Chapter 4

Coset Cayley Digraph Structures

This chapter deals with a class of Cayley digraph structures induced by groups, whose vertices are cosets. By taking \(G/H\), the collection of all left cosets of \(H\) in \(G\) and defining relations \(E_1 \ E_2 \ldots \ E_n\) suitably we prove that the Cayley digraph structure \((G/H; E_1, E_2, \ldots, E_n)\) is vertex transitive. We include various graph properties in terms of algebraic properties. Moreover we prove that every Cayley digraph structure \((V; W_1, W_2, \ldots, W_n)\) is isomorphic to \((G/H; E_1, E_2, \ldots, E_n)\).

4.1 Definitions

Definition 4.1.1. Let \(G\) be a group and \(S_1, S_2, \ldots, S_n\) be mutually disjoint subsets of \(G\) and \(H\) be a subgroup of \(G\). Then coset Cayley digraph structure of \(G\) with respect to \(S_1, S_2, \ldots, S_n\) is defined as the digraph structure \((G/H; E_1, E_2, \ldots, E_n)\), where

\[
E_i = \{(xH, yH) : x^{-1}y \in HS_iH\}.
\]

The sets \(S_1, S_2, \ldots, S_n\) are called connection sets of \((G/H; E_1, E_2, \ldots, E_n)\).

We denote the coset Cayley digraph structure \(\text{Cay}(G/H; HS_1H, HS_2H, \ldots, HS_nH)\) induced by the group \(G\) by \(\mathcal{C}\).

Here we may use the following notations.
Let Cay\((G/H; HS_1 H, HS_2 H, \ldots, HS_n H)\) be a coset Cayley digraph structure induced by the group \(G\).

(1) Let \(A_k\) be the union of set of all \(k\) products of the form \((HS_i H)(HS_j H)\cdots(HS_k H)\). Then \(\bigcup_k A_k\) is denoted by \([HSH]\).

(2) Let \(A_k^{-1}\) be the union of set of all \(k\) products of the form:

\((HS_{i_1}^{-1} H)(HS_{i_2}^{-1} H)\cdots(HS_{i_k}^{-1} H)\).

Then \(\bigcup_k A_k^{-1}\) is denoted by \([HS^{-1} H]\).

4.1.1 Main theorems

**Theorem 4.1.2.** If \(G\) is a group and let \(S_1, S_2, \ldots, S_n\) are mutually disjoint subsets of \(G\) and \(H\) is a subgroup of \(G\), then the coset Cayley digraph structure \(\mathcal{C}\) is vertex transitive.

**Proof.** To see that Cay\((G/H; HS_1 H, HS_2 H, \ldots, HS_n H)\) is a vertex transitive digraph structure, we first need only show that \(E_i\)'s are well defined. Let \(x, y, x', y'\) be any four elements of \(G\) with \(xH = x'H\) and \(yH = y'H\). Then \(x = x'h_1\) and \(y = y'h_2\) for some \(h_1, h_2 \in H\). Observe that

\[
(xH, yH) \in E_i \Leftrightarrow x^{-1}y \in HS_i H \\
\Leftrightarrow (x'h_1)^{-1}(y'h_2) \in HS_i H \\
\Leftrightarrow h_1^{-1}(x')^{-1}y' \in HS_i H \\
\Leftrightarrow (x')^{-1}y' \in HS_i H \\
\Leftrightarrow (x'H, y'H) \in HS_i H.
\]

Hence each \(E_i\)'s are well defined and hence \(\mathcal{C}\) is a digraph structure. Let \(aH\) and \(bH\) be any two arbitrary elements in \(G/H\). Define a mapping \(\varphi : G \rightarrow G\) by

\[
\varphi(xH) = ba^{-1}xH \text{ for all } xH \in G/H.
\]

This mapping defines a permutation of the vertices of \(\mathcal{C}\). It is also an auto-
morphism. Note that
\[(xH, yH) \in E_i \iff x^{-1}y \in HS_iH \]
\[\iff (ba^{-1}x)^{-1}(ba^{-1}y) \in HS_iH \]
\[\iff (ba^{-1}x, ba^{-1}y) \in E_i \]
\[\iff (\varphi(xH), \varphi(yH)) \in E_i.\]

