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CHAPTER 4

Fuzzy Dimension of N-Groups with DCC on Ideals

It is well known that the dimension of a vector space is defined as the number of elements in its basis. One can define a basis of a vector space as a maximal set of linearly independent vectors or a minimal set of vectors, which spans the space. The former when generalized to modules over rings becomes the concept of Goldie dimension. Goldie [9] introduced the concept: finite Goldie dimension in modules over rings. A module is said to have finite Goldie dimension if it contains no infinite direct sum of non-zero submodules. The concept of Goldie dimension in N-groups was introduced by Reddy and Satyanarayana [18]. Later the authors Satyanarayana and Syam Prasad [33, 34] studied the concepts linearly independent elements, u-linearly independent elements in N-groups and obtained structure theorems for N-groups which contains no infinite direct sum of non-zero ideals.

In this chapter we consider the fuzzy ideals of N-group G where N is a zero-symmetric right near-ring. We introduce the concepts: minimal elements, fuzzy linearly independent elements, and fuzzy basis of an N-group G and obtained several important related results. This chapter is divided into three sections.

In section-1, we introduce the concept "minimal element", obtained some results on minimal elements, u-elements, linearly independent elements. We proved that G has DCCI, then there exists n (here n = dim G) linearly independent minimal elements in G, which form an essential.

In section-2, we introduce the concept of fuzzy linearly independent elements with respect to a fuzzy ideal μ of G. We prove that for a fuzzy ideal μ on G, if $x_1, x_2, \ldots, x_n$ are minimal elements in G with distinct μ-values, then $x_1, x_2, \ldots, x_n$ are (i). linearly independent; and (ii). fuzzy μ-linearly independent.

In section-3, we introduce the notion "fuzzy pseudo basis" and "fuzzy dimension". We proved that if G has DCCI, then there exists an essential ideal A of G and a fuzzy ideal μ of A such that S-dim (μ) = dim G.

Section-4.1: Minimal Elements

We start this section by introducing the new concept "minimal element" in N-group G.

4.1.1 Definition: An element $x \in G$ is said to be a *minimal element* if <x> is minimal in the set of all non-zero ideals of G.

4.1.2 Theorem: If G has DCC on ideals, then every nonzero ideal of G contains a minimal element.

Proof: Let K be a nonzero ideal of G.

Since G has DCC on its ideals, it follows that the set of all ideals of G contained in K has a minimal element.

So K contains a minimal ideal A (that is, A is minimal in the set of all non-zero ideals of G contained in K).

Let $0 \neq a \in A$.

Then $0 \neq \langle a \rangle \subseteq A$ and so $\langle a \rangle = A$.

Since $\langle a \rangle$ is a minimal ideal, we have that 'a' is a minimal element.

4.1.3 Note: There are N-groups, which do not satisfy DCC on its ideals, but contains a minimal element. For this, we observe the following example.

4.1.4 Example: Write $N = Z$, $G = Z \oplus Z_6$.

Now G is direct sum of two N-groups, and so G is an N-group.

Since $2Z \oplus Z_6 \supsetneq 2^2Z \oplus Z_6 \supsetneq \ldots \supsetneq 2^nZ \oplus Z_6 \supsetneq \ldots$ is strictly decreasing sequence of ideals, it follows that G has no DCC on its ideals.

Consider $g = (0, 2) \in G$.

Now the ideal generated by g,

that is, $\langle g \rangle = Zg = \{(0, 0), (0, 2), (0, 4)\}$ is a minimal element in the set of all non-zero ideals of G.

Hence g is a minimal element.

Thus G has no DCC on its ideals, but it contains a minimal element.

4.1.5 Theorem: Every minimal element is an u-element.

(For the definition of u-element please refer Section -1.2)

Proof: Let $0 \neq a \in G$ be a minimal element.

Consider Na.

Let $(0) \neq L$ and I be ideals of G such that $L \subseteq$ <a>, $I \subseteq$ <a> and $L \cap I = (0)$.

