Chapter 3

Cayley digraph structures induced by loops

The aim of this chapter is to generalize the results obtained in chapter 2 to a weaker algebraic structure, viz. loop. By introducing $R$ associative subsets of a loop, we prove that a bigger class of Cayley digraph structures could be induced by loops. Moreover we discuss the relation between properties digraph structures and those of loops.

3.1 Cayley digraph structures and loops

**Definition 3.1.1.** Let $G$ be a loop, and let $A$ be a subset of $G$. Then $A$ said to be a $R$ associative (right associative) subset of $G$, if for every $x, y \in G$, $(xy)A = x(yA)$.

This means, if $x, y \in G$ and $a \in A$, then $(xy)a = x(ya')$ for some $a' \in A$.

**Definition 3.1.2.** Let $G$ be a loop and $S_1, S_2, \ldots, S_n$ be mutually disjoint $R$ associative subsets of $G$. Then Cayley digraph structure of $G$ with respect to $S_1, S_2, \ldots, S_n$ is defined as the digraph structure $X = (G; E_1, E_2, \ldots, E_n)$, where

$$E_i = \{(x, y) : z \in S_i\},$$

where $z$ denotes the solution of the equation $y = xz$.

The sets $S_1, S_2, \ldots, S_n$ are called connection sets of $X$. The Cayley digraph
structure of $G$ with respect to $S_1, S_2, \ldots, S_n$ is denoted by $\text{Cay}(G; S_1, S_2, \ldots, S_n)$.

### 3.1.1 Example

**Example 3.1.3.** Let $G = \{1, 2, 3, 4, 5, 6\}$. Define a binary operation in $G$ as follows:

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Obviously $G$ is a loop. Let $S_1 = \{1\}, S_2 = \{2\}, S_3 = \{3\}, S_4 = \{4\}$. Then the graphical representation of $\text{Cay}(G; S_1, S_2, S_3, S_4)$ is shown in figure 3.1.

![Figure 3.1: Cay($G = \{1, 2, 3, 4, 5, 6\}; S_1 = \{1\}, S_2 = \{2\}, S_3 = \{3\}, S_4 = \{4\}$)](image)

In this chapter we may use the following notations. Let $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ be a Cayley digraph structure induced by the loop $G$. 44
(1) Let $S_1, S_2, \ldots, S_n$ be subsets of a loop $G$, then we may define the product $S_1, S_2, \ldots, S_n$ as follows:

$$S_1S_2 \cdots S_n = \{(s_1s_2)\ldots s_n : s_i \in S_i, i = 1, 2, \ldots, n\}$$

If $S_1 = S_2 = \cdots = S$, we denote the above product as $S^n$.

(2) Let $A_k$ be the union of set of all $k$ products of the form $S_{i_1}S_{i_2} \cdots S_{i_k}$ from the set $\{S_1, S_2, \ldots, S_n\}$. Then $\bigcup_k A_k$ is denoted by $[S]$.

(3) Let $D$ be a subset of $G$. We define $(D)\ell = \{z\ell : z\ell z = 1 \text{ for some } z \in D\}$.

(4) Let $A$ be a subset of a loop $G$, then the semigroup generated by $A$ is denoted by $<A>$.

(5) Let $A = \{S_i \cup S_{i_\ell} : i = 1, 2, \ldots, n\}$ and $B_k$ be the set of all finite products of elements from $A$ taken $k$ at a time. Then we define $[[S]] = \bigcup_k B_k$

### 3.1.2 Main theorem

**Theorem 3.1.4.** If $G$ is a loop and let $S_1, S_2, \ldots, S_n$ are mutually disjoint associative subsets of $G$, then the Cayley digraph structure $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is vertex transitive.

