideal.

Let $G = Z_5$, the additive group of integers modulo 5 and $N = Z$, the near-ring of integers. Then G is an N-group.

Write $I = \{1, 2, 3\} \subseteq G$.

Define $\mu: G \to [0, 1]$ by $\mu(x) = \begin{cases} \alpha & \text{if } x \in I \\ \beta & \text{if } x \notin I \end{cases}$.

If $\alpha = \beta$, then μ is a fuzzy ideal but I is not an ideal of G.

Thus if $\alpha = \beta$ then the statement of 2.1.15 is not true.

2.1.19 Remark: If we replace the condition "$\alpha > \beta$" of the statement of 2.1.15 by "$\beta > \alpha$", then the new statement is not true.

Suppose $\beta > \alpha$. Let I be an ideal of a given N-group G with $I \neq G$.

Define $\mu: G \to [0, 1]$ by $\mu(x) = \begin{cases} \alpha & \text{if } x \in I \\ \beta & \text{if } x \notin I \end{cases}$.

If possible suppose μ is an ideal.

Then by Proposition 2.1.9, it follows that $\mu(0) \geq \mu(x)\ \forall\ x \in G$.

Let $y \in G \setminus I$. Now $\alpha = \mu(0)$ (since $0 \in I$)

$\geq \mu(y)$ (by Proposition 2.1.9)

$= \beta$ (by definition of μ)

$\Rightarrow \alpha \geq \beta$, a contradiction to the fact $\beta > \alpha$.

This shows that μ is not a fuzzy ideal.

Now I is an ideal but μ is not a fuzzy ideal.

So if $\beta > \alpha$, the statement of 2.1.15 is not true.

2.1.20 Proposition: Let μ be a fuzzy ideal of G and μ_t, μ_s (with $t < s$) be two level ideals of μ. Then the following two conditions are equivalent:

(i) $\mu_t = \mu_s$; and

(ii) there is no $x \in G$ such that $t \leq \mu(x) < s$.

Proof: (i) $\Rightarrow$ (ii): In a contrary way, suppose that there exists an element $x \in G$ such that $t \leq \mu(x) < s$.

Then $x \in \mu_t$ and $x \notin \mu_s$ and so $\mu_t \neq \mu_s$, a contradiction.

Hence we get (ii).

(ii) $\Rightarrow$ (i): Since $t < s$ we have $\mu_t \geq \mu_s$.

Let $x \in \mu_t \Rightarrow \mu(x) \geq t$.

By given condition (ii), there is no y such that $s > \mu(y) \geq t$ and so $\mu(x) \geq s$ which implies $x \in \mu_s$. Thus $\mu_t \leq \mu_s$.

2.1.21 Theorem: Suppose $\mu: G \to [0, 1]$ is a fuzzy ideal. Then $\mu(g - g^1) = \mu(0)$ implies $\mu(g) = \mu(g^1)$ for all $g, g^1 \in G$. Converse is not true [For this, observe the Example 2.1.12(ii). Here $\mu(1) = \mu(3)$ but $\mu(3 - 1) = \mu(2) = \frac{1}{2} \neq \mu(0)$].

2.1.22 Definition: Let $\mu: G \rightarrow [0, 1]$ be a fuzzy ideal and $x \in G$. Then the fuzzy subset $x + \mu: G \rightarrow [0, 1]$ defined by $(x + \mu)(y) = \mu(y - x)$ is called a *fuzzy coset* of the fuzzy ideal μ.

2.1.23 Proposition: Suppose $\mu: G \rightarrow [0,1]$ is a fuzzy ideal and $x, y \in G$. Then $x + \mu = y + \mu \Leftrightarrow \mu(x - y) = \mu(0)$.

Proof: Suppose $x + \mu = y + \mu$.

This implies $(x + \mu)(a) = (y + \mu)(a)$ for all $a \in M$

$\Rightarrow (x + \mu)(x) = (y + \mu)(x)$

$\Rightarrow \mu(x - x) = \mu(x - y)$ (by the definition of fuzzy coset of fuzzy submodule μ)

$\Rightarrow \mu(0) = \mu(x - y)$.

