Chapter 2

Cayley digraph structures
induced by groups

This chapter contains two sections. In first section we introduce the concept digraph structure and provide example of digraph structures. We also discuss different types of digraph structures. In second section, we give a class of Cayley digraph structures induced by groups. These Cayley digraph structures can be viewed as a generalization of Cayley digraphs. Many graph theoretic properties are included in terms of algebraic properties. Some of the results in this chapter are revised version of the results in [13].

2.1 Digraph structure

In this section we extend the definition of a digraph as follows:

Definition 2.1.1. Let $V$ be a nonempty set and let $E_1, E_2, \ldots, E_n$ be mutually disjoint binary relations on $V$. Then the $(n+1)$-tuple $G = (V; E_1, E_2, \ldots, E_n)$ is called a digraph structure[13]. The elements of $V$ are called vertices and the elements of $E_i$ are called $E_i$-edges.

2.1.1 Example of a digraph structure

The following is an example of a digraph structure.
Example 2.1.2. Let $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$. Define the following relations on $V$.

$E_1 = \{(1, 2), (2, 4), (4, 3), (1, 3)\}$,
$E_2 = \{(5, 6), (6, 8), (8, 7), (5, 7)\}$,
$E_3 = \{(9, 10), (10, 12), (12, 11), (9, 11)\}$,
$E_4 = \{(13, 14), (14, 16), (16, 15), (13, 15)\}$, and
$E_5 = \{(4, 13), (10, 7)\}$.

Then $G = (V; E_1, E_2, E_3, E_4, E_5)$ is a digraph structure.

![Figure 2.1: A digraph structure](image)

2.1.2 Different types of digraph structures

We now define some special types of digraph structures.

Definition 2.1.3. A digraph structure $G = (V; E_1, E_2, \ldots, E_n)$ is called $E_1E_2 \cdots E_n$-trivial if $E_i = \emptyset$ for all $i$, and $E_i$-trivial if $E_i = \emptyset$.

Definition 2.1.4. A digraph structure $G = (V; E_1, E_2, \ldots, E_n)$ is called $E_1E_2 \cdots E_n$-reflexive, if for all $x \in G$, $(x, x) \in E_i$ for some $i$, and $E_i$-reflexive if for all $x \in G$, $(x, x) \in E_i$.

Definition 2.1.5. A digraph structure $G = (V; E_1, E_2, \ldots, E_n)$ is called $E_1E_2 \cdots E_n$-symmetric, if $E_i = E_i^{-1}$ for all $i$, and $E_i$-symmetric if $E_i = E_i^{-1}$.
Definition 2.1.6. A digraph structure $G = (V; E_1, E_2, \ldots, E_n)$ is called $E_1 E_2 \cdots E_n$-anti symmetric, if $(x, y) \in E_i$ and $(y, x) \in E_i$, implies $x = y$ for all $i$, and $E_i$-anti symmetric, if $(x, y) \in E_i$ and $(y, x) \in E_i$, implies $x = y$.

Definition 2.1.7. A digraph structure $G = (V; E_1, E_2, \ldots, E_n)$ is called $E_1 E_2 \cdots E_n$-transitive if for every $i$ and $j$, $E_i \circ E_j \subseteq E_k$ for some $k$, and $E_i$-transitive if $E_i \circ E_i \subseteq E_i$.

Definition 2.1.8. A digraph structure $G = (V; E_1, E_2, \ldots, E_n)$ is called $E_1 E_2 \cdots E_n$-Hasse diagram if for every positive integer $n \geq 2$ and every $v_0, v_1, \ldots, v_n$ of $V$, $(v_i, v_{i+1}) \in \cup E_i$ for all $i = 0, 1, 2, \ldots, n-1$, implies $(v_0, v_n) \notin E_i$ for all $i$, and $E_i$-Hasse diagram if for every positive integer $n \geq 2$ and every $v_0, v_1, \ldots, v_n$ of $V$, $(v_i, v_{i+1}) \in E_i$ for all $i = 0, 1, 2, \ldots, n-1$, implies $(v_0, v_n) \notin E_i$.

Definition 2.1.9. A digraph structure $G = (V; E_1, E_2, \ldots, E_n)$ is called $E_1 E_2 \cdots E_n$-complete if $\cup E_i = V \times V$, and $E_i$-complete if $E_i = V \times V$.

Definition 2.1.10. A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called an $E_1 E_2 \cdots E_n$-quasi ordered set if it is both $E_1 E_2 \cdots E_n$-reflexive and $E_1 E_2 \cdots E_n$-transitive.

Definition 2.1.11. A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called an $E_1 E_2 \cdots E_n$-partially ordered set if it is $E_1 E_2 \cdots E_n$-anti symmetric and $E_1 E_2 \cdots E_n$-quasi ordered set. Similarly, we can define $E_i$-quasi ordered set and $E_i$-partially ordered set as in the case of ordinary relations.

Definition 2.1.12. An $E_1 E_2 \cdots E_n$-walk of length $k$ in a digraph structure is an alternating sequence $W = v_0, e_0, v_1, \ldots, e_{k-1}, v_k$, where $e_i = (v_i, v_{i+1}) \in \cup E_i$.