**Proof.** Let $a$ and $b$ be any two arbitrary elements in $G$. Define a mapping $\varphi : G \to G$ by

$$\varphi(x) = (b/a)x \text{ for all } x \in G.$$ 

where $(b/a)$ denotes the solution of the equation $b = za$. This mapping defines a permutation of the vertices of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$. It is also an automorphism. Let $x, y \in G$ such that $y = xz$. Note that

$$(x, y) \in E_i \iff z \in S_i \text{ for some } i.$$

The equation $y = xz$ can be written as

$$(b/a)y = (b/a)(xz)$$

45
\[(b/a)xz' \quad \text{for some } z' \in S_i.\]

The above equation tells us that \(((b/a)x, (b/a)y) \in E_i\). That is, \((\varphi(x), \varphi(y)) \in E_i\). Similarly, we assume that \((\varphi(x), \varphi(y)) \in E_i\). Then \((b/a)y = ((b/a)x)z\) for some \(z \in S_i\). This implies that \((b/a)y = (b/a)(xz')\) for some \(z' \in S_i\). By left cancellation law, we obtain \(y = xz'\). This tells us that \((b/a)y = (b/a)(xz')\) for some \(z' \in S_i\).

By left cancellation law, we obtain \(y = xz'\). This tells us that \((x,y) \in E_i\).

Also we note that \(\varphi(a) = (b/a)a = b\). Hence Cay\((G; S_1, S_2, \ldots, S_n)\) is vertex transitive.

### 3.1.3 Corollaries

**Corollary 3.1.5.** Cay\((G; S_1, S_2, \ldots, S_n)\) is an \(E_1E_2\cdots E_n\)-trivial digraph structure \(\iff S_i = \emptyset\) for all \(i\).

**Proof.** By definition, Cay\((G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2\cdots E_n\)-trivial \(\iff E_i = \emptyset\) for all \(i\). This implies that \(S_i = \emptyset\) for all \(i\).

**Corollary 3.1.6.** Cay\((G; S_1, S_2, \ldots, S_n)\) is an \(E_i\)-trivial digraph structure \(\iff S_i = \emptyset\).

**Corollary 3.1.7.** Cay\((G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2\cdots E_n\)-reflexive \(\iff 1 \in S_i\) for some \(i\).

**Proof.** Assume that Cay\((G; S_1, S_2, \ldots, S_n)\) is an \(E_1E_2\cdots E_n\)-reflexive digraph structure. Then for every \(x \in G\), \((x, x) \in E_i\) for some \(i\). This implies that the equation \(x = xz\) has a unique solution in \(S_i\) for some \(i\). That is, \(1 \in S_i\) for some \(i\).

Conversely, assume that \(1 \in S_i\) for some \(i\). This implies for each \(x \in G\), \((x, x) \in E_i\) for some \(i\). That is, \((x, x) \in \cup E_i\) for all \(x \in G\).

**Corollary 3.1.8.** Cay\((G; S_1, S_2, \ldots, S_n)\) is \(E_1E_2\cdots E_n\)-symmetric if and only if \(S_i = S_{i'}\) for all \(i\).

**Proof.** First, assume that Cay\((G; S_1, S_2, \ldots, S_n)\) is an \(E_1E_2\cdots E_n\)-symmetric digraph structure. Let \(a \in S_i\). Then \((1, a) \in E_i\). Since Cay\((G; S_1, S_2, \ldots, S_n)\) is symmetric \((a, 1) \in E_i\). This implies that the equation \(1 = at\) has a solution in \(S_i\). That is \(a \in S_{i'}\). Hence \(S_i \subseteq S_{i'}\). Similarly, we can prove that \(S_{i'} \subseteq S_i\).
Conversely, assume that $S_i = S_{i'}$ for all $i$. Suppose that $(x, y) \in E_i$. Then the equation $y = xz$ has a solution in $S_i$. That is $z \in S_i$. Consider the equation $x = yt$. This equation can be written as:

\[
xz = (yt)z
\]
\[
y = y(tz') \text{ for some } z' \in S_i
\]
\[
y1 = y(tz')
\]

*i.e.,* $1 = tz'$ (by left cancellation law).

The above equation tells us that $t \in S_i$. Since $S_i = S_{i'}$, it follows that $t \in S_i$. Hence the equation $x = yt$ has a solution in $S_i$. That is $(y, x) \in E_i$. \hfill \square

**Corollary 3.1.9.** Cay($G; S_1, S_2, \ldots, S_n$) is an $E_1E_2 \cdots E_n$-transitive if and only if for every $i, j$, $S_iS_j \subseteq S_k$ for some $k$.