Since $L \neq (0)$, $(0) \subseteq L \subseteq$ <a>, and a is minimal, it follows that L = <a>.

Now $I = I \cap$ <a> $= I \cap L = (0)$.

This shows that L is essential in <a>.

Hence <a> is uniform ideal and so a is an u-element.

4.1.6 Note: The converse of Theorem 4.1.5 is not true.

For this observe the example given here.

Write G = Z, the additive group of integers, and N = Z, the near-ring of integers.

Since Z is an uniform N-group, and 1 is a generator, we have that 1 is an u-element.

But 2Z is a proper ideal of 1.Z = Z = G.

Hence 1 cannot be a minimal element.

Thus 1 is an u-element but not a minimal element.

4.1.7 Theorem: Suppose μ is a fuzzy ideal of G.

(i) If $g \in G$, then for any $x \in$ <g> we have $\mu(x) \geq \mu(g)$; and

(ii) If g is a minimal element, then for any $0 \neq x \in$ <g> we have $\mu(x) = \mu(g)$.

Proof: (i) By straightforward verification, we conclude that for $g \in G$, <g> $= \bigcup_{i=0}^{\infty} A_i$

where $A_{k+1} = A_k^* \cup A_k^+ \cup A_k^0$, $A_0 = \{g\}$ and

$A_k^* = \{y + x - y \mid y \in G, x \in A_k\}$,

$A_k^+ = \{n(y + x) - ny \mid n \in N, y \in G, x \in A_k\}$,

$A_k^0 = \{x - y \mid x, y \in A_k\}$.

We prove that $\mu(y) \geq \mu(g)$ for all $y \in A_m$ for $m \geq 1$.

For this, we use induction on m.

It is obvious if $m = 0$.

Suppose the induction hypothesis for k.

That is, $\mu(y) \geq \mu(g)$ for all $y \in A_k$.

Now let $v \in A_k^* \cup A_k^+ \cup A_k^0$.

Suppose $v \in A_k^*$. Then $v = z + y - z$ for some $y \in A_k$.

Now $\mu(v) = \mu(z + y - z) \geq \mu(y)$ (since μ is a fuzzy ideal of G)

$\geq \mu(g)$ (since $y \in A_k$).

Let $v \in A_k^0$.

Then $v = y_1 - y_2$ for some $y_1, y_2 \in A_k$.

Now $\mu(v) = \mu(y_1 - y_2) \geq \min\{\mu(y_1), \mu(y_2)\}$

$\geq \mu(g)$, by induction hypothesis.

$\geq \min\{\mu(g), \mu(g)\}$ (since $y_1, y_2 \in A_k$ and $\mu(y_1) \geq \mu(g)$, $\mu(y_2) \geq \mu(g)$)

$= \mu(g)$.

Suppose $v \in A_k^+$.

Then $v = n(y + x) - ny$ for some $n \in N$, $y \in G$, $x \in A_k$.

Now $\mu(v) = \mu(n(y + x) - ny) \geq \mu(x)$ (since μ is a fuzzy ideal)

$\geq \mu(g)$ (by induction hypothesis).

Thus in all cases, we proved that $\mu(v) \geq \mu(g)$ for all $v \in A_{k+1}$.

Hence by the principle of mathematical induction, we conclude that $\mu(v) \geq \mu(g)$ for all $v \in A_m$ and for all positive integers m.

We proved that $\mu(v) \geq \mu(g)$ for all $v \in A_m$ and for all positive integers m.

Hence $\mu(x) \geq \mu(g)$ for all $x \in \langle g \rangle$.

(ii) Let $g \in G$ be a minimal element.

Let $0 \neq x \in \langle g \rangle$.

Now $0 \neq \langle x \rangle \subseteq \langle g \rangle$.

Since g is a minimal element, we have $\langle x \rangle = \langle g \rangle$.

Therefore $g \in \langle x \rangle$ and by (i), we have $\mu(g) \geq \mu(x)$.

Thus $\mu(x) = \mu(g)$.