Converse: Suppose that $\mu(x - y) = \mu(0)$.

Now we prove that $x + \mu = y + \mu$.

Let $z \in M$.

Now $(x + \mu)(z) = \mu(z - x)$

$= \mu(z - y + y - x)$

$\geq \min\{\mu(z - y), \mu(y - x)\}$

$= \min\{\mu(z - y), \mu(0)\}$ (Since $\mu(y - x) = \mu(0)$)

$= \mu(z - y)$ (by Proposition 2.1.9)

$= (y + \mu)(z)$.

Therefore $(x + \mu)(z) \geq (y + \mu)(z)$ for all $z \in M$.

Similarly, we can show that $(y + \mu)(z) \geq (x + \mu)(z)$.

Hence $x + \mu = y + \mu$.

The proof is complete.

2.1.24 Note: We write $G/\mu = \{ x + \mu / x \in G\}$. Define $(x + \mu) + (y + \mu) = (x + y) + \mu$ and $n(x + \mu) = nx + \mu$ for $x, y \in G$ and $n \in N$.

Saikia and Bharthakur [19] proved that G/μ is an N-group with respect to the operations defined above.

Section - 2.2: Fuzzy ideals of G/μ

In this section, we introduce fuzzy ideal θ_μ of the quotient N-group G/μ with respect to a fuzzy ideal μ of G. If μ is a fuzzy ideal of G and θ a fuzzy ideal of G/μ such that $\theta_\mu \subseteq \theta$ and $\theta_\mu(0) = \theta(0)$, then the corresponding σ_θ: $G \rightarrow [0, 1]$ is defined and proved that it is a fuzzy ideal of G such that $\mu \subseteq \sigma_\theta$ and $\mu(0) = \sigma_\theta(0)$.

2.2.1 Definitions: (i) If μ is a fuzzy ideal of G, then we define a *fuzzy set* θ_μ on G/μ corresponding to μ by $\theta_\mu(x + \mu) = \mu(x)$ for all $x \in G$.

(ii) If θ is a fuzzy ideal of G/μ, then we define a *fuzzy set* σ_θ on G by $\sigma_\theta(x) = \theta(x + \mu)$ for all $x \in G$.

2.2.2 Theorem: If μ is a fuzzy ideal of G, then the fuzzy set $\theta_\mu: G/\mu \to [0, 1]$ defined above is a fuzzy ideal of G/μ.

Proof: Suppose μ is a fuzzy ideal of G.

Now we verify that θ_μ is a fuzzy ideal of G/μ.

Let $x + \mu, y + \mu \in G/\mu$.

Now $\theta_\mu((x + \mu) - (y + \mu)) = \theta_\mu((x - y) + \mu)$

$= \mu(x - y)$ (by definition of θ_μ)

$\geq \min\{\mu(x), \mu(y)\}$ (since μ is a fuzzy ideal)

$= \min\{\theta_\mu(x + \mu), \theta_\mu(y + \mu)\}$

Let $n \in N$ and $x + \mu, g + \mu \in G/\mu$.

Now $\theta_\mu(n((g + \mu) + (x + \mu)) - n(g + \mu)) = \theta_\mu((n(g + x) + \mu) - (ng + \mu))$

$= \theta_\mu((n(g + x) - ng) + \mu)$

$= \mu(n(g + x) - ng)$ (by the definition of θ_μ)

$\geq \mu(x)$ (since μ is a fuzzy ideal of G)

$= \theta_\mu(x + \mu)$.

Therefore θ_μ is a fuzzy ideal of G/μ.

2.2.3 Corollary: If μ and σ are two fuzzy ideals of G such that $\mu \subseteq \sigma$ and $\sigma(0) = \mu(0)$, then the mapping $h_\sigma: G/\mu \to [0, 1]$ defined by $h_\sigma(x + \mu) = \sigma(x)$ for all $x + \mu \in G/\mu$, is a fuzzy ideal. Also $\theta_\mu \subseteq h_\sigma$ on G/μ and $\theta_\mu(0) = h_\sigma(0)$.