Definition 2.1.13. An $E_1 E_2 \cdots E_n$-walk $w = v_0, e_0, v_1, \ldots, e_{k-1}, v_k$ is called an $E_1 E_2 \cdots E_n$-path if the internal vertices $v_1, v_2, \ldots, v_{k-1}$ are distinct. We use the notation $(v_0, v_1, v_2, \ldots, v_n)$ for the $E_1 E_2 \cdots E_n$-path $w$. As in digraphs, we define $E_i$-walk and $E_i$ path. For example, an $E_i$-path between two vertices $u$ and $v$ consists of only $E_i$ edges.

Definition 2.1.14. A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called $E_1 E_2 \cdots E_n$-connected if there exists at least one $E_1 E_2 \cdots E_n$-path from $v$ to $u$ for all $u, v \in V$. 

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Definition 2.1.15. A digraph structure \((V; E_1, E_2, \ldots, E_n)\) is called \(E_1E_2\cdots E_n\)-locally connected if and only if for every pair of vertices \(u, v \in V\) there is an \(E_1E_2\cdots E_n\)-path from \(v\) to \(u\) whenever there is an \(E_1E_2\cdots E_n\)-path from \(u\) to \(v\).

Definition 2.1.16. A digraph structure \((V; E_1, E_2, \ldots, E_n)\) is called \(E_1E_2\cdots E_n\)-semi connected for every pair of vertices \(u, v\), there is an \(E_1E_2\cdots E_n\)-path from \(u\) to \(v\) or an \(E_1E_2\cdots E_n\)-path from \(v\) to \(u\).

Definition 2.1.17. A digraph structure \((V; E_1, E_2, \ldots, E_n)\) is called \(E_1E_2\cdots E_n\)-quasi connected if for every pair of vertices \(x, y\) there is a vertex \(z\) such that there is an \(E_1E_2\cdots E_n\)-path from \(z\) to \(x\) and an \(E_1E_2\cdots E_n\)-path from \(z\) to \(y\).

Definition 2.1.18. A digraph structure \((V; E_1, E_2, \ldots, E_n)\) is called \(E_i\)-connected if there exists at least one \(E_i\)-path from \(v\) to \(u\) for all \(u, v \in V\). Similarly, we can define \(E_i\)-locally connected, \(E_i\)-semi connected and \(E_i\)-quasi connected digraph structures.

2.1.3 Distance and diameter

Definition 2.1.19. \(E_1E_2\cdots E_n\)-distance between two vertices \(x\) and \(y\) in a digraph structure \(G\) is the length of the shortest \(E_1E_2\cdots E_n\)-path between \(x\) and \(y\), denoted by \(d_{1,2,3,\ldots,n}(x,y)\). Let \(G = (V; E_1, E_2, \ldots, E_n)\) be a finite \(E_1E_2\cdots E_n\)-connected digraph structure. Then \(E_1E_2\cdots E_n\)-diameter of \(G\) is defined as:

\[
d(G) = \max_{x,y \in G} \{d_{1,2,3,\ldots,n}(x,y)\}.
\]

Similarly we can define \(E_i\)-distance and \(E_i\)-diameter as in digraphs.

2.1.4 Isomorphic digraph structures

Definition 2.1.20. Two digraph structures \((V_1; E_1, E_2, \ldots, E_n)\) and \((V_2; R_1, \ldots, R_m)\) are said to be isomorphic if (i) \(m = n\) and (ii) there exists a bijective function \(f: V_1 \rightarrow V_2\) such that \((x, y) \in E_i \iff (f(x), f(y)) \in R_i\).

This concept of isomorphism is a generalization of isomorphism between two digraphs. The function \(f\) is called an isomorphism.
Definition 2.1.21. An isomorphism of a digraph structure onto itself is called an automorphism.

2.1.5 Vertex transitive digraph structure

Definition 2.1.22. A graph structure \((V; E_1, E_2, \ldots, E_n)\) is said to be vertex-transitive if, given any two vertices \(a\) and \(b\) of \(V\), there is some automorphism \(f : V \to V\) such that \(f(a) = b\).

2.1.6 Vertex degrees of digraph structure

Definition 2.1.23. Let \((V; E_1, E_2, \ldots, E_n)\) be a graph structure and let \(v \in V\). Then the \(E_1 E_2 \cdots E_n\)-out-degree of \(u\) is \(|\{v \in V : (u, v) \in \cup E_i\}|\) and \(E_1 E_2 \cdots E_n\)-in-degree of \(u\) is \(|\{v \in V : (v, u) \in \cup E_i\}|\). Similarly we can define the \(E_i\)-out-degree and \(E_i\)-in-degree as in the case of digraphs.

2.1.7 Source and sink of digraph structure

Definition 2.1.24. Let \((V; E_1, E_2, \ldots, E_n)\) be a digraph structure. A vertex \(v \in G\) is called an \(E_1 E_2 \cdots E_n\)-source if for every vertex \(x \in G\), there is an \(E_1 E_2 \cdots E_n\)-path from \(v\) to \(x\).

Definition 2.1.25. A vertex \(u \in G\) is called an \(E_1 E_2 \cdots E_n\)-sink if for every vertex \(y \in G\) there is an \(E_1 E_2 \cdots E_n\)-path from \(y\) to \(u\). As in digraphs, we define \(E_i\)-source and \(E_i\)-sink.