**Proof.** First, assume that Cay($G; S_1, S_2, \ldots, S_n$) is $E_1E_2 \cdots E_n$-transitive. Let $x \in S_iS_j$. Then $x = z_1z_2$ for some $z_1 \in S_i$ and $z_2 \in S_j$. This implies that $(1, z_1) \in E_i$ and $(z_1, z_1z_2) \in E_j$. Since Cay($G, S_1, S_2, \ldots, S_n$) is transitive $(1, z_1z_2) \in E_k$ for some $k$. That is $z_1z_2 \in S_k$. Hence $S_iS_j \subseteq S_k$ for some $k$.

Conversely assume that for each $i, j$, $S_iS_j \subseteq S_k$ for some $k$. Let $x, y$ and $z \in G$ such that $y = xt_1$ and $z = yt_2$. If $(x, y) \in E_i$ and $(y, z) \in E_j$, then $t_1 \in S_i$ and $t_2 \in S_j$. Note that the equation $z = yt_2$ can be written as:

\[
z = (xt_1)t_2
\]
\[
= x(t_1t_2') \text{ for some } t_2' \in S_j
\]
\[
= xt_3 \text{ where } t_3 = t_1t_2'.
\]

Note that $t_3 \in S_iS_j$. Since $S_iS_j \subseteq S_k$, $t_3 \in S_k$. That the equation $z = xt$ has a solution $t_3$ in $S_k$. Hence Cay($G; S_1, S_2, \ldots, S_n$) is transitive. \hfill \square

**Corollary 3.1.10.** Cay($G; S_1, S_2, \ldots, S_n$) is $E_1E_2 \cdots E_n$-complete if and only if $G = \cup S_i$.

**Proof.** Suppose Cay($G; S_1, S_2, \ldots, S_n$) is $E_1E_2 \cdots E_n$-complete. Then for every $x \in G$, we have $(1, x) \in \cup E_i$. This implies that $x \in S_i$ for some $i$. This implies that $G = \cup S_i$. 47
Conversely, assume that $G = \cup S_i$. Let $x$ and $y$ be two arbitrary elements in $G$ such that $y = xz$. Then $z \in G$. This implies that $z \in S_i$ for some $i$. That is, $(1, z) \in \cup E_i$. That is $(x, xz) = (x, y) \in \cup E_i$. This shows that $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is $E_1E_2\cdots E_n$-complete. □

**Corollary 3.1.11.** $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is $E_i$-complete if and only if $G = S_i$.

**Corollary 3.1.12.** $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is $E_1E_2\cdots E_n$-connected if and only if $G = [S]$.

**Proof.** Suppose $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is $E_1E_2\cdots E_n$-connected and let $x \in G$. Let

$$(1, y_1, y_2, \ldots, y_n, x)$$

be a $E_1E_2\cdots E_n$-path leading from 1 to $x$. Then we have,

$$y_1 = z_1, y_2 = y_1 z_2, \ldots, y_k = y_{k-1} z_k, x = y_k z_{k+1}$$

for some $z_j \in S_{i_j}, j = 1, 2, \ldots, k + 1$. Note that the equation $x = y_k z_{k+1}$ can be written as

$$x = (y_{k-1} z_k) z_{k+1}$$

$$= ((y_{k-2} z_{k-1}) y_{k-1} z_k) z_{k+1}$$

$$= (z_1 z_2) \cdots z_{k+1}$$

The last equation tells us that $x \in S_{i_1} S_{i_2} \cdots S_{i_{k+1}}$. This implies that $x \in A$ for some $A \in [S]$. Since $x$ is arbitrary, $G = [S]$.

Conversely, assume that $G = [S]$. Let $x$ and $y$ be any arbitrary elements in $G$. Let $y = xz$. Then $z \in G$. Then $z \in S_i S_j \cdots S_k$ for some $i, j, \ldots$ and $k$. This implies that $z = s_i s_j \cdots s_k$ for some $i, j, \ldots$ and $k$. Then clearly, $(1, s_i, s_is_j, \ldots, s_is_j \cdots s_k)$ is an $E_1E_2\cdots E_n$-path from 1 to $z$. That is

$$(x, xs_i, xs_is_j, \ldots, xs_is_j \cdots s_k)$$

is a $E_1E_2\cdots E_n$-path from $x$ to $y$. Hence $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is connected. □
Corollary 3.1.13. Cay($G; S_1, S_2, \ldots, S_n$) is $E_i$-connected if and only if $G = \langle S_i \rangle$, where $\langle S_i \rangle$ is the semigroup generated by $S_i$.