Proof: First we verify that h_σ is well defined.

Let $x + \mu, y + \mu \in G/\mu$ and $x + \mu = y + \mu$.

This implies $\mu(0) = \mu(x - y)$ (by Proposition 2.1.23)

$\Rightarrow \sigma(0) \geq \sigma(x - y)$

$\geq \mu(x - y)$ (since $\sigma \geq \mu$)

$= \mu(0)$

$= \sigma(0)$

$\Rightarrow \sigma(0) = \sigma(x - y)$

$\Rightarrow x + \sigma = y + \sigma$ (by Proposition 2.1.23)

$\Rightarrow (x + \sigma)(0) = (y + \sigma)(0)$

$\Rightarrow \sigma(0 - x) = \sigma(0 - y)$ (by Definition 2.1.22)

$\Rightarrow \sigma(-x) = \sigma(-y)$

$\Rightarrow \sigma(x) = \sigma(y)$ (since σ is a fuzzy ideal)

$\Rightarrow h_\sigma(x + \mu) = h_\sigma(y + \mu)$ (by the definition of h_σ in Corollary 2.2.3)

Therefore h_σ is well defined.

Now we verify that h_σ is a fuzzy ideal of G/μ.

Let $x + \mu, y + \mu \in G/\mu$.

Now $h_\sigma((x + \mu) - (y + \mu)) = h_\sigma((x - y) + \mu)$

$= \sigma(x - y)$ (by definition of h_σ)

$\geq \min\{\sigma(x), \sigma(y)\}$

$= \min\{h_\sigma(x + \mu), h_\sigma(y + \mu)\}$ (by the definition of h_σ)

Let $x + \mu$, $g + \mu \in G/\mu$ and $n \in N$.

Now $h_\sigma(n((g + \mu) + (x + \mu)) - n(g + \mu))$

$= h_\sigma(n(g + x) + \mu) - (ng + \mu))$

$= h_\sigma((n(g + x) - ng) + \mu)$

$= \sigma(n(g + x) - ng))$ (by the definition of h_σ)

$\geq \sigma(x)$ (since σ is a fuzzy ideal)

$= h_\sigma(x + \mu)$ (by the definition of h_σ)

Therefore h_σ is a fuzzy ideal of G/μ.

Let $x \in G$.

Now $h_\sigma(x + \mu) = \sigma(x)$ (by the definition of h_σ)

$\geq \mu(x)$

$= \theta_\mu(x + \mu)$ (by the definition of θ_μ)

This true for all $x \in G$. So $\theta_\mu \subseteq h_\sigma$.

Also $\theta_\mu(0) = \mu(0) = \sigma(0) = h_\sigma(0)$.

2.2.4 Theorem: Let μ be a fuzzy ideal of G and θ a fuzzy ideal of G/μ such that $\theta_\mu \subseteq \theta$ and $\theta_\mu(0) = \theta(0)$. Then $\sigma_\theta: G \to [0, 1]$ (please refer Definition 2.2.1 (ii)), σ_θ is defined by $\sigma_\theta(x) = \theta(x + \mu)$, is a fuzzy ideal of G such that $\mu \subseteq \sigma_\theta$ and $\mu(0) = \sigma_\theta(0)$.

Proof: Part (i): First we show that σ_θ is a fuzzy ideal of G.

Let x, y $\in$ G and n $\in$ N.

Now $\sigma_\theta(x - y) = \theta((x - y) + \mu)$ (by the definition of σ_θ)

$= \theta((x + \mu) - (y + \mu))$ (by the definition of '+' on G/μ)

$\geq \min\{\theta(x + \mu), \theta(y + \mu)\}$ (since θ is a fuzzy ideal)

$= \min\{\sigma_\theta(x), \sigma_\theta(y)\}$ (by the definition of σ_θ)

Therefore $\sigma_\theta(x - y) \geq \min\{\sigma_\theta(x), \sigma_\theta(y)\}$.