2.1.8 Reachable sets and antecedent sets of a vertex

Definition 2.1.26. Let \((V; E_1, E_2, \ldots, E_n)\) be a digraph structure and let \(u \in G\). Then the \(E_1 E_2 \cdots E_n\)-reachable set of \(u\) is defined as:

\[ R_{1,2,3,\ldots,n}(u) = \{x \in G : \text{there is an } E_1 E_2 \cdots E_n\text{-path from } u \text{ to } x\} \]

Similarly, the \(E_1 E_2 \cdots E_n\)-antecedent set of \(v\) is defined as:

\[ Q_{1,2,3,\ldots,n}(u) = \{x \in G : \text{there is an } E_1 E_2 \cdots E_n\text{-path from } x \text{ to } u\} \]
As in the case of digraphs, we can define the $E_i$-reachable set and $E_i$-antecedent set of a vertex.

In this chapter we may use the following notations. Let $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ Cayley digraph structure.

(1) Let $S_1, S_2, \ldots, S_n$ be subsets of a group $G$, then we may define the product $S_1, S_2, \ldots, S_n$ as follows:

$$S_1S_2\ldots S_n = \{(s_1s_2)s_3\ldots s_n : s_i \in S_i, i = 1, 2, \ldots, n\}.$$ 

If $S_1 = S_2 = \cdots = S$, we denote the above product as $S^n$.

(2) Let $A_k$ be the union of set of all $k$ products of the form $S_{i_1}S_{i_2}\cdots S_{i_k}$ from the set $\{S_1, S_2, \ldots, S_n\}$. Then $\bigcup_k A_k$ is denoted by $[S]$.

(3) Let $A = \{S_i \cup S_i^{-1} : i = 1, 2, \cdots, n\}$ and $B_k$ be the set of all finite products of elements from $A$ taken $k$ at time. We define $[[S]] = \bigcup_k B_k$.

(4) Let $A$ be a subset of a group $G$, then the semigroup generated by $A$ is denoted by $<A>$.

### 2.2 Cayley digraph structure

In this section we generalize the definition of Cayley digraphs as follows:

**Definition 2.2.1.** Let $G$ be a group and $S_1, S_2, \ldots, S_n$ be mutually disjoint subsets of $G$. Then Cayley digraph structure of $G$ with respect to $S_1, S_2, \ldots, S_n$ is defined as the digraph structure $X = (G; E_1, E_2, \ldots, E_n)$, where

$$E_i = \{(x, y) : x^{-1}y \in S_i\}, i = 1, 2, 3, \ldots, n.$$ 

The sets $S_1, S_2, \ldots, S_n$ are called connection sets of $X$. The Cayley digraph structure of $G$ with respect to $S_1, S_2, \ldots, S_n$ is denoted by $\text{Cay}(G; S_1, S_2, \ldots, S_n)$.

A digraph structure with only one connection set is the usual Cayley digraph.

So a Cayley digraph structure is a generalization of Cayley digraph.
2.2.1 Examples of Cayley digraph structures

The following are some examples of Cayley digraph structures.

**Example 2.2.2.** Let $G = \mathbb{Z}$, the additive group of integers and let $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$. Then the Cayley digraph structure $\text{Cay}(G; S_1, S_2, S_3)$ is shown in figure 2.2.

![Figure 2.2: Cay(\mathbb{Z}; S_1 = \{1\}, S_2 = \{2\}, S_3 = \{3\})](image)

**Example 2.2.3.** Let $G = \mathbb{Z} \oplus \mathbb{Z}$, the direct sum of the group of integers and let $S_1 = \{(1,0)\}$, $S_2 = \{(0,1)\}$, $S_3 = \{(1,1)\}$. Then the Cayley digraph structure $\text{Cay}(G; S_1, S_2, S_3)$ is shown in figure 2.3.

![Figure 2.3: Cay(\mathbb{Z} \oplus \mathbb{Z}; S_1 = \{(1,0)\}, S_2 = \{(0,1)\}, S_3 = \{(1,1)\})](image)

**Example 2.2.4.** Consider the permutation group $S_4$ and the following connection sets:

$S_1 = \{(12), (13)\}$ and $S_2 = \{(14)\}$

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The Cayley digraph structure $\text{Cay}(S_4; S_1, S_2)$ is shown in figure 2.4.

![Cayley digraph](image)

Figure 2.4: The Cayley digraph structure $\text{Cay}(S_4; \{(12), (13)\}, \{(14)\})$.

**Example 2.2.5.** Consider the permutation group $S_4$ and the following connection sets:

$$S_1 = \{(14), (23)\} \text{ and } S_2 = \{(13)(24)\}.$$  

The Cayley digraph structure $\text{Cay}(S_4; S_1, S_2)$ is shown in figure 2.5.
Figure 2.5: The Cayley digraph structure Cay($S_4; \{(14), (23)\}, \{(13), (24)\}$).

2.2.2 Main Theorem

Theorem 2.2.6. If $G$ is a group and let $S_1, S_2, \ldots, S_n$ are mutually disjoint subsets of $G$, then the Cayley digraph structure Cay($G; S_1, S_2, \ldots, S_n$) is vertex transitive.