Also we note that
\[\varphi(aH) = ba^{-1}aH = bH.\]

Hence \(\mathcal{C}\) is vertex transitive digraph structure. \(\square\)

**Theorem 4.1.3.** Let \((V; W_1, W_2, \cdots, W_n)\) be any vertex transitive digraph structure such that \(|V| \geq n\). Then the Cayley digraph structure \((V; W_1, W_2, \cdots, W_n)\) is isomorphic to \(\text{Cay}(G/H; HS_1H, HS_2H, \ldots, HS_nH)\).

**Proof.** Let \(G\) be the automorphism group of the digraph structure \((V; W_1, W_2, \cdots, W_n)\). Let \(q_1, q_2, \cdots, q_n\) be fixed elements in \(V\). For \(i = 1, 2, \ldots, n\), define the following:

\[H_i := \{\theta \in G : \theta(q_i) = q_i\},\]
\[S_i := \{\theta \in G : (q_i, \theta(q_i)) \in W_i\}.\]

Note that \(H = \cap_{i=1}^n H_i\) is a subgroup of \(G\). Construct the Cayley digraph structure \(\text{Cay}(G/H; HS_1H, HS_2H, \ldots, HS_nH)\) as in theorem 2.2.1.

Define a map \(\varphi : G/H \longrightarrow V\) by
\[(xH)\varphi = x(q_i)\text{ for all } xH \in G/H.\]

where \(q_i\) is a fixed element in the set \(\{q_1, q_2, \ldots, q_n\}\).

(i) \(\varphi\) is well defined:
Let \(xH = yH\). Then \(y = xh_1\), for some \(h_1 \in H\). Observe that
\[\varphi(yH) = y(q_i)\]
\[= (xh_1)(q_i)\]
\[
= x[h_1(q_i)] \\
= x(q_i) \\
= \varphi(xH).
\]

(ii) \( \varphi \) is one to one:

\[
\varphi(xH) = \varphi(yH) \iff x(q_i) = y(q_i) \\
\iff y^{-1}x(q_i) = q_i \\
\iff y^{-1}x \in H \\
\iff xH = yH.
\]

(iii) \( \varphi \) is onto:
Let \( v \) be any element in \( V \). Since \((V; W_1, W_2, \cdots, W_n)\) is vertex transitive, there exists an automorphism \( \theta \) such that \( \theta(v) = q_i \). This implies that \( v = \theta^{-1}(q_i) \). That is, \( v = \varphi(\theta^{-1}H) \).

(iv) \( \varphi \) preserves adjacency relation:

Observe that

\[
(xH, yH) \in E_i \iff x^{-1}y \in HS_iH \\
\iff x^{-1}y = h_1s_ih_2 \\
\iff h_1^{-1}x^{-1}y^{-1}h_2^{-1} = s_i \in S_i \\
\iff (q_i, (h_1^{-1}x^{-1}y^{-1}h_2^{-1})(q_i)) \in W_i \\
\iff (h_1(q_i), x^{-1}y(q_i)) \in W_i \\
\iff (x(q_i), y(q_i)) \in W_i \\
\iff (\varphi(xH), \varphi(yH)) \in W_i.
\]

This completes the proof.

\[\square\]

4.1.2 Corollaries

**Corollary 4.1.4.** The coset Cayley graph structure \( \mathcal{C} \) is an \( E_1E_2\cdots E_n \) -trivial digraph structure \( \iff S_i = \emptyset \) for all \( i \).

**Proof.** By definition, \( \mathcal{C} \) is \( E_1E_2\cdots E_n \)-trivial \( \iff E_i = \emptyset \) for all \( i \). This implies
that $S_i = \emptyset$ for all $i$.  \hfill $\square$

**Corollary 4.1.5.** The coset Cayley graph structure $\mathcal{C}$ is an $E_i$-trivial digraph structure $\iff S_i = \emptyset$.

**Corollary 4.1.6.** The coset Cayley graph structure $\mathcal{C}$ is $E_1 E_2 \cdots E_n$-reflexive $\iff 1 \in S_i$ for some $i$.