Let g $\in$ G.

Now $\sigma_\theta(n(g + x) - ng) = \theta(n(g + x) - ng) + \mu)$ (by the definition of σ_θ)

$= \theta(n((g + \mu) + (x + \mu)) - n(g + \mu))$

$\geq \theta(x + \mu)$ (since θ is a fuzzy ideal)

$= \sigma_\theta(x)$ (by the definition of σ_θ).

Therefore σ_θ is a fuzzy ideal of G.

Part (ii): For any x $\in$ G, we have that $\sigma_\theta(x) = \theta(x + \mu) \geq \theta_\mu(x + \mu)$ (since $\theta \supseteq \theta_\mu$)

$= \mu(x)$ (by the definition of θ_μ)

This shows that $\mu \subseteq \sigma_\theta$.

Also $\sigma_\theta(0) = \theta(0 + \mu)$ (by the definition of σ_θ)

$= \theta(0) = \theta_\mu(0) = \theta_\mu(0 + \mu) = \mu(0)$.

Therefore $\sigma_\theta(0) = \mu(0)$.

Section-2.3: Homomorphisms and Fuzzy ideals of N-groups

In this section, we proved some important results related to homomorphisms and fuzzy ideals of N-groups. We start this section with the following proposition.

2.3.1 Proposition: Suppose $\mu: G \rightarrow [0, 1]$ is a fuzzy ideal of G.

(i) The mapping $\Phi: G \rightarrow G/\mu$ defined by $\Phi(x) = x + \mu$ is an onto homomorphism with $\ker \Phi = G_\mu$. Hence the N-group G/μ is isomorphic to the N-group G/G_μ under the mapping $f: G/G_\mu \rightarrow G/\mu$ defined by $f(x + G_\mu) = x + \mu$, where

$G_\mu = \{x \in G \,/\, \mu(x) = \mu(0)\}$.

(ii) Suppose μ and σ are two fuzzy ideals of G such that $G_\mu = G_\sigma$. Then the mapping $g: G/\mu \rightarrow G/\sigma$ defined by $g(x + \mu) = x + \sigma$ is an isomorphism.

(iii) If $G/\mu \cong G/\sigma$ under the isomorphism $g(x + \mu) = x + \sigma$, then $G_\mu = G_\sigma$.

Proof: (i) Define $\Phi: G \rightarrow G/\mu$ by $\Phi(x) = x + \mu$ for all $x \in G$.

Clearly Φ is well defined.

Let $x_1, x_2 \in G$ and $n \in N$.

Then $\Phi(x_1 + x_2) = (x_1 + x_2) + \mu$ (by the definition of Φ)

$= (x_1 + \mu) + (x_2 + \mu) = \Phi(x_1) + \Phi(x_2)$.

$\Phi(nx_1) = nx_1 + \mu$ (by the definition of Φ)

$= n(x_1 + \mu)$

$= n\Phi(x_1)$ (by the definition of Φ)

Therefore Φ is a N-group homomorphism.

To verify that Φ is onto, let us consider an element $x + \mu \in G/\mu$.

Now $x \in G$ and by the definition of Φ, we have that $\Phi(x) = x + \mu$.

This shows that Φ is onto.

Hence Φ is a N-group epimorphism.

So by the Fundamental Theorem of homomorphisms,

$G/\ker \Phi \cong \Phi(G)$ and so $G/\ker \Phi \cong G/\mu$.

It remains to show that $\ker \Phi = G_\mu$.

Now $x \in \ker \Phi \Leftrightarrow \Phi(x) = 0$

$\Leftrightarrow x + \mu = 0 + \mu$ (by the definition of Φ)

$\Leftrightarrow \mu(x) = \mu(0)$ (by Proposition 2.1.23)

$\Leftrightarrow x \in G_\mu$.