Proof. Let $a$ and $b$ be any two arbitrary elements in $G$. Define a mapping $\varphi : G \rightarrow G$ by

$$\varphi(x) = ba^{-1}x \text{ for all } x \in G.$$ 

This mapping defines a permutation of the vertices of Cay($G; S_1, S_2, \ldots, S_n$). It is also an automorphism. Let $x, y \in G$ such that $y = xz$. Note that

$$(x, y) \in E_i \iff x^{-1}y \in S_i \text{ for some } i$$

$$\iff (ba^{-1}x)^{-1}(ba^{-1}y) \in S_i \text{ for some } i$$

$$\iff (\varphi(x), \varphi(y)) \in E_i$$

We note that

$$\varphi(a) = ba^{-1}a = b$$
Hence Cay\((G; S_1, S_2, \ldots, S_n)\) is vertex transitive.

\[\text{2.2.3 Corollaries}\]

**Corollary 2.2.7.** Cay\((G; S_1, S_2, \ldots, S_n)\) is an \(E_1E_2 \cdots E_n\)-trivial digraph structure ⇔ \(S_i = \emptyset\) for all \(i\).

*Proof.* By definition, Cay\((G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2 \cdots E_n\)-trivial ⇔ \(E_i = \emptyset\) for all \(i\). This implies that \(S_i = \emptyset\) for all \(i\). □

**Corollary 2.2.8.** Cay\((G; S_1, S_2, \ldots, S_n)\) is an \(E_i\)-trivial digraph structure ⇔ \(S_i = \emptyset\).

**Corollary 2.2.9.** Cay\((G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2 \cdots E_n\)-reflexive ⇔ \(1 \in S_i\) for some \(i\).

*Proof.* Assume that Cay\((G; S_1, S_2, \ldots, S_n)\) is an \(E_1E_2 \cdots E_n\)-reflexive digraph structure. Then for every \(x \in G\), \((x, x) \in E_i\) for some \(i\). This implies that \(x^{-1}x \in S_i\) for some \(i\). That is, \(1 \in S_i\) for some \(i\).

Conversely, assume that \(1 \in S_i\) for some \(i\). This implies for each \(x \in G\), \((x, x) \in E_i\) for some \(i\). That is, \((x, x) \in \bigcup E_i\) for all \(x \in G\). □

**Corollary 2.2.10.** Cay\((G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2 \cdots E_n\)-symmetric if and only if \(S_i = S_i^{-1}\) for all \(i\).

*Proof.* First, assume that Cay\((G; S_1, S_2, \ldots, S_n)\) is an \(E_1E_2 \cdots E_n\)-symmetric digraph structure. Let \(a \in S_i\). Then \((1, a) \in E_i\). Since Cay\((G; S_1, S_2, \ldots, S_n)\) is symmetric, \((a, 1) \in E_i\). This implies that the equation \(1 = at\) has a solution in \(S_i\). That is, \(a \in S_i^{-1}\). Hence \(S_i \subseteq S_i^{-1}\). Similarly, we can prove that \(S_i^{-1} \subseteq S_i\).

Conversely, assume that \(S_i = S_i^{-1}\) for all \(i\). Suppose that \((x, y) \in E_i\). Then the equation \(y = xz\) has a solution in \(S_i\). That is, \(z \in S_i\). Consider the equation \(x = yt\). Observe that

\[
(x, y) \in E \iff x^{-1}y \in S_i \iff x^{-1}y \in S_i^{-1} \iff (x^{-1}y)^{-1} \in S_i
\]
This completes the proof.}

**Corollary 2.2.11.** Cay$(G; S_1, S_2, \ldots, S_n)$ is $E_1 E_2 \cdots E_n$-transitive if and only if for every $i, j$, $S_i S_j \subseteq S_k$ for some $k$.

**Proof.** First, assume that Cay$(G; S_1, S_2, \ldots, S_n)$ is $E_1 E_2 \cdots E_n$-transitive. Let $x \in S_i S_j$. Then $x = z_1 z_2$ for some $z_1 \in S_i$ and $z_2 \in S_j$. This implies that $(1, z_1) \in E_i$ and $(z_1, z_1 z_2) \in E_j$. Since Cay$(G, S_1, S_2, \ldots, S_n)$ is transitive $(1, z_1 z_2) \in E_k$ for some $k$. That is $z_1 z_2 \in S_k$. Hence $S_i S_j \subseteq S_k$ for some $k$.

Conversely assume that for each $i, j$, $S_i S_j \subseteq S_k$ for some $k$. Let $(1, x), (x, y) \in E_i$. Then $x \in S_i$ for some $i$ and $x^{-1} y \in S_j$ for some $j$. This implies that $y \in S_i S_j$. Since $S_i S_j \subseteq S_k$ for some $k$, $(1, y) \in E_k$.

Hence Cay$(G; S_1, S_2, \ldots, S_n)$ is transitive.}

**Corollary 2.2.12.** Cay$(G; S_1, S_2, \ldots, S_n)$ is $E_1 E_2 \cdots E_n$-complete if and only if $G = \cup S_i$.

**Proof.** Suppose Cay$(G; S_1, S_2, \ldots, S_n)$ is $E_1 E_2 \cdots E_n$-complete. Then for every $x \in G$, we have $(1, x) \in \cup E_i$. This implies that $x \in S_i$ for some $i$. This implies that $G = \cup S_i$.