*Proof.* Assume that $\mathcal{C}$ is an $E_1 E_2 \cdots E_n$-reflexive digraph structure. Then for every $xH \in G/H$, $(xH, xH) \in E_i$ for some $i$. This implies that $1 \in HS_i H$ for some $i$. Conversely, assume that $1 \in S_i$ for some $i$. This implies for each $xH \in G/H$, $(xH, xH) \in E_i$ for some $i$. That is, $(xH, xH) \in \cup E_i$ for all $x \in G$. \hfill $\square$

**Corollary 4.1.7.** The coset Cayley graph structure $\mathcal{C}$ is $E_i$-reflexive $\iff 1 \in HS_i H$.

**Corollary 4.1.8.** The coset Cayley graph structure $\mathcal{C}$ is $E_1 E_2 \cdots E_n$-symmetric if and only if $HS_i H = HS_i^{-1} H$ for all $i$.

*Proof.* First, assume that $\mathcal{C}$ is an $E_1 E_2 \cdots E_n$-symmetric digraph structure. Let $a \in HS_i H$. Then $(H, aH) \in E_i$. Since $\mathcal{C}$ is symmetric $(a, 1) \in E_i$. This implies that $a^{-1} \in HS_i H$. That is $a \in HS_i^{-1} H$. Hence $HS_i H \subseteq HS_i^{-1} H$. Similarly, we can prove that $HS_i^{-1} H \subseteq HS_i H$.

Conversely, if $HS_i H = HS_i^{-1} H$, we can prove that $\mathcal{C}$ is an $E_1 E_2 \cdots E_n$-symmetric digraph structure. \hfill $\square$

**Corollary 4.1.9.** $\mathcal{C}$ is $E_i$-symmetric if and only if $HS_i H = HS_i^{-1} H$.

**Corollary 4.1.10.** $\mathcal{C}$ is an $E_1 E_2 \cdots E_n$-transitive if and only if for every $i, j$, $HS_i HS_j H \subseteq HS_k H$ for some $k$.

*Proof.* First, assume that $\mathcal{C}$ is an $E_1 E_2 \cdots E_n$-transitive. We will show that for all $(i, j)$, $HS_i HS_j H \subseteq HS_k H$ for some $k$. Let $x \in HS_i HS_j H = HS_i H HS_j H$. Then

$$x = z_1 z_2 \text{ for some } z_1 \in HS_i H, z_2 \in HS_j H.$$  

This implies that $(H, z_1 H) \in E_i$ and $(z_1 H, z_1 z_2 H) \in E_j$. Since $\mathcal{C}$ is $E_1 E_2 \cdots E_n$-transitive, $(H, z_1 z_2 H) \in HS_k H$ for some $k$. That is $z_1 z_2 \in HS_k H$. Hence $HS_i HS_j H \subseteq HS_k H$.  

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Conversely, assume that all \((i, j), HS_iHS_jH \subseteq HS_kH\) for some \(k\). We will show that \(C\) is \(E_1E_2\cdots E_n\)-transitive. Let \((H, xH) \in E_i, (xH, yH) \in E_j\). Then \(x \in HS_iH\) and \(x^{-1}y \in HS_jH\). This implies that \(y = xx^{-1}y \in HS_iHS_jH\). Since \(HS_iHS_jH \subseteq HS_kH\), we have \(y \in HS_kH\). It follows that \((H, yH) \in E_k\).

\[\square\]

**Corollary 4.1.11.** \(C\) is \(E_1E_2\cdots E_n\)-complete if and only if \(G = \cup HS_iH\).

**Proof.** Suppose \(C\) is \(E_1E_2\cdots E_n\)-complete. Then for every \(xH \in G/H\), we have \((H, xH) \in \cup E_i\). This implies that \(x \in HS_iH\) for some \(i\). This implies that \(G = \cup HS_iH\). Conversely, assume that \(G = \cup HS_iH\). Let \(xH\) and \(yH\) be two arbitrary elements in \(G/H\) such that \(y = xz\). Then \(z \in G\). This implies that \(z \in HS_iH\) for some \(i\). That is, \((H, zH) \in \cup E_i\). That is \((xH, xzH) = (xH, yH) \in \cup E_i\). This shows that \(C\) is complete.