This shows that $G/G_\mu \cong G/\mu$.

(ii) Given that $G_\mu = G_\sigma$

$\Rightarrow G/G_\mu = G/G_\sigma$

By above Part (i), we get that $G/\mu \cong G/G_\mu = G/G_\sigma \cong G/\sigma$.

So $G/\mu \cong G/\sigma$ under the isomorphism g.

(iii) Suppose $G/\mu \cong G/\sigma$ under the isomorphism g defined by $g(x + \mu) = x + \sigma$.

Now we have to show that $G_\mu = G_\sigma$.

Let $x \in G_\mu$.

Then $\mu(x) = \mu(0)$

$\Rightarrow \mu(x - 0) = \mu(0)$

$\Rightarrow x + \mu = 0 + \mu$ (by Proposition 2.1.23)

$\Rightarrow g(x + \mu) = g(0 + \mu)$

$\Rightarrow x + \sigma = 0 + \sigma$ (by the definition of g)

$\Rightarrow \sigma(x) = \sigma(0)$ (by proposition 2.1.23)

$\Rightarrow x \in G_\sigma$ (by the definition of G_σ)

Therefore $G_\mu \subseteq G_\sigma$.

Similarly it can be verified easily that $G_\sigma \subseteq G_\mu$.

Thus we can conclude that $G_\mu = G_\sigma$.

2.3.2 Example: Write $G = Z$ and $G^1 = Z_6$.

Then G and G^1 are N-groups over Z, the near-ring of integers. Let f: $Z \rightarrow Z_6$ be the cannonical epimorphism.

Define μ: $G \rightarrow [0, 1]$ by $\mu(x) = \chi_{2Z}(x)$ for all $x \in G$.

Note that χ_{2Z} is the characteristic function of 2Z.

By Corollary 2.1.17, we have that, μ is a fuzzy ideal of G.

Now we verify that $f(\mu)$ is a fuzzy ideal of G^1.

Now $f(\mu)(0) = \sup\{\mu(x) / x \in f^{-1}(0)\}$

$= \sup\{\mu(x) / x \in 6Z\} \geq 1$.

So $f(\mu)(0) = 1$.

Similarly, we have $f(\mu)(2) = 1$, $f(\mu)(4) = 1$, $f(\mu)(1) = 0$, $f(\mu)(3) = 0$, $f(\mu)(5) = 0$.

The level sets of $f(\mu)$ are given by $(f(\mu))_0 = G$ and $(f(\mu))_1 = \{0, 2, 4\}$, each of which is an ideal of $G = Z_6$.

Therefore by Remark 2.1.10, $f(\mu)$ is a fuzzy ideal of Z_6.

2.3.3 Proposition: Suppose that $f: G \rightarrow G^1$ is a N-group epimorphism; and $\mu: G \rightarrow [0, 1]$ is a fuzzy ideal of G. Then $f(\mu)$ is a fuzzy ideal of G^1.

Proof: Suppose that $f: G \rightarrow G^1$ is an epimorphism.

Now we verify that $f(\mu)$ is a fuzzy ideal of G^1.

Let $u, v \in G^1$ and $n \in N$.

Since $f: G \rightarrow G^1$ is onto, there exist $x, y \in G$ such that $f(x) = u$, $f(y) = v$.

Now $f(x - y) = f(x) - f(y) = u - v$

$\Rightarrow x - y \in f^{-1}(u - v)$ and so $f^{-1}(u - v) \neq \phi$.

$$
\begin{aligned}
\text{Now } f(\mu)(u - v) &= \sup\{\mu(z) / z \in f^{-1}(u - v)\} \\
&\geq \sup\{\mu(x - y) / x \in f^{-1}(u),\ y \in f^{-1}(v)\} \\
&\geq \sup\{\min\{\mu(x), \mu(y)\} / x \in f^{-1}(u),\ y \in f^{-1}(v)\} \\
&= \min\{\sup\{\mu(x) / x \in f^{-1}(u)\}, \sup\{\mu(y) / y \in f^{-1}(v)\}\} \\
&= \min\{f(\mu)(u), f(\mu)(v)\}
\end{aligned}
$$

Therefore $f(\mu)(u - v) \geq \min\{f(\mu)(u),\ f(\mu)(v)\}$.