Conversely, assume that $G = \cup S_i$. Let $x$ and $y$ be two arbitrary elements in $G$ such that $y = xz$. Then $z \in G$. This implies that $z \in S_i$ for some $i$. That is, $(1, z) \in \cup E_i$. That is $(x, xz) = (x, y) \in \cup E_i$. This shows that Cay$(G; S_1, S_2, \ldots, S_n)$ is $E_1 E_2 \cdots E_n$-complete.}

**Corollary 2.2.13.** Cay$(G; S_1, S_2, \ldots, S_n)$ is a union of complete graph if and only if each $S_i$ is a subgroup of $G$.

**Corollary 2.2.14.** Cay$(G; S_1, S_2, \ldots, S_n)$ is $E_1 E_2 \cdots E_n$-connected if and only if $G = [S]$.

**Proof.** Suppose Cay$(G; S_1, S_2, \ldots, S_n)$ is $E_1 E_2 \cdots E_n$-connected and let $x \in G$. Let

$$(1, y_1, y_2, \ldots, y_n, x)$$
be a $E_1E_2\cdots E_n$-path leading from 1 to $x$. Then we have,

$$y_1 \in S_i, y_1^{-1}y_2 \in S_j, \ldots, y_n^{-1}x \in S_k.$$ 

This implies that $x \in A$ for some $A \in [S]$. Since $x$ is arbitrary, $G = [S]$.

Conversely, assume that $G = [S]$. Let $x$ any arbitrary elements in $G$. Then $x \in S_iS_j\cdots S_k$ for some $i, j, \ldots$ and $k$. This implies that $x = s_is_j\ldots s_k$ for some $i, j, \ldots$ and $k$. Then clearly, $(1, s_is_j, \ldots, s_is_j\ldots s_k)$ is an $E_1E_2\cdots E_n$-path from 1 to $x$. That is

$$(x, xs_1, xs_is_j, \ldots, xs_is_j\ldots s_k)$$

is a $E_1E_2\cdots E_n$-path from $x$ to $y$. Hence $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is connected.

\[\square\]

**Corollary 2.2.15.** $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is $E_i$-connected if and only if $G = [S_i]$.

**Corollary 2.2.16.** $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is weakly connected if and only if $G = [[S]]$.

**Proof.** Suppose $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is weakly connected. Then there exists a weak path, say

$$(1, x_1, x_2, \ldots, x_n, x)$$

from 1 to $x$. This implies that

$$x_1 \in S_i \cup S_i^{-1} \text{ for some } i$$
$$x_1^{-1}x_2 \in S_j \cup S_j^{-1} \text{ for some } j$$
$$\vdots$$
$$x_n^{-1}x \in S_k \cup S_k^{-1} \text{ for some } k.$$ 

This implies that $x \in [[S]]$. Since $x$ is arbitrary $G = [[S]]$.

Conversely assume that $G = [[S]]$. Let $x$ and $y$ be any two elements in $G$. Then the equation $y = xz$ has a unique solution $z \in G$. This implies that
\[ z \in (S_i \cup S_i^{-1})(S_j \cup S_j^{-1}) \cdots (S_k \cup S_k^{-1}) \text{ for some } i, j, \ldots, k. \]That is
\[ z = x_1 x_2 x_3 \cdots x_k \]
where \( x_i \in (S_i \cup S_i^{-1}) \). This implies that
\[ (1, x_1, x_1 x_2, x_1 x_2 x_3, \ldots, x_1 x_2 x_3 \cdots x_k) \]
is weak path from 1 to \( z \). That is
\[ (x, xx_1, xx_1 x_2, xx_1 x_2 x_3, \ldots, xx_1 x_2 x_3 \cdots x_k) \]
is a weak path from \( x \) to \( y \). Hence \( G \) is weakly connected.

**Corollary 2.2.17.** \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is \( E_1 E_2 \cdots E_n \)-quasi connected if and only if \( G = [S]^{-1}[S] \).

**Proof.** First, assume that \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is quasi connected. Let \( x \) be any arbitrary element in \( G \). Then there exists a vertex \( y \in G \) such that there is a path from \( y \) to 1, say:
\[ (y, y_1, y_2, \cdots, y_n, 1) \]
and a path from \( y \) to \( x \), say:
\[ (y, x_1, x_2, \ldots, x_m, x). \]
Then we have the following system of equations:
\[ y_1 = yz_1 \text{ for some } z_1 \in S_1 \]
\[ y_2 = y_1 z_2 \text{ for some } z_2 \in S_2 \]
\[ y_3 = y_2 z_3 \text{ for some } z_3 \in S_3 \]
\[ \vdots \]
\[ 1 = y_n z_{n+1} \text{ for some } z_{n+1} \in S_{n+1} \] (2.1)
and

\[ x_1 = y t_1 \text{ for some } z_1 \in S_{i_1} \]
\[ x_2 = x_1 t_2 \text{ for some } z_2 \in S_{i_2} \]
\[ x_3 = x_2 t_3 \text{ for some } z_3 \in S_{i_3} \]
\[ \vdots \]
\[ x = x_m t_{m+1} \text{ for some } z_{m+1} \in S_{i_{m+1}}. \]

Observe that equation (5.1) can be written as:

\[ 1 = y(w_1 w_2 \ldots w_{n+1}) \text{ for some } w_k \in S_{i_k}, k = 1, 2, \ldots, n + 1. \] (2.3)