\[\square\]

**Corollary 4.1.12.** \(C\) is \(E_i\)-complete if and only if \(G = HS_iH\).

**Corollary 4.1.13.** \(C\) is \(E_1E_2\cdots E_n\)-connected if and only if \(G = [HSH]\).

**Proof.** Suppose \(C\) is \(E_1E_2\cdots E_n\)-connected and let \(xH \in G/H\). Let

\[(H, y_1H, y_2H, \ldots, y_nH, xH)\]

be a \(E_1E_2\cdots E_n\)-path leading from \(H\) to \(xH\). Then we have,

- \(y_1 \in HS_{i_1}H\) for some \(i_1\)
- \(y_1^{-1}y_2 \in HS_{i_2}H\) for some \(i_2\)
- \(y_2^{-1}y_3 \in HS_{i_3}H\) for some \(i_3\)
- \(\vdots\)
- \(y_n^{-1}x \in HS_{i_{n+1}}H\) for some \(i_{n+1}\).

The above equation tells us that

\[x = y_1y_1^{-1}y_2y_2^{-1}y_3 \cdots y_n^{-1}x \in (HS_{i_1}H)(HS_{i_2}H)(HS_{i_3}H)\cdots(HS_{i_n}H) \subseteq [HSH].\]

Since \(x\) is arbitrary, \(G = [HSH]\).
Conversely, assume that \( G = [HSH] \). Let \( x \) and \( y \) be any arbitrary elements in \( G \). Let \( y = xz \). Then \( z \in G \). That is; \( z \in (HS_iH)(HS_jH) \cdots (HS_kH) \) for some \( i, j, \ldots \) and \( k \). This implies that \( z = s_is_j \ldots s_k \) for some \( i, j, \ldots \) and \( k \). Then clearly, \((H, s_is_js_H, \ldots, s_is_j \ldots s_kH)\) is an \( E_1E_2\cdots E_n \)-path from \( H \) to \( zH \). That is

\[
(xH, xs_is_jH, \ldots, xs_is_j \ldots s_kH)
\]
is a \( E_1E_2\cdots E_n \)-path from \( xH \) to \( yH \). Hence \( \mathcal{C} \) is connected. \( \square \)

**Corollary 4.1.14.** \( \mathcal{C} \) is \( E_i \)-connected if and only if \( G = < HS_iH > \), where \(< HS_iH >\) is the semigroup generated by \( HS_iH \).

**Corollary 4.1.15.** \( \mathcal{C} \) is \( E_1E_2\cdots E_n \)-quasi connected if and only if \( G = [HS^{-1}H][HSH] \).

**Proof.** First, assume that \( \mathcal{C} \) is quasi strongly connected. Let \( xH \) be any arbitrary element in \( G/H \). Then there exists a vertex \( yH \in G \) such that there is a path from \( yH \) to \( H \), say:

\[
(yH, y_1H, y_2H, \ldots, y_nH, H)
\]
and a path from \( yH \) to \( xH \), say:

\[
(yH, x_1H, x_2H, \ldots, x_mH, xH).
\]

Then we have the following system of equations:

\[
\begin{align*}
y^{-1}y_1 & \in HS_{i_1}H \\
y_1^{-1}y_2 & \in HS_{i_2}H \\
y_2^{-1}y_3 & \in HS_{i_3}H \\
& \quad \vdots \\
y_n^{-1} & \in HS_{i_{n+1}}H.
\end{align*}
\]
and

\[ y^{-1}x_1 \in HS_{i_1}H \]
\[ x_1^{-1}x_2 \in HS_{i_2}H \]
\[ x_2^{-1}x_3 \in HS_{i_3}H \]
\[ \vdots \]
\[ x_m^{-1}x \in HS_{m+1}H. \]  

(4.2)

From equation (5.1) we obtain the following:

\[ y^{-1} = (y^{-1}y_1)(y_1^{-1}y_2)(y_2^{-1}y_3) \cdots (y_n^{-1}) \in S_{i_2} \in (HS_{i_1}H)(HS_{i_2}H) \cdots (HS_{i_{n+1}}H). \]