Corollary 3.1.14. Cay($G; S_1, S_2, \ldots, S_n$) is $E_1E_2\cdots E_n$-quasi connected if and only if $G = [S][S]$.

Proof. First, assume that Cay($G; S_1, S_2, \ldots, S_n$) is quasi connected. Let $x$ be any arbitrary element in $G$. Then there exists a vertex $y \in G$ such that there is a path from $y$ to 1, say:

$$(y, y_1, y_2, \ldots, y_n, 1)$$

and a path from $y$ to $x$, say:

$$(y, x_1, x_2, \ldots, x_m, x).$$

Then we have the following system of equations:

\begin{align*}
y_1 &= yz_1 \text{ for some } z_1 \in S_1, \\
y_2 &= y_1z_2 \text{ for some } z_2 \in S_2, \\
y_3 &= y_2z_3 \text{ for some } z_3 \in S_3, \\
& \vdots \\
1 &= y_nz_{n+1} \text{ for some } z_{n+1} \in S_{n+1} \\
\end{align*} \tag{3.1}

and

\begin{align*}
x_1 &= yt_1 \text{ for some } z_1 \in S_1, \\
x_2 &= x_1t_2 \text{ for some } z_2 \in S_2, \\
x_3 &= x_2t_3 \text{ for some } z_3 \in S_3, \\
& \vdots \\
x &= x_mt_{m+1} \text{ for some } z_{m+1} \in S_{m+1}. \\
\end{align*} \tag{3.2}

Observe that equation (5.1) can be written as:

$$1 = y(w_1w_2 \cdots w_{n+1}) \text{ for some } w_k \in S_{i_k}, k = 1, 2, \ldots, n + 1. \tag{3.3}$$
This implies that
\[ y \in [S]_\ell. \]  \hfill (3.4)

Similarly, equation (5.2) can be written as:
\[ x = y(v_1 v_2 \ldots v_{m+1}) \text{ for some } v_k \in S_{i_k}, k = 1, 2, \ldots, m + 1. \]  \hfill (3.5)

From equations (5.3) and (5.4), we have
\[ x \in [S]_\ell[S]. \]  \hfill (3.6)

Since \( x \) is arbitrary, \( G = [S]_\ell[S] \).

Conversely, assume that \( G = [S]_\ell[S] \). Let \( x \) and \( y \) be two arbitrary vertices in \( G \). Let \( y = xz \). Then \( z \in G \). This implies that \( z \in [S]_\ell[S] \). Then there exists \( z_1 \in [S]_\ell \) and \( z_2 \in [S] \) such that \( z = z_1 z_2 \). \( z_1 \in [S]_\ell \) implies that there exists \( t_k \in S_{i_k} \) such that
\[ 1 = z_1(t_1 t_2 \ldots t_m) \]
i.e., \( 1 = (((z_1 r_1) r_2) \ldots r_m) \) for some \( r_k \in S_{i_k}, k = 1, 2, \ldots, m \).

This implies that
\[ (z_1, z_1 r_1, z_1 r_1 r_2, \ldots, 1) \]
is a path from \( z_1 \) to 1. That is
\[ (yz_1, yz_1 r_1, yz_1 r_1 r_2, \ldots, y) \]
is a path from \( yz_1 \) to \( y \).

Similarly, \( z_2 \in [S] \) implies that there exists \( a_k \in S_{i_k} \) such that
\[ z_2 = a_1 a_2 \ldots a_m. \]

Observe that
\[ (z_2, a_1 a_2, a_1 a_2 a_3, \ldots, 1) \]
is a path from $z_2$ to 1. That is,

$$ (z_1 z_2, z_1 a_1 a_2, a_1 a_2 a_3, \ldots, z_1) $$

is a path from $z$ to $yz_1$. That is

$$ (yz, yz_1 a_1 a_2, ya_1 a_2 a_3, \ldots, yz_1) $$

is a path from $x$ to $yz_1$. This implies that $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is $E_1 E_2 \cdots E_n$-quasi connected.

**Corollary 3.1.15.** $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is $E_i$-quasi connected if and only if $G = < S_i >_\ell < S_i >$.

**Corollary 3.1.16.** $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is $E_1 E_2 \cdots E_n$-locally connected if and only if $[S] = [S]_\ell$.