This implies that

\[ y \in [S]^{-1}. \] (2.4)

Similarly, equation (5.2) can be written as:

\[ x = y(v_1 v_2 \ldots v_{m+1}) \text{ for some } v_k \in S_{i_k}, k = 1, 2, \ldots, m + 1. \] (2.5)

From equations (5.3) and (5.4), we have

\[ x \in [S]^{-1}[S]. \] (2.6)

Since \( x \) is arbitrary, \( G = [S]^{-1}[S] \).
Conversely, assume that \( G = [S]^{-1}[S] \). Let \( x \) and \( y \) be two arbitrary vertices in \( G \). Let \( y = xz \). Then \( z \in G \). This implies that \( z \in G = [S]^{-1}[S] \). Then there exists \( z_1 \in [S]^{-1} \) and \( z_2 \in [S] \) such that \( z = z_1 z_2 \). \( z_1 \in [S]^{-1} \) implies that there exists \( t_k \in S_{i_k} \) such that

\[ 1 = z_1(t_1 t_2 \ldots t_n) \]

\[ i.e., \ 1 = (((z_1 r_1) r_2) \ldots r_n) \text{ for some } r_k \in S_{i_k}, k = 1, 2, \ldots, n. \]

This implies that

\[ (z_1, z_1 r_1, z_1 r_1 r_2, \ldots, 1) \]
is a path from $z_1$ to 1. That is

$$(yz_1, yz_1r_1, yz_1r_1r_2, \ldots, y)$$

is a path from $yz_1$ to $y$.

Similarly, $z_2 \in [S]$ implies that there exists $a_k \in S_{ik}$ such that

$$z_2 = a_1a_2 \ldots a_m.$$ 

Observe that

$$(z_2, a_1a_2, a_1a_2a_3, \ldots, 1)$$

is a path from $z_2$ to 1. That is,

$$(z_1z_2, z_1a_1a_2, a_1a_2a_3, \ldots, z_1)$$

is a path from $z$ to $z_1$. That is

$$(yz, yz_1a_1a_2, ya_1a_2a_3, \ldots, z_1)$$

is a path from $x$ to $yz_1$.

\[\square\]

Corollary 2.2.18. \text{Cay}(G; S_1, S_2, \ldots, S_n) is $E_i$-quasi connected if and only if $G = [S_i][S_i]^{-1}$.

Corollary 2.2.19. \text{Cay}(G; S_1, S_2, \ldots, S_n) is $E_1E_2 \cdots E_n$-locally connected if and only if $[S] = [S]^{-1}$.

\textbf{Proof.} Assume that \text{Cay}(G; S_1, S_2, \ldots, S_n) is $E_1E_2 \cdots E_n$-locally connected. Let $x \in [S]$. Then $x \in A_m$ for some $m$. Then $x = s_is_j \ldots s_n$. Let $x_0 = 1, x_1 = s_i, x_2 = s_is_j, \ldots, x_n = s_is_j \ldots s_n$. Then

$$(x_0, x_1, x_2, \ldots, x_n)$$

is a path leading from 1 to $x$. Since \text{Cay}(G; S_1, S_2, \ldots, S_n) is locally connected, there exists a path from $x$ to 1, say:

$$(x, y_1, y_2, \ldots, y_m, 1).$$
This implies that

\begin{align*}
y_1 &= xt_1 \text{ for some } t_1 \in S_i, \\
y_2 &= y_1t_2 \text{ for some } t_2 \in S_i, \\
&\vdots \\
1 &= y_m t_{m+1} \text{ for some } t_{m+1} \in S_i.
\end{align*}

This implies that \(1 = x(z_1z_2 \cdots z_n)\) for some \(z_k \in S_i, k = 1, 2, 3, \ldots (m + 1)\). That is \(x \in [S]^{-1}\). Hence \([S] = [S]^{-1}\). \hfill \Box

**Corollary 2.2.20.** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_i\)-locally connected if and only if \(G = [S_i]\).

**Corollary 2.2.21.** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2 \cdots E_n\)-semi connected if and only if \(G = [S] \cup [S]^{-1}\).

**Proof.** Assume that \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2 \cdots E_n\)-semi connected and let \(x \in G\). Then there is a path from 1 to \(x\), say

\[(1, x_1, x_2, \cdots, x_n, x)\]

or a path from \(x\) to 1, say

\[(x, y_1, y_2, \cdots, y_m, 1).\]

This implies that \(x \in [S]\) or \(x \in [S]^{-1}\). This implies that \(G = [S] \cup [S]^{-1}\). Similarly, if \(G = [S] \cup [S]^{-1}\), then one can prove that \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2 \cdots E_n\)-semi connected. \hfill \Box

**Corollary 2.2.22.** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is \(E_i\)-semi connected if and only if \(G = <S_i> \cup <S_i>^{-1}\).

**Corollary 2.2.23.** \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is an \(E_1E_2 \cdots E_n\)-quasi ordered set if and only if

\begin{align*}
(i) &\ 1 \in S_1 \cup S_2 \cdots \cup S_n, \\
(ii) &\ \text{for every } (i,j), S_iS_j \subseteq S_k \text{ for some } k.
\end{align*}
Corollary 2.2.24. Cay($G; S_1, S_2, \ldots, S_n$) is an $E_i$-quasi ordered set if and only if

(i) $1 \in S_i$,
(ii) $S_i^2 \subseteq S_i$.