This implies that

\[ y \in (HS_{i_1}^{-1}H)(HS_{i_2}^{-1}H) \cdots (HS_{i_{n+1}}^{-1}H) \in [HS^{-1}H]. \]  

(4.3)

Similarly, from equation (5.2) we obtain the following:

\[ y^{-1}x = (y^{-1}x_1)(x_1^{-1}x_2) \cdots (x_m^{-1}x) \in (HS_{i_1}H)(HS_{i_2}H) \cdots (HS_{i_{m+1}}H). \]  

(4.4)

That is

\[ y^{-1}x \in [HSH]. \]

That is

\[ x \in y[HSH] \subseteq [HS^{-1}H][HSH]. \]

Since \( x \) is arbitrary, we have

\[ G = [HS^{-1}H][HSH]. \]

Conversely, assume that \( G = [HS^{-1}H][HSH] \). Let \( x \) and \( y \) be two arbitrary vertices in \( G \). Let \( y = xz \). Then \( z \in G \). This implies that \( z \in [HS^{-1}H][HSH] \). Then there exists \( z_1 \in [HS^{-1}H] \) and \( z_2 \in [HSH] \) such that \( z = z_1z_2 \). \( z_1 \in [HS^{-1}H] \) implies that there exists \( t_k \in S_{i_k} \) such that

\[ z_1 = t_1t_2 \cdots t_n \text{ for some } t_k \in S_{i_k}^{-1}, k = 1, 2, \ldots, n. \]
This implies that
\[(z_1H, t_1t_2H \ldots t_{n-1}, \ldots, H)\]
is a path from \(z_1H\) to \(H\). That is
\[(yz_1H, yt_1t_2H \ldots t_{n-1}H, \ldots, yH)\]
is a path from \(yz_1H\) to \(yH\).
Similarly, \(z_2 \in [HS]\) implies that there exists \(a_k \in S_{ik}\) such that
\[z_2 = a_1a_2 \ldots a_m.\]
Observe that
\[(z_2H, a_1a_2H, a_1a_2a_3H, \ldots, H)\]
is a path from \(z_2H\) to \(H\). That is,
\[(z_1z_2H, z_1a_1a_2H, a_1a_2a_3H, \ldots, z_1H)\]
is a path from \(zH\) to \(z_1H\). That is
\[(yzH, yz_1a_1a_2H, ya_1a_2a_3H, \ldots, z_1H)\]
is a path from \(xH\) to \(z_1H\).

\[\square\]

**Corollary 4.1.16.** \(C\) is \(E_i\)-quasi connected if and only if \(G = \langle HS_i^{-1}H \rangle < HS_iH \rangle\).

**Corollary 4.1.17.** \(C\) is \(E_1E_2 \cdots E_n\)-locally connected if and only if \([HSH] = [HS^{-1}H]\).

**Proof.** Assume that \(C\) is \(E_1E_2 \cdots E_n\)-locally connected. Let \(x \in [S]\). Then \(x \in A_m\) for some \(m\). Then \(x = s_is_j \ldots s_m\). Let \(x_0 = 1, x_1 = s_i, x_2 = s_is_j, \ldots, x_m = s_is_j \ldots s_m\). Then
\[(x_0H, x_1H, x_2H, \ldots, x_mH)\]
is a path leading from 1 to \(x\). Since \(C\) is locally connected, there exists a path from \(x\) to 1, say:
\[(xH, y_1H, y_2H, \ldots, y_mH, H)\]
This implies that

\[ x^{-1}y_1 \in S_{i_1}, \]
\[ y_1^{-1}y_2 \in S_{i_2}, \]
\[ \vdots \]
\[ y_m^{-1} \in S_{i_m}. \]

The above equations tells us that \( x^{-1} \in [HS] \). That is \( x \in [HS^{-1}H] \). Hence \( [HS] \subseteq [HS^{-1}H] \). Similarly one can prove that \([HS] \supseteq [HS^{-1}H]\). This implies \([HS] = [HS^{-1}H]\). Conversely, if \([HS] = [HS^{-1}H]\), one can easily verify that \( C \) is \( E_1 E_2 \cdots E_n \) -locally connected.