**Proof.** Assume that $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is $E_1 E_2 \cdots E_n$-locally connected. Let $x \in [S]$. Then $x \in A_m$ for some $m$. Then $x = s_is_j \ldots s_m$. Let $x_0 = 1, x_1 = s_i, x_2 = s_is_j, \ldots, x_m = s_is_j \ldots s_m$. Then

$$ (x_0, x_1, x_2, \ldots, x_m) $$

is a path leading from 1 to $x$. Since $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is locally connected, there exists a path from $x$ to 1, say:

$$ (x, y_1, y_2, \ldots, y_m, 1). $$

This implies that

$$ y_1 = xt_1 \text{ for some } t_1 \in S_i $$

$$ y_2 = y_1 t_2 \text{ for some } t_2 \in S_{i_2} $$

$$ \vdots $$

$$ 1 = y_m t_{m+1} \text{ for some } t_{m+1} \in S_{i_n}. $$

This implies that $1 = x(z_1 z_2 \cdots z_n)$ for some $z_k \in S_{i_k}, k = 1, 2, 3, \ldots, (m + 1)$. That is $x \in [S]_\ell$. Hence $[S] \subseteq [S]_\ell$. Similarly, one can prove that $[S]_\ell \subseteq [S]$. 

51
Hence $|S| = |S|_\ell$. Conversely, if $|S| = |S|_\ell$, one can easily verify that $Cay(G; S_1, S_2, \ldots, S_n)$ is $E_i$-locally connected.

**Corollary 3.1.17.** $Cay(G; S_1, S_2, \ldots, S_n)$ is $E_i$-locally connected if and only if $<S_i> = <S_i>_\ell$.

**Corollary 3.1.18.** $Cay(G; S_1, S_2, \ldots, S_n)$ is $E_1E_2\cdots E_n$-semi connected if and only if $G = [S] \cup [S]_\ell$.

**Proof.** Assume that $Cay(G; S_1, S_2, \ldots, S_n)$ is $E_1E_2\cdots E_n$-semi connected and let $x \in G$. Then there is a path from 1 to $x$, say

$$(1, x_1, x_2, \cdots, x_n, x)$$

or a path from $x$ to 1, say

$$(x, y_1, y_2, \cdots, y_m, 1)$$

This implies that $x \in [S]$ or $x \in [S]_\ell$. This implies that $G = [S] \cup [S]_\ell$. Similarly, if $G = [S] \cup [S]_\ell$, then one can prove that $Cay(G; S_1, S_2, \ldots, S_n)$ is $E_1E_2\cdots E_n$-semi connected.

**Corollary 3.1.19.** $Cay(G; S_1, S_2, \ldots, S_n)$ is $E_i$-semi connected if and only if $G = <S_i> \cup <S_i>_\ell$.

**Corollary 3.1.20.** $Cay(G; S_1, S_2, \ldots, S_n)$ is an $E_1E_2\cdots E_n$-quasi ordered set if and only if

$$(i)1 \in S_1 \cup S_2 \cdots \cup S_n,$$

$$(ii)\text{for every } (i, j), \ S_iS_j \subseteq S_k \text{ for some } k.$$
Corollary 3.1.22. Cay\((G; S_1, S_2, \ldots, S_n)\) if an \(E_1 E_2 \cdots E_n\)-partially ordered set if and only if

\[
(i) 1 \in S_1 \cup S_2 \cdots \cup S_n,
(ii) \text{for every } (i, j), \ S_i S_j \subseteq S_k \text{ for some } k
(iii) \cup (S_i \cap S_i^{-1}) = \{1\}.
\]

Proof. Observe that

\[
x \in \cup(S_i \cap S_{i_\ell}) \iff x \in (S_i \cap S_{i_\ell}) \text{ for some } i
\]
\[
\iff x \in S_i \text{ and } x \in S_{i_\ell}
\]
\[
\iff (1, x) \in E_i \text{ and } (x, 1) \in E_i
\]
\[
\iff x = 1.
\]

From these equivalences, the result follows. \(\square\)

Corollary 3.1.23. Cay\((G; S_1, S_2, \ldots, S_n)\) if an \(E_i\)-partially ordered set if and only if

\[
(i) 1 \in S_i,
(ii) S_i \subseteq S_i
(iii) S_i \cap S_i = \{1\}.
\]

Corollary 3.1.24. Let \(A_m\) (\(m \geq 2\))is the set of \(m\) products of the form \(S_1 S_2 \cdots S_m\). Then Cay\((G; S_1, S_2, \ldots, S_n)\) is an \(E_1 E_2 \cdots E_n\)-Hasse diagram if and only if \(C \cap S_i = \emptyset\) for all \(i\) and for all \(C \in A_m\).