4.1.8 Note: If G satisfies the descending chain condition on its ideals, then we say that, "G has DCCI". Let K be an ideal of G. If the set $\{J / J \text{ is an ideal of } G, J \subseteq K\}$ has the descending chain condition, then we say that K has DCC on the ideals of G (we write DCCIG, in short).

4.1.9 Lemma: If x is a u-element in G and G has DCCI, then there exist minimal element $y \in \langle x \rangle$ such that $\langle y \rangle \leq_e \langle x \rangle$ (that is $\langle y \rangle$ is essential in $\langle x \rangle$).

Proof: Consider the ideal $\langle x \rangle$.

Since x is a u-element $x \neq 0$.

By Theorem 4.1.2, there exists a minimal element $y \in \langle x \rangle$.

Since $\langle y \rangle$ is a non-zero ideal of $\langle x \rangle$, and $\langle x \rangle$ is uniform ideal,

it follows that $\langle y \rangle \leq_e \langle x \rangle$.

Now we recollect the following definitions.

4.1.10 Definitions (Satyanarayana & Syam Prasad [34]): (i) Let X be a subset of G. X is said to be a *linearly independent* (l. i., in short) set if the sum $\sum_{a \in X} \langle a \rangle$ is direct.

If $\{a_i / 1 \leq i \leq n\}$ is a l. i. set, then we say that the elements a_i, $1 \leq i \leq n$ are *linearly independent*. If X is not an l. i. set then we say that X is a *linearly dependent* (l. d. in short) set.

(ii) A subset X of G is said to be *u-linearly independent* (u-l.i., in short) set if every element of X is an u-element and X is a l.i. set.

(iii) A l.i. set X in G is said to be an *essential basis* for G if $\sum_{a \in X} \langle a \rangle \leq_e G$. We also say that X forms an essential basis for G.

4.1.11 Example: Let V be a finite dimensional vector space over a field F.

Suppose dim V = n.

Then V contains a basis, say $\{v_1, v_2, \ldots, v_n\}$.

(i) Let us view V as N-group where N = F.

Now $\langle v_1 \rangle = Fv, \ldots, \langle v_i \rangle = Fv_i, \ldots, \langle v_n \rangle = Fv_n$.

The sum of the subspaces $Fv_1, \ldots, Fv_n$ of the vector space V is direct.

So $\sum_{i=1}^{n} \langle v_i \rangle$ is direct.

Hence $X = \{v_1, v_2, \ldots, v_n\}$ is a linearly independent set in the N-group V.

(ii) Let $v^* \in \langle \{v_1, v_2, \ldots, v_n\} \rangle$, the subspace of V spanned by $v_1, v_2, \ldots, v_n$.

Then $v^*, v_1, \ldots, v_n$ are linearly dependent elements in the vector space V.

So the sum of $Fv^*, Fv_1, \ldots, Fv_n$ cannot be a direct sum in the vector space V.

Thus the sum of $\langle v^* \rangle = Fv^*$, $\langle v_1 \rangle = Fv_1, \ldots, \langle v_n \rangle = Fv_n$ cannot be a direct sum in the N-group V.

Hence $\{v^*, v_1, \ldots, v_n\}$ is a linearly dependent set in the N-group V.

(iii) Write $G = Z \oplus Z$; the direct sum of N-groups Z and Z where N = Z, the near-ring of integers.

Consider the elements $(2, 0), (3, 0) \in G$.

Now <(2 , 0) > = 2Z ⊕ 0 and <(0, 3)> = 0 ⊕ 3Z.

Since (2Z ⊕ 0) ∩ (0 ⊕ 3Z) = {(0, 0)}.

It follows that X = {(2, 0), (3, 0)} is a linearly independent set.

Since 2Z ⊕ 0 is an uniform ideal, (2, 0) is a u-element.

Since 0 ⊕ 3Z is an uniform ideal, (0, 3) is a u-element.

Therefore X = {(2, 0), (0, 3)} is a u-linearly independent set.

(iv) Consider G, N, X as in above (iii).