**Corollary 4.1.18.** \( C \) is \( E_i \) -locally connected if and only if \( <HS_i^{-1}H> =<HS_iH> \).

**Corollary 4.1.19.** \( C \) is \( E_1 E_2 \cdots E_n \) -semi connected if and only if \( G = [HS] \cup [HS^{-1}H] \).

**Proof.** Assume that \( C \) is \( E_1 E_2 \cdots E_n \) -semi connected and let \( xH \in G/H \). Then there is a path from \( H \) to \( xH \), say

\[ (H, x_1 H, x_2 H, \cdots, x_n H, xH) \]

or a path from \( xH \) to \( H \), say

\[ (xH, y_1 H, y_2 H, \cdots, y_m H, H) \]

This implies that \( x \in [HS] \) or \( x \in [HS^{-1}H] \). This implies that \( G = [HS] \cup [HS^{-1}H] \). Similarly, if \( G = [HS] \cup [H^{-1}H] \), then one can prove that \( C \) is \( E_1 E_2 \cdots E_n \) -semi connected.

**Corollary 4.1.20.** \( C \) is \( E_i \) -semi connected if and only if \( G =<HS_iH> \cup <HS_i^{-1}H> \).

**Corollary 4.1.21.** \( C \) is an \( E_1 E_2 \cdots E_n \) -quasi ordered set if and only if

(i)\(1 \in (HS_1 H) \cup (HS_2 H) \cdots \cup (HS_n H),\)

(ii)for every \((i, j), (HS_i H)(HS_j H) \subseteq (HS_k H), \) for some \( k.\)
Corollary 4.1.22. \( C \) is an \( E_i \)-quasi ordered set if and only if

(i) \( 1 \in HS_i H \),

(ii) \( HS_i^2 H \subseteq HS_i H \).

Corollary 4.1.23. \( C \) if an \( E_1 E_2 \cdots E_n \)-partially ordered set if and only if

(i) \( 1 \in (HS_1 H) \cup (HS_2 H) \cdots \cup (HS_n H) \),

(ii) for every \( (i, j) \), \( (HS_i H)(HS_j H) \subseteq (HS_k H) \) for some \( k \)

(iii) \( \cup (HS_i H) \cap (HS_i^{-1} H) = \{1\} \).

Proof. Observe that

\[ x \in \cup (HS_i H) \cap H(S_i)^{-1} H \iff x \in (HS_i H) \cap (H(S_i)^{-1} H) \text{ for some } i \]

\[ \iff x \in HS_i H \text{ and } x \in H(S_i)^{-1} H \]

\[ \iff (H, xH) \in E_i \text{ and } (xH, H) \in E_i \]

\[ \iff x = 1. \]

This completes the proof.

Corollary 4.1.24. \( C \) if an \( E_i \)-partially ordered set if and only if

(i) \( 1 \in HS_i H \),

(ii) \( (HS_i H)^2 \subseteq HS_i H \)

(iii) \( (HS_i H) \cap (H(S_i)^{-1} H) = \{1\} \).

Corollary 4.1.25. Let \( A_m(m \geq 2) \) is the set of \( m \) products of the form \( S_{i_1}, S_{i_2}, \ldots, S_{i_m} \). Then \( C \) is an \( E_1 E_2 \cdots E_n \)-Hasse diagram if and only if \( C \cap S_i = \emptyset \) for all \( i \) and for all \( C \in A_m \).

Proof. Suppose the condition holds. Let \( x_0 H, x_1 H, \ldots, x_m H \) be \( (m + 1) \) elements in \( G/H \) such that \( (x_i H, x_{i+1} H) \in \cup E_i \) for \( i = 0, 1, \ldots, m - 1 \). This implies that

\[ x_0^{-1} x_1 \in S_{i_1} \]
\[x_1^{-1} x_2 \in S_{i_2}\]
\[x_2^{-1} x_3 \in S_{i_3}\]
\[\vdots\]
\[x_{m-1}^{-1} x_m \in S_{i_m}.
\]

The above equation tells us that \(x_0^{-1} x_m \in A_m\). Since \(C \cap S_i = \emptyset\) for all \(i\) and for all \(C \in A_m\), \((x_0, x_m) \notin E_i\).