Proof. Suppose the condition holds. Let \(x_0, x_1, \ldots, x_m\) be \((m + 1)\) elements in \(G\) such that \((x_i, x_{i+1}) \in \cup E_i\) for \(i = 0, 1, \ldots, m - 1\). This implies that

\[
x_1 = x_0 t_1 \text{ for some } t_1 \in S_1
x_2 = x_1 t_2 \text{ for some } t_2 \in S_2
x_3 = x_2 t_3 \text{ for some } t_3 \in S_3
\]
\[
\vdots
x_m = x_{m-1} t_m \text{ for some } t_m \in S_m.
\]
The last equation can be written as:

\[ x_m = ((x_{m-2}t_{m-1}))t_m = (x_0t_1t_2)\cdots t_m = x_0(z_1z_2\ldots z_m) \text{ for some } z_k \in S_{i_k}, k = 1, 2, \ldots, m = x_0t, \text{ where } t = z_1z_2\ldots z_m \in A_m. \]

Since \( C \cap S_i = \emptyset \) for all \( i \) and for all \( C \in A_m, (x_0, x_m) \notin \bigcup E_i \).

Conversely assume that \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is an \( E_1E_2\cdots E_n \)-Hasse diagram. We will show that \( C \cap S_i = \emptyset \) for all \( i \) and for all \( C \in A_m \). Let \( S_{i_1}S_{i_2}S_{i_3}\cdots S_{i_m} \) be any element in \( A_m \). Let \( x \in S_{i_1}S_{i_2}S_{i_3}\cdots S_{i_m} \). Then \( x = s_{i_1}s_{i_2}s_{i_3}\cdots s_{i_m} \) for some \( s_{i_k} \in S_{i_k} \). This implies that

\[(1, s_{i_1}, s_{i_2}s_{i_3}, \ldots, x)\]

is a path from 1 to \( x \). Since \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is an \( E_1E_2\cdots E_n \)-Hasse diagram, \( x \notin S_i \) for any \( i \). That is, \( A_m \cap S_i = \emptyset \) for all \( i \).

**Corollary 3.1.25.** Let \( A_m (m \geq 2) \) be the set of all \( m \) products of the form \( S_{i_1}S_{i_2}\cdots S_{i_m} \). Then \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is an \( E_i \)-Hasse diagram if and only if \( S_{i_m}^2 \cup S = \emptyset \), for all \( m \geq 2 \).

**Corollary 3.1.26.** The \( E_1E_2\cdots E_n \)-out-degree of \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is the cardinal number \( |S_1 \cup S_2 \cup \cdots \cup S_n| \).

**Proof.** Since \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is vertex transitive it suffices to consider the out degree of the vertex 1 \( \in G \). Observe that

\[ \rho(1) = \{u : (1, u) \in \bigcup E\} = \{u : u \in S_i \text{ for some } i\} = S_1 \cup S_2 \cup \cdots \cup S_n. \]

Hence \( |\rho(1)| = |S_1 \cup S_2 \cup \cdots \cup S_n| \).

**Corollary 3.1.27.** The \( E_i \)-out-degree of \( \text{Cay}(G; S_1, S_2, \ldots, S_n) \) is the cardinal number \( |S_i| \).
Corollary 3.1.28. The $E_1 E_2 \cdots E_n$ -in-degree of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is the cardinal number $|S_{1\ell} \cup S_{2\ell} \cup \cdots \cup S_{n\ell}|$.