Since 2Z is essential in Z and 3Z is essential in Z.

It follows that <(2, 0)> = 2Z ⊕ 0 is essential in Z ⊕ 0 and <(0, 3)> = 0 ⊕ 3Z is essential in 0 ⊕ 3Z.

Now by Result 1.2.19, it follows that <(2, 0)> + <(0, 3)> is essential in (Z ⊕ 0) + (0 ⊕ Z) = Z ⊕ Z = G.

This shows that X forms an essential basis for G.

To get the familiarity with the concepts "FGD" and "l.i.subset" we include the proof for the following note.

4.1.12 Note: G has FGD ⇔ every l. i. subset X of G is a finite set.

Verification: Suppose G has FGD.

We have to prove that every linearly independent subset X of G is a finite set.

In a contrary way, suppose that there exists an infinite subset X of G which is a linearly independent set.

Then there exist infinite distinct elements $a_1, a_2, \ldots, a_n, \ldots$ in X.

Since X is a linearly independent set $\sum_{i=1}^{\infty} < a_i >$ is a direct sum.

Since each $a_i \neq 0$, each $<a_i>$ is a nonzero ideal.

Hence G contains direct sum of infinite number of nonzero ideals, a contradiction.

Therefore X is a finite subset of G.

Conversely, suppose that every linearly independent set X of G is finite.

We have to verify that G has FGD.

In a contrary way assume that G has no FGD.

Then G contains nonzero ideals $\{H_i\}_{i=1}^{\infty}$ whose sum is direct.

Now take $0 \neq a_i \in H_i$ for all positive integer i.

Now $0 \neq <a_i> \subseteq H_i$.

Since $\sum_{i=1}^{\infty} H_i$ is direct, it follows that $\sum_{i=1}^{\infty} < a_i >$ is a direct sum.

This shows that $X = \{a_1, a_2, \ldots, \ldots\}$ is a linearly independent set.

By supposition X is finite.

So there exist two positive integers i and j such that $i \neq j$ and $a_i = a_j$.

Now $a_i = a_j \in H_i \cap H_j = (0)$ (since $i \neq j$ and $\sum_{i=1}^{\infty} H_i$ is direct)

$\Rightarrow a_i = 0$, a contradiction to the selection of a_i .

Hence G has FGD.

4.1.13 Note: Suppose that dim G = n and X ⊆ G, and X is a linearly independent set.

Consider the following conditions:

(i) |X| = n;

(ii) X is a maximal linearly independent set; and

(iii) X is essential basis for G.

Then the implications (i) ⇒ (iii) ⇒ (i) ⇒ (ii) are true, but (ii) ⇒ (iii) need not be true.

Verification: (i) ⇒ (iii): Suppose dim G = n and |X| = n, X is linearly independent set.

By definition X is an essential basis for G.

(i) ⇒ (ii): Suppose (i).

Then X is linearly independent set.

If X is not maximal, then there exist a ∈ G such that X ∪ {a} is linearly dependent set.

That implies there exist Y such that X ⊆ Y, Y is linearly independent set and Y contains at least (n + 1) linearly independent elements, which is contradiction.

Hence X is maximal linearly independent set.

(ii) ⇒ (iii) is not true.

For this, observe the following example.

4.1.14 Example: Write $G = Z_2 \oplus Z_6$.

G is a N-group where N = Z.

Since G is a finite N-group it has FGD.

(i) Write u = (1, 0). Then 2u = (0, 0).

So <u> = {(1, 0), (0, 0)} is an ideal of the N-group G.

Also u is an u-element.

Write v = (0, 2).

Then <v> = {(0, 2), (0, 4), (0, 0)} is an ideal of N-group G.

Also v is an u-element.

Write w = (0, 3).

Then <w> = {(0, 3), (0, 0)} is an ideal of an N-group G.

Also w is an u-element.

(ii) Now it is easy to verify that the sum of G = <u> ⊕ <v> ⊕ <w> is direct.

Thus dim G = 3.

Write X = {(1, 0), (0, 1)}.