Conversely assume that \(C\) is an \(E_1 E_2 \cdots E_n\)-Hasse diagram. We will show that \(C \cap S_i = \emptyset\) for all \(i\) and for all \(C \in A_m\). Let \(S_{i_1} S_{i_2} S_{i_3} \cdots S_{i_m}\) be any element in \(A_m\). Let \(x \in S_{i_1} S_{i_2} S_{i_3} \cdots S_{i_m}\). Then \(x = s_{i_1} s_{i_2} s_{i_3} \cdots s_{i_m}\) for some \(s_{i_k} \in S_{i_k}\). This implies that

\[(H, s_{i_1} H, s_{i_2} s_{i_3} H, \ldots, xH)\]

is a path from \(H\) to \(xH\). Since \(C\) is an \(E_1 E_2 \cdots E_n\)-Hasse diagram, \(x \notin S_i\) for any \(i\). That is, \(A_m \cap S_i = \emptyset\) for all \(i\). \(\square\)

**Corollary 4.1.26.** The \(E_1 E_2 \cdots E_n\)-out-degree of \(C\) is the cardinal number \(|S_1 \cup S_2 \cup \cdots \cup S_n/H|\).

**Proof.** Since \(C\) is vertex transitive it suffices to consider the out degree of the vertex \(1 \in G\). Observe that

\[\rho(H) = \{uH : (H, uH) \in E\}\]
\[= \{uH : u \in HS_i H, \text{ for some } i\}\]
\[= (HS_1 H) \cup (HS_2 H) \cup \cdots \cup (HS_n H)/H.\]

Hence \(|\rho(H)| = |(HS_1 H) \cup (HS_2 H) \cup \cdots \cup (HS_n H)/H|\). \(\square\)

**Corollary 4.1.27.** The \(E_i\)-out-degree of \(C\) is the cardinal number \(|HS_i H/H|\).

**Corollary 4.1.28.** The \(E_1 E_2 \cdots E_n\)-in-degree of \(C\) is the cardinal number \(|(HS_1^{-1} H) \cup (HS_2^{-1} H) \cup \cdots \cup (HS_n^{-1} H)/H|\).

**Proof.** Since \(C\) is vertex transitive it suffices to consider the in degree of the vertex \(H \in G/H\). Observe that

\[\sigma(H) = \{uH : (uH, H) \in E\}\]

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\[ E_i = \{ uH : (uH, H) \in E_i, \text{ for some } i \} \]
\[ E_i = \{ uH : u^{-1} \in HS_i H, \text{ for some } i \} \]
\[ E_i = \{ uH : u \in HS_i^{-1} H, \text{ for some } i \} \].

Hence \( |\sigma(H)| = |(HS_1^{-1} H) \cup (HS_2^{-1} H) \cup \ldots \cup (HS_n^{-1} H)/H| \). \[ \square \]

**Corollary 4.1.29.** The \( E_i \) in-degree of \( C \) is the cardinal number \( |HS_i^{-1} H/H| \).

**Corollary 4.1.30.** For \( k = 1, 2, 3, \ldots \) let \( A_k \) be the set of all \( k \) products of the form \((HS_{i_1} H)(HS_{i_2} H) \cdots (HS_{i_k} H)\). If \( C \) has finite diameter, then the diameter of \( C \) is the least positive integer \( m \) such that

\[ G = A_m. \]

**Proof.** Let \( m \) be the smallest positive integer such that \( G = A_m \). We will show that the diameter of \( C \) is \( n \). Let \( xH \) and \( yH \) be any two arbitrary elements in \( G \) such that \( y = xz \). Then \( z \in G \). This implies that \( x \in A_m \). But then \( z \) has a representation of the form \( x = s_{i_1}s_{i_2} \cdots s_{i_m} \). This implies that