Let $g^1, x^1 \in G^1$.

Since f is onto, $f(g) = g^1$ and $f(x) = x^1$ for some $g, x \in G$.

Now $(f(\mu))(n(g^1 + x^1) - ng^1)$

$$= \sup\{\mu(z) / z \in f^{-1}(n(g^1 + x^1) - ng^1)\}$$

$\geq \sup\{\mu(n(g + x) - ng) / nx \in f^{-1}(n(g^1 + x^1) - ng^1)\}$ where $f(g) = g^1$ and $f(x) = x^1\}$

$\geq \sup\{\mu(x) / x \in f^{-1}(n(g^1+x^1)-ng^1)\}$ (since μ is a fuzzy ideal of G)

$= f(\mu)(x^1)$.

Therefore $f(\mu)(n(g^1 + x^1) - ng^1) \geq f(\mu)(x^1)$.

This is true for all $x^1, g^1 \in G$.

Hence $f(\mu)$ is a fuzzy ideal of G^1.

2.3.4 Example: Write $G = Z$ and $G^1 = Z_6$. Then G and G^1 are N-groups over Z, the near-ring of integers. Let $f: G \rightarrow G^1$ be the canonical homomorphism.

Define $\mu^1: G^1 \rightarrow [0, 1]$ by $\mu^1(x) = \begin{cases} 1 & \text{if } x \in \{0, 2, 4\} \\ 0 & \text{otherwise} \end{cases}$

Note that μ^1 is the characteristic function of the ideal $\{0, 2, 4\}$.

So by Corollary 2.1.17, it follows that μ^1 is a fuzzy ideal of G^1.

We know that $f^{-1}(\mu^1)(x) = \mu^1(f(x))$ for all $x \in G$.

If x is even, then $f(x) \in \{0, 2, 4\}$.

So $f^{-1}(\mu^1)(x) = \mu^1(f(x)) = 1$.

If x is odd, then $f(x) \in \{1, 3, 5\}$

So $f^{-1}(\mu^1)(x) = \mu^1(f(x)) = 0$.

Therefore the level sets of $f^{-1}(\mu^1)$ are $(f^{-1}(\mu^1))_0 = G$ and $(f^{-1}(\mu^1))_1 = 2Z$, each of which is an ideal of $G = Z$.

Therefore by Theorem 2.1.10, we have that $f^{-1}(\mu^1)$ is a fuzzy ideal of Z.

2.3.5 Proposition: Suppose that $f: G \to G^1$ is a N-group homomorphism; and $\mu^1: G^1 \to [0, 1]$ is a fuzzy ideal of G^1. Then $f^{-1}(\mu^1)$ is a fuzzy ideal of G, which is constant on ker f.

Proof: Part (i): First we show that $f^{-1}(\mu^1)$ is a fuzzy ideal of G.

Let $x, y \in G$ and $n \in N$.

Now $f^{-1}(\mu^1)(x - y) = \mu^1(f(x - y))$ (by the definition of $f^{-1}(\mu^1)$).

$= \mu^1(f(x) - f(y))$ (since f is a N-group homomorphism)

$\geq \min\{\mu^1(f(x)), \mu^1(f(y))\}$ (since μ^1 is a fuzzy ideal of G^1)

$= \min\{f^{-1}(\mu^1)(x), f^{-1}(\mu^1)(y)\}$ (by the definition of $f^{-1}(\mu^1)$)

and $f^{-1}(\mu^1)(n(g + x) - ng)$

$= \mu^1(f(n(g + x) - ng))$ (by the definition of $f^{-1}(\mu^1)$)

$= \mu^1(n(f(g) + f(x)) - nf(g))$ (since f is a N-group homomorphism)

$\geq \mu^1(f(x)$ (since μ^1 is a fuzzy ideal of G^1)

$= f^{-1}(\mu^1)(x)$ (by the definition of $f^{-1}(\mu^1)$)

Therefore $f^{-1}(\mu^1)$ is a fuzzy ideal of G.