Now X is a linearly independent set.

There is no Y ⊆ G such that X ⊊ Y and Y is a linearly independent set.

Thus X is a maximal linearly independent set with |X| = 2, but dim G = 3 ≠ 2 = |X|.

4.1.15 Proposition (2.2 of Satyanarayana & Syam Prasad [32]) : The implication (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) is true where

(i) G has DCCI

(ii) Every inverse system of non - zero ideals of G is bounded below by a non - zero ideal of G.

(iii) There exists a finite sum of simple ideals of G, whose sum is direct and essential in G.

(iv) G has FGD.

4.1.16 Theorem: If G has DCCI, then there exist linearly independent minimal elements $x_1, x_2, \ldots, x_n$ in G where n = dim G, and the sum $<x_1> + \ldots + <x_n>$ is direct and essential in G. Also $B = \{x_1, x_2, \ldots, x_n\}$ forms an essential basis for G.

Proof: Since G has DCCI; by the Proposition 4.1.15, G has FGD.

Suppose n = dim G.

Then by the Theorem 1.2.32, there exist u-linearly independent elements $u_1, u_2, \ldots, u_n$ such that the sum $<u_1> + \ldots + <u_n>$ is direct and essential in G.

Since G has DCCI, by Lemma 4.1.9, there exist minimal elements

$x_i \in <u_i>$ such that $<x_i> \leq_e <u_i>$ for $1 \leq i \leq n$.

Since $u_1, u_2, \ldots, u_n$ are linearly independent, it follows that $x_1, x_2, \ldots, x_n$ are also linearly independent.

Thus we have linearly independent minimal elements $x_1, x_2, \ldots, x_n$ in G where n = dim G.

Since $<x_i> \leq_e <u_i>$ by Result 1.2.19, it follows that

$<x_1> \oplus \ldots \oplus <x_n> \leq_e <u_1> \oplus \ldots \oplus <u_n> \leq_e G$ and so $<x_1> \oplus \ldots \oplus <x_n> \leq_e G$.

Thus $B = \{x_1, x_2, \ldots, x_n\}$ forms an essential basis for G.

Section-4.2: Fuzzy Linearly Independent Elements

We start this section with the following definition.

4.2.1 Definition: Let G be an N-group and μ be a fuzzy ideal of G. $x_1, x_2, \ldots, x_n \in G$ are said to be *fuzzy μ -linearly independent* (or *fuzzy linearly independent with respect to μ*) if it satisfies the following two conditions:

(i) $x_1, x_2, \ldots, x_n$ are linearly independent; and

(ii) $\mu(y_1 + \ldots + y_n) = \min\{\mu(y_1), \ldots, \mu(y_n)\}$ for any $y_i \in \langle x_i \rangle$, $1 \le i \le n$.

4.2.2 Example: Consider Example 4.1.11(iii) and (iv).

Let $G = Z \oplus Z$ and $N = Z$, as in Example 4.1.11(iii) and (iv).

Define μ: $G \to [0, 1]$ by $\mu(x, y) = \begin{cases} 0.5 & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$.

Then μ is a fuzzy ideal of G.

Write $x_1 = (2, 0)$, $x_2 = (0, 3)$.

We know that $x_1 = (2, 0)$, $x_2 = (0, 3)$ are linearly independent elements.

Take $y_1 \in \langle x_1 \rangle$ and $y_2 \in \langle x_2 \rangle$.

Then $y_1 = (2n, 0)$ for some $n \in Z$, and $y_2 = (0, 3m)$ for some $m \in Z$

Now $\mu(y_1 + y_2) = \mu((2n, 0) + (0, 3m)) = \mu(2n, 3m)$.

Case (i): Suppose $3m = 0$.

Then $\mu(y_1 + y_2) = \mu(2n, 3m) = \mu(2n, 0) = 0.5$;

$\mu(y_1) = \mu(2n, 0) = 0.5$; and

$\mu(y_2) = \mu(0, 3m) = \mu(0, 0) = 0.5$.