Corollary 2.2.25. Cay($G; S_1, S_2, \ldots, S_n$) is an $E_1 E_2 \cdots E_n$-partially ordered set if and only if

(i) $1 \in S_1 \cup S_2 \cdots \cup S_n$,
(ii) for every $(i, j)$, $S_i S_j \subseteq S_k$ for some $k$
(iii) $\cup (S_i \cap S_i^{-1}) = \{1\}$.

Proof. Observe that

\[ x \in \cup (S_i \cap (S_i)^{-1}) \iff x \in (S_i \cap (S_i)^{-1}) \text{ for some } i \]
\[ \iff x \in S_i \text{ and } x \in (S_i)^{-1} \]
\[ \iff (1, x) \in E_i \text{ and } (x, 1) \in E_i \]
\[ \iff x = 1. \]

This completes the proof. \qed

Corollary 2.2.26. Cay($G; S_1, S_2, \ldots, S_n$) is an $E_i$-partially ordered set if and only if

(i) $1 \in S_i$
(ii) $S_i^2 \subseteq S_i$
(iii) $S_i \cap (S_i)^{-1} = \{1\}$.

Corollary 2.2.27. Let $A_m$ is the set of $m$ products of the form $S_{i_1} S_{i_2} \cdots S_{i_m}$. Then Cay($G; S_1, S_2, \ldots, S_n$) is an $E_1 E_2 \cdots E_n$-Hasse diagram if and only if $C \cap S_i = \emptyset$ for all $i$ and for all $C \in A_m$.

Proof. Suppose the condition holds. Let $x_0, x_1, \ldots, x_m$ be $(m + 1)$ elements in
such that \((x_i, x_{i+1}) \in \cup E_i\) for \(i = 0, 1, \ldots, m - 1\). This implies that

\[
\begin{align*}
x_1 &= x_0 t_1 \text{ for some } t_1 \in S_{i_1} \\
x_2 &= x_1 t_2 \text{ for some } t_2 \in S_{i_2} \\
x_3 &= x_2 t_3 \text{ for some } t_3 \in S_{i_3} \\
& \vdots \\
x_m &= x_{m-1} t_m \text{ for some } t_m \in S_{i_m}.
\end{align*}
\]

The last equation can be written as:

\[
x_m = ((x_{n-2} t_{m-1})) t_m = ((x_0 t_1) t_2) \cdots t_m = x_0 (z_1 z_2 \ldots z_m) \text{ for some } z_k \in S_{i_k}, k = 1, 2, \ldots, m = x_0 t, \text{ where } t = z_1 z_2 \ldots z_m \in A_m.
\]

Since \(C \cap S_i = \emptyset\) for all \(i\) and for all \(C \in A_n\), \((x_0, x_m) \notin \cup E_i\).

Conversely assume that \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is an \(E_1 E_2 \cdots E_n\) - Hasse diagram. We will show that \(C \cap S_i = \emptyset\) for all \(i\) and for all \(C \in A_m\). Let \(S_{i_1} S_{i_2} S_{i_3} \cdots S_{i_m}\) be any element in \(A_m\). Let \(x \in S_{i_1} S_{i_2} S_{i_3} \cdots S_{i_m}\). Then \(x = s_{i_1} s_{i_2} s_{i_3} \cdots s_{i_m}\) for some \(s_{i_k} \in S_{i_k}\). This implies that

\[
(1, s_{i_1}, s_{i_2} s_{i_3}, \ldots, x)
\]

is a path from 1 to \(x\). Since \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is an \(E_1 E_2 \cdots E_n\) - Hasse diagram, \(x \notin S_i\) for any \(i\). That is, \(A_m \cap S_i = \emptyset\) for all \(i\).

\[\Box\]

**Theorem 2.2.28.** The \(E_1 E_2 \cdots E_n\)- out-degree of \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is the cardinal number \(|S_1 \cup S_2 \cup \cdots \cup S_n|\).

**Proof.** Since \(\text{Cay}(G; S_1, S_2, \ldots, S_n)\) is vertex- transitive it suffices to consider the out degree of the vertex 1 \(\in G\). Observe that

\[
\rho(1) = \{u : (1, u) \in \cup E_i\} = \{u : u \in S_i \text{ for some } i\} = S_1 \cup S_2 \cup \cdots \cup S_n,
\]

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Hence $|\rho(1)| = |S_1 \cup S_2 \cup \cdots \cup S_n|$. 

**Theorem 2.2.29.** The $E_i$-out-degree of Cay($G; S_1, S_2, \ldots, S_n$) is the cardinal number $|S_i|$. 

**Theorem 2.2.30.** The $E_iE_2 \cdots E_n$ -in-degree of Cay($G; S_1, S_2, \ldots, S_n$) is the cardinal number $|(S_1)^{-1} \cup (S_2)^{-1} \cup \cdots \cup (S_n)^{-1}|$. 