Part (ii): For any $x \in \ker f$, we have that

$f^{-1}(\mu^1)(x) = \mu^1(f(x))$ (by the definition of $f^{-1}(\mu^1)$)

$= \mu^1(0^1)$ (since $x \in \ker f$).

This is true for all $x \in \ker f$.

This shows that $f^{-1}(\mu^1)$ is constant on ker f.

The proof is complete.

2.3.6 Proposition: Suppose that f: $G \to G^1$ is a N-group homomorphism; and $\mu: G \to [0, 1]$ and $\mu^1: G^1 \to [0, 1]$ are fuzzy ideals of G and G^1 respectively. Then the following statements (i) - (v) are true.

(i) $f(\mu)(0^1) = \mu(0)$;

(ii) If μ is constant on ker f, then $(f(\mu))(f(x)) = \mu(x)$ for all $x \in G$;

(iii) $f^{-1}(M^1_{\mu^1}) = M_{f^{-1}(\mu^1)}$;

(iv) If f is an epimorphism, then $f f^{-1}(\mu^1) = \mu^1$; and

(v) If μ is constant on ker f, then $f^{-1}f(\mu) = \mu$.

Proof: (i) Now $f(\mu)(0^1) = \sup\{\mu(x) / x \in f^{-1}(0^1)\}$

$= \mu(0)$ (since $0 \in f^{-1}(0^1)$ and $\mu(0) \geq \mu(x)$ for all $x \in G$)

Therefore $f(\mu)(0^1) = \mu(0)$.

(ii) Suppose μ is constant on ker f.

Now we verify that $(f(\mu))(f(x)) = \mu(x)$ for all $x \in G$.

Let $x \in G$. Write $f(x) = x^1$.

For any $z \in f^{-1}(x^1)$, we have $f(z) = x^1 = f(x)$

$\Rightarrow f(z - x) = 0^1$ (since f is a homomorphism)

$\Rightarrow z - x \in \ker f$ (by the definition of ker f)

$\Rightarrow \mu(z - x) = \mu(0)$ (since μ is constant on ker f)

$\Rightarrow z + \mu = x + \mu$ (by Proposition 2.1.23)

$\Rightarrow (z + \mu)(0) = (x + \mu)(0)$

$\Rightarrow \mu(0 - z) = \mu(0 - x)$

$\Rightarrow \mu(-z) = \mu(-x)$

$\Rightarrow \mu(z) = \mu(x)$.

Now $(f(\mu))(f(x)) = f(\mu)(x^1)$

$= \sup\{\mu(z) / z \in f^{-1}(x^1)\}$

$= \sup\{\mu(x)\} = \mu(x)$.

This shows that $(f(\mu))(f(x)) = \mu(x)$ for all $x \in G$.

(iii) Now we show that $f^{-1}\left(M^1_{\mu^1}\right) = M_{f^{-1}(\mu^1)}$

Now $x \in f^{-1}\left(M^1_{\mu^1}\right)$

$\Leftrightarrow f(x) \in M^1_{\mu^1}$

$\Leftrightarrow \mu^1(f(x)) = \mu^1(0^1)$ (by the definiton of $M^1_{\mu^1}$)

$\Leftrightarrow \mu^1(f(x)) = \mu^1(f(0))$ (since $f(0) = 0^1$)

$\Leftrightarrow (f^{-1}\mu^1)(x) = (f^{-1}\mu^1)(0)$ (by the definition of $f^{-1}(\mu^1)$)

$\Leftrightarrow x \in M_{f^{-1}(\mu^1)}$

(iv) Suppose f is an epimorphism

Now we verify that $f(f^{-1}(\mu^1)) = \mu^1$

Let $x^1 \in G^1$.