Therefore $\mu(y_1 + y_2) = 0.5 = \min\{0.5, 0.5\}$

$$= \min\{\mu(y_1), \mu(y_2)\}.$$

Case (ii): Suppose $3m \neq 0$.

Then $\mu(y_1 + y_2) = \mu(2n, 3m) = 0$.

$\mu(y_1) = 0.5$; and $\mu(y_2) = \mu(0, 3m) = 0$.

Therefore $\mu(y_1 + y_2) = 0 = \min\{0.5, 0\} = \min\{\mu(y_1), \mu(y_2)\}$.

Hence $\{x_1, x_2\}$ is a fuzzy μ-linearly independent set.

4.2.3 Theorem: Let μ be a fuzzy ideal of G. If $x_1, x_2, \ldots, x_n$ are minimal elements in G with distinct μ-values, then $x_1, x_2, \ldots, x_n$ are

(i) linearly independent; and

(ii) fuzzy μ-linearly independent.

Proof: The proof is by induction on n.

If $n = 1$, then x_1 is linearly independent and also fuzzy linearly independent.

Let us assume that the statement is true for $(n - 1)$.

Now suppose $x_1, x_2, \ldots, x_n$ are minimal elements with distinct μ values.

By induction hypothesis $x_1, x_2, \ldots, x_{n-1}$ are linearly independent and fuzzy linearly independent.

Part (i): If $x_1, \ldots, x_n$ are not linearly independent, then the sum of $<x_1>, <x_2>, \ldots, <x_n>$ is not direct.

This means $<x_i> \cap (<x_1> \oplus \ldots <x_{i-1}> \oplus <x_{i+1}> \oplus \ldots \oplus <x_n>) \neq \{0\}$.

This implies $0 \neq y_i = y_1 + \ldots + y_{i-1} + y_{i+1} + \ldots + y_n$ for some $y_j \in <x_j>$ for $1 \leq j \leq n$.

Now $\mu(x_i) = \mu(y_i)$ (by Theorem 4.1.7)

$= \mu(y_1 + \ldots + y_{i-1} + y_{i+1} + \ldots + y_n)$

$= \min\{\mu(y_1), \ldots, \mu(y_{i-1}), \mu(y_{i+1}), \ldots, \mu(y_n)\}$ (by induction hypothesis)

$= \mu(y_k)$ for some $k \in \{1, 2, \ldots, i - 1, i + 1, \ldots, n\}$

$= \mu(x_k)$ (by Theorem 4.1.7).

Thus $\mu(x_i) = \mu(x_k)$ for some $i \neq k$, a contradiction.

This shows that $x_1, x_2, \ldots, x_n$ are linearly independent.

Part (ii): Now we prove that $x_1, x_2, \ldots, x_n$ are fuzzy linearly independent.

Suppose $y_i \in <x_i>$, $1 \leq i \leq n$.

$\mu(y_1 + y_2 + \ldots + y_{n-1}) = \min\{\mu(y_1), \ldots, \mu(y_{n-1})\}$ (by the induction hypothesis)

$= \mu(y_j)$, for some j with $1 \leq j \leq n - 1$

$= \mu(x_j)$ (by the Theorem 4.1.7).

Now $\mu(x_j) \neq \mu(x_n) \Rightarrow \mu(y_1 + y_2 + \ldots + y_{n-1})$

$= \mu(x_j) \neq \mu(x_n) = \mu(y_n)$

$\Rightarrow \mu(y_1 + y_2 + \ldots + y_{n-1} + y_n) = \min\{\mu(y_1 + \ldots + y_{n-1}), \mu(y_n)\}$ (by Proposition 2.1.5)

$= \min\{\min\{\mu(y_1), \ldots, \mu(y_{n-1}), \mu(y_n)\}\}$

$= \min\{\mu(y_1), \ldots, \mu(y_n)\}$.

This shows that $x_1, x_2, \ldots, x_n$ are fuzzy linearly independent with respect to μ.

Section-4.3: Fuzzy Dimension

We start this section with the following definition.