\((H, s_{i_1} H, s_{i_1}s_{i_2} H, \ldots, zH)\)

is path of \( m \) edges from \( H \) to \( zH \). That is

\((xH, xs_{i_1} H, xs_{i_1}s_{i_2} H, \ldots, yH)\)

is a path of length \( m \) from \( xH \) to \( yH \). This shows that \( d(xH, yH) \leq m \). Since \( xH \) and \( yH \) are arbitrary,

\[ \max_{xH,yH \in G} \{d_{1,2,\ldots,n}(xH, yH)\} \leq m. \]

Therefore the diameter of \( C \) is less than or equal to \( m \). On the other hand let the diameter of \( C \) be \( k \). Let \( x \in G \) and \( d_{1,2,\ldots,n}(H, xH) = k \). Then we have \( x \in B \) for some \( B \in A_k \). That is

\[ G = A_k. \]

Now by the minimality of \( k \), we have \( m \leq k \). Hence \( k = m \). \[ \square \]
Corollary 4.1.31. The vertex $H$ is an $E_1E_2\cdots E_n$ - source of $\mathcal{C}$ if and only if $G = [HSH]$.

Proof. First, assume that $H$ is an $E_1E_2\cdots E_n$ -source of $\mathcal{C}$. Then for any vertex $xH \in G/H$, there is an $E_1E_2\cdots E_n$ -path from $H$ to $xH$. This implies that $G = [S]$. Conversely, if $G = [HSH]$, one can prove that $H$ is an $E_1E_2\cdots E_n$ -source.

Corollary 4.1.32. The vertex $H$ is an $E_i$ -source of $\mathcal{C}$ if and only if $G =< HS_iH >$.

Corollary 4.1.33. The vertex $1$ is an $E_1E_2\cdots E_n$ -sink of $\mathcal{C}$ if and only if $G = [HS^{-1}H]$.

Proof. First, assume that $H$ is an $E_1E_2\cdots E_n$ -sink of $\mathcal{C}$. Then for each $xH \in G/H$, there is an $E_1E_2\cdots E_n$ - path from $xH$ to $H$. This implies that $x \in [HS^{-1}H]$. Hence $G = [HS^{-1}H]$. Conversely, if $G = [HS^{-1}H]$, one can easily prove that $H$ is an $E_1E_2\cdots E_n$ -sink of $\mathcal{C}$.

Corollary 4.1.34. The vertex $H$ is an $E_i$ -sink of $\mathcal{C}$ if and only if $G =< HS_i^{-1}H >$.

Corollary 4.1.35. The $E_1E_2\cdots E_n$ -reachable set $R_{1,2,\ldots,n}(H)$ of the vertex $H$ is the set $[HSH]$.

Proof. By definition,

$$R(H) = \{x : \text{there exists an } E_1E_2\cdots E_n \text{-path from } H \text{ to } xH\}.$$  

Observe that

$$x \in R_{1,2,\ldots,n}(H) \iff \text{there exists an } E_1E_2\cdots E_n \text{-path from } H \text{ to } xH, \text{ say } (H, x_1H, x_2H, \ldots, x_nH, xH) \iff x \in [HSH].$$

Therefore, $R_{1,2,3,\ldots,n}(H) = [HSH]$.  

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Corollary 4.1.36. The $E_i$ -reachable set $R_i(H)$ of the vertex $H$ is the set
$< S_i >$.

Corollary 4.1.37. The $E_1E_2\cdots E_n$ -antecedent set $Q_{1,2,\ldots,n}(1)$ of the vertex $H$ is the set $[HS^{-1}H]$.

Proof. Observe that

$x \in Q_{1,2,\ldots,n}(H) \iff$ there exists an $E_1E_2\cdots E_n$ -path from $xH$ to $H$, say

$(xH, x_1H, x_2H, \ldots, x_nH, H)$

$\iff x \in [HS^{-1}H]$.

Therefore,

$Q_{1,2,\ldots,n}(H) = [HS^{-1}H]$. 

\hfill \Box

Corollary 4.1.38. The $E_i$ -antecedent set $Q_i(H)$ of the vertex $H$ is the set $< S_i^{-1} >$. 

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