Then $x^1 = f(x)$ for some $x \in G$.

Now $(f(f^{-1}(\mu^1)))(x^1)$

$= (f(f^{-1}(\mu^1)))(f(x))$

$= (f(\beta))(f(x))$ where $\beta = f^{-1}(\mu^1)$ which is constant on ker f (by Proposition 2.3.5)

$= \beta(x)$ (by (ii))

$= f^{-1}(\mu^1)(x)$ (since $\beta = f^{-1}(\mu^1)$)

$= \mu^1(f(x)) = \mu^1(x^1)$ (since $f(x) = x^1$)

This is true for all $x^1 \in G^1$.

Hence $f(f^{-1}(\mu^1)) = \mu^1$.

(v) Suppose μ is constant on ker f.

Now we verify that $f^{-1}(f(\mu)) = \mu$.

Let $x \in G$.

Now $(f^{-1}(f(\mu)))(x) = (f^{-1}(\beta))(x)$ where $\beta = f(\mu)$

$= \beta(f(x))$

$= (f(\mu))(f(x)) = \mu(x)$ (by (ii))

This is true for all $x \in G$.

Hence $(f^{-1}f)(\mu) = \mu$

2.3.7 Proposition: If f: $G \to G^1$ is an epimorphism of N-groups, then there is an order preserving bijection between the fuzzy ideals of G^1 and the fuzzy ideals of G that are constant on ker f.

Proof: Part (i): Let $F(G) = \{\mu / \mu$ is a fuzzy ideal of G such that μ is constant on ker f$\}$, and $F(G^1) = \{ \mu^1 / \mu^1$ is a fuzzy ideal of $G^1 \}$.

Define ϕ: $F(G) \to F(G^1)$ by $\phi(\mu) = f(\mu)$ for all $\mu \in F(G)$.

Since f is well defined, it follows that ϕ is well defined.

Let $\mu_1, \mu_2 \in F(G)$ and $\phi(\mu_1) = \phi(\mu_2)$

$\Rightarrow f(\mu_1) = f(\mu_2)$ (by the definition of ϕ)

$\Rightarrow f^{-1}(f(\mu_1)) = f^{-1}(f(\mu_2))$

$\Rightarrow \mu_1 = \mu_2$ (by Proposition 2.3.6 (v)).

Therefore ϕ is one-one.

To verify that ϕ is onto, take $\mu^1 \in F(G^1)$.

Write $\mu = f^{-1}(\mu^1)$

By Proposition 2.3.5, μ is a fuzzy ideal of G which is constant on ker f.

So $\mu \in F(G)$.

Now $\phi(\mu) = f(\mu) = f(f^{-1}(\mu^1)) = \mu^1$ (by Proposition 2.3.6(v)).

This shows that ϕ is onto.

Hence ϕ is a bijection from F(G) to $F(G^1)$.

Part (ii): Now we verify that ϕ is order preserving.

Let $\mu_1, \mu_2 \in F(G)$ satisfying the property $\mu_1 \leq \mu_2$.

Let $x^1 \in G^1$.

Since f is onto, there exists $x \in G$ such that $f(x) = x^1$.

Now $f(\mu_1)(x^1) = \sup\{\mu_1(t) / t \in f^{-1}(x^1)\}$

$\leq \sup\{\mu_2(t) / t \in f^{-1}(x^1)\}$ (since $\mu_1 \leq \mu_2$)

$= f(\mu_2)(x^1)$.

So $(f(\mu_1))(x^1) \leq (f(\mu_2))(x^1)$. This is true for all $x^1 \in G^1$.

This shows that $f(\mu_1) \leq f(\mu_2)$.

Now $\mu_1 \leq \mu_2 \Rightarrow \phi(\mu_1) = f(\mu_1) \leq f(\mu_2) = \phi(\mu_2)$.

Thus we can get that there exists an order preserving bijection between the fuzzy ideals of G^1 and the fuzzy ideals of G that are constant on ker f.