**Proof.** Since Cay($G; S_1, S_2, \ldots, S_n$) is vertex transitive it suffices to consider the in degree of the vertex $1 \in G$. Observe that 

$$\sigma(1) = \{u : (u, 1) \in \cup E_i\}$$

$$= \{u : u^{-1} \in S_i \text{ for some } i\}$$

$$= (S_1)^{-1} \cup (S_2)^{-1} \cup \cdots \cup (S_n)^{-1}.$$ 

Hence $|\sigma(1)| = |(S_1)^{-1} \cup (S_2)^{-1} \cup \cdots \cup (S_n)^{-1}|$. 

**Theorem 2.2.31.** The $E_i$ in-degree of Cay($G; S_1, S_2, \ldots, S_n$) is the cardinal number $|S_i|^{-1}$. 

**Theorem 2.2.32.** For $k = 1, 2, 3, \ldots$ let $A_k$ be the set of all $k$ products of the form $S_{i_1}S_{i_2}S_{i_3}\cdots S_{i_k}$. If Cay($G; S_1, S_2, \ldots, S_n$) has finite diameter, then the diameter of Cay($G; S_1, S_2, \ldots, S_n$) is the least positive integer $m$ such that 

$$G = A_m.$$ 

**Proof.** Let $m$ be the smallest positive integer such that $G = A_m$. We will show that the diameter of Cay($G; S_1, S_2, \ldots, S_n$) is $m$. Let $x$ and $y$ be any two arbitrary elements in $G$ such that $y = xz$. Then $z \in G$. This implies that $x \in A_m$. But then $z$ has a representation of the form $x = s_{i_1}s_{i_2}\cdots s_{i_m}$. This implies that 

$$(1, s_{i_1}, s_{i_1}s_{i_2}, \ldots, z)$$ 

is path of $n$ edges from 1 to $z$. That is 

$$(x, xs_{i_1}, xs_{i_1}s_{i_2}, \ldots, y)$$
is a path of length $m$ from $x$ to $y$. This shows that $d(x, y) \leq m$. Since $x$ and $y$ are arbitrary,

$$\max_{x,y \in G}\{d_{1,2,...,n}(x,y)\} \leq m.$$  

Therefore the diameter of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is less than or equal to $m$. On the other hand let the diameter of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ be $k$. Let $x \in G$ and $d_{1,2,...,n}(1, x) = k$. Then we have $x \in B$ for some $B \in A_k$. That is

$$G = A_k.$$ 

Now by the minimality of $k$, we have $m \leq k$. Hence $k = m$. \hfill \Box

**Corollary 2.2.33.** The vertex 1 is an $E_1E_2\cdots E_n$-source of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ if and only if $G = [S]$.

*Proof.* First, assume that 1 is an $E_1E_2\cdots E_n$-source of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$. Then for any vertex $x \in G$, there is an $E_1E_2\cdots E_n$-path from 1 to $x$. This implies that $G = [S]$.

Conversely, if $G = [S]$, one can prove that 1 is an $E_1E_2\cdots E_n$-source. \hfill \Box

**Corollary 2.2.34.** The vertex 1 is an $E_i$-source of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ if and only if $G = <S_i>$. 

**Corollary 2.2.35.** The vertex 1 is an $E_1E_2\cdots E_n$-sink of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ if and only if $G = [S]^{-1}$.

*Proof.* First, assume that 1 is an $E_1E_2\cdots E_n$-sink of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$. Then for each $x \in G$, there is an $E_1E_2\cdots E_n$-path from $x$ to 1. This implies that $x \in [S]^{-1}$. Hence $G = [S]^{-1}$.

Conversely, if $G = [S]^{-1}$, one can easily prove that 1 is an $E_1E_2\cdots E_n$-sink of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$. \hfill \Box

**Corollary 2.2.36.** The vertex 1 is an $E_i$-sink of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ if and only if $G = <S_i>^{-1}$.

**Corollary 2.2.37.** The $E_1E_2\cdots E_n$-reachable set $R_{1,2,...,n}(1)$ of the vertex 1 is the set $[S]$. 

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Proof. By definition,

\[ R(1) = \{ x : \text{there exists an } E_1E_2\cdots E_n \text{-path from 1 to } x \}. \]

Observe that

\[ x \in R_{1,2,\ldots,n}(1) \iff \text{there exists an } E_1E_2\cdots E_n \text{-path from 1 to } x, \text{ say } (1, x_1, x_2, \ldots, x_n, x) \]
\[ \iff x \in [S]. \]

Therefore, \( R_{1,2,3,\ldots,n}(1) = [S] \).

**Corollary 2.2.38.** The \( E_i \)-reachable set \( R_i(1) \) of the vertex 1 is the set \( <S_i> \).

**Corollary 2.2.39.** The \( E_1E_2\cdots E_n \)-antecedent set \( Q_{1,2,\ldots,n}(1) \) of the vertex 1 is the set \( [S]^{-1} \).

Proof. Observe that

\[ x \in Q_{1,2,\ldots,n}(1) \iff \text{there exists an } E_1E_2\cdots E_n \text{-path from } x \text{ to 1, say } (x, x_1, x_2, \ldots, x_n, 1) \]
\[ \iff x \in [S]^{-1}. \]

Therefore,

\[ Q_{1,2,\ldots,n}(1) = [S]^{-1}. \]

This completes the proof.

**Corollary 2.2.40.** The \( E_i \)-antecedent set \( Q_i(1) \) of the vertex 1 is the set \( <S_i>^{-1} \).