4.3.1 Definitions: (i). Let μ be a fuzzy ideal of G. A subset B of G is said to be a *fuzzy pseudo basis* for μ if B is a maximal subset of G such that $x_1, x_2, \ldots, x_k$ are fuzzy linearly independent for any finite subset $\{x_1, x_2, \ldots, x_k\}$ of B.

(ii). Consider the set **B** = {k | there exist a fuzzy pseudo basis B for μ with |B| = k}.

If B has no upper bound, then we say that the *fuzzy dimension of μ* is infinite.

We denote this fact by S-dim(μ) = ∞. If **B** has an upper bound, then the *fuzzy dimension of μ* is sup **B**. We denote this fact by S-dim(μ) = sup **B**. If m = S-dim(μ) = sup **B**, then a fuzzy pseudo basis B for μ with |B| = m, is called as *fuzzy basis* for the fuzzy ideal μ.

4.3.2 Proposition: Suppose G has FGD and μ is a fuzzy ideal of G. Then

(i). |B| ≤ dim G for any fuzzy pseudo basis B for μ; and

(ii). S-dim (μ) ≤ dim G.

Proof: Suppose n = dim G.

(i). Suppose B is a fuzzy pseudo basis for μ.

If |B| > n, then B contain distinct elements $x_1, x_2, \ldots, x_{n+1}$.

Since B is a fuzzy pseudo basis, the elements $x_1, x_2, \ldots, x_{n+1}$ are linearly independent; and by Theorem 1.2.32, it follows that n + 1 ≤ n, a contradiction.

Therefore |B| ≤ n = dim G.

(ii). From (i) it is clear that dim M is an upper bound for the set

B = {k | there exist a fuzzy pseudo basis B for μ with $|B| = k$}.

Therefore S-dim(μ) = sup **B** $\leq$ dim G.

4.3.3 Definition: An N-group G is said to have a *fuzzy basis* if there exists an essential ideal A of G and a fuzzy ideal μ of A such that S-dim(μ) = dim G. The fuzzy pseudo basis of μ is called as *fuzzy basis* for G.

4.3.4 Remark: If G has FGD, then every fuzzy basis for G is a basis for G.

Verification: Suppose dim G = m.

Let B be a fuzzy basis for G.

This means that there exists an essential ideal A of G; fuzzy ideal μ of A such that

S - dim (μ) = dim G = m and B is a fuzzy basis for the fuzzy ideal μ.

Now B is a linearly independent set in G with maximum number of linearly independent elements.

Then $|B|$ = max{k / there exist a l.i. set A in G such that $|A| = k$}

= dim G (by Result 1.2.26)

Thus B is a basis for G.

Hence every fuzzy basis for G is a basis for G if G has FGD.

4.3.5 Theorem: Suppose that G has DCCI. Then G has a fuzzy basis.

(In other words, there exists an essential ideal A of G and a fuzzy ideal μ of A such that

S - dim(μ) = dim G).

Proof: Since G has DCCI, it has FGD.

Suppose dim G = n.

By Theorem 4.1.16, there exist linearly independent minimal elements $x_1, x_2, \ldots, x_n$ such that $\{x_1, x_2, \ldots\ldots x_n\}$ forms an essential basis for G.

Take $0 \le t_1 < t_2 < \ldots < t_n \le 1$.

Define $\mu(y_i) = t_i$ for $y_i \in \langle x_i \rangle$, $1 \le i \le n$.

Then μ is a fuzzy ideal on $A = \langle x_1 \rangle + \langle x_2 \rangle + \ldots + \langle x_n \rangle \le_e G$.

By the Theorem 4.2.3, $x_1, x_2, \ldots, x_n$ are fuzzy μ-linearly independent.

So $\{x_1, x_2, \ldots, x_n\}$ is a pseudo basis for μ.

Now dim M = n ≤ sup B ≤ dim G (by the Proposition 4.3.2)

and hence S-dim μ = dim G.

This shows that G has a fuzzy basis.