Proof. Since $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is vertex transitive it suffices to consider the in degree of the vertex $1 \in G$. Observe that

$$
\sigma(1) = \{ u : (u, 1) \in \cup E \} = \{ u : (u, 1) \in E_i \} = \{ u : 1 = uz \text{ for some } z \in S_i \} = \{ z : z \ell \in S_{i\ell} \text{ for some } i \} = S_{1\ell} \cup S_{2\ell} \cup \cdots \cup S_{n\ell}.
$$

Hence $|\sigma(1)| = |S_{1\ell} \cup S_{2\ell} \cup \cdots \cup S_{n\ell}|$. \(\square\)

Corollary 3.1.29. The $E_i$ -in-degree of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is the cardinal number $|S_{i\ell}|$.

Corollary 3.1.30. For $k = 1, 2, 3, \ldots$ let $A_k$ be the set of all $k$ products of the form $S_{i_1} S_{i_2} S_{i_3} \cdots S_{i_k}$. If $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ has finite diameter, then the diameter of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is the least positive integer $m$ such that

$$
G = A_m.
$$

Proof. Let $m$ be the smallest positive integer such that $G = A_m$. We will show that the diameter of $\text{Cay}(G; S_1, S_2, \ldots, S_n)$ is $m$. Let $x$ and $y$ be any two arbitrary elements in $G$ such that $y = xz$. Then $z \in G$. This implies that $x \in A$. But then $z$ has a representation of the form $x = s_{i_1} s_{i_2} \cdots s_{i_m}$. This implies that

$$(1, s_{i_1}, s_{i_1} s_{i_2}, \ldots, z)$$

is path of $m$ edges from 1 to $z$. That is

$$(x, xs_{i_1}, xs_{i_1} s_{i_2}, \ldots, y)$$

is a path of length $m$ from $x$ to $y$. This shows that $d(x, y) \leq m$. Since $x$ and $y$ are arbitrary,

$$
\max_{x, y \in G} \{ d_1, 2, \ldots, n(x, y) \} \leq m.
$$

55
Therefore the diameter of Cay($G; S_1, S_2, \ldots, S_n$) is less than or equal to $m$. On the other hand let the diameter of Cay($G; S_1, S_2, \ldots, S_n$) be $k$. Let $x \in G$ and $d_{1,2,\ldots,n}(1, x) = k$. Then we have $x \in B$ for some $B \in A_k$. That is

\[ G = A_k. \]

Now by the minimality of $k$, we have $m \leq k$. Hence $k = m$. \hfill \Box

**Corollary 3.1.31.** If Cay($G; S_1, S_2, \ldots, S_n$) has finite, then $E_i$ -diameter of the Cayley digraph structure Cay($G; S_1, S_2, \ldots, S_n$) is the least positive integer $m$ such that

\[ G = S_i^m. \]

**Corollary 3.1.32.** The vertex 1 is an $E_1E_2 \cdots E_n$ -source of Cay($G; S_1, S_2, \ldots, S_n$) if and only if $G = [S]$. 

*Proof.* First, assume that 1 is an $E_1E_2 \cdots E_n$ -source of Cay($G; S_1, S_2, \ldots, S_n$). Then for any vertex $x \in G$, there is an $E_1E_2 \cdots E_n$ -path from 1 to $x$. This implies that $G = [S]$. Conversely, if $G = [S]$, one can prove that 1 is an $E_1E_2 \cdots E_n$ -source. \hfill \Box

**Corollary 3.1.33.** The vertex 1 is an $E_i$ -source of Cay($G; S_1, S_2, \ldots, S_n$) if and only if $G = <S_i>$. 

**Corollary 3.1.34.** The vertex 1 is an $E_1E_2 \cdots E_i$ -sink of Cay($G; S_1, S_2, \ldots, S_n$) if and only if $G = [S]_\ell$. 

*Proof.* First, assume that 1 is an $E_1E_2 \cdots E_n$ -sink of Cay($G; S_1, S_2, \ldots, S_n$). Then for each $x \in G$, there is an $E_1E_2 \cdots E_n$ -path from $x$ to 1. This implies that $x \in [S]_\ell$. Hence $G = [S]_\ell$. Conversely, if $G = [S]_\ell$, one can easily prove that 1 is an $E_1E_2 \cdots E_n$ -sink of Cay($G; S_1, S_2, \ldots, S_n$). \hfill \Box

**Corollary 3.1.35.** The vertex 1 is an $E_i$ -sink of Cay($G; S_1, S_2, \ldots, S_n$) if and only if $G = <S_i>_{\ell}$. 

**Corollary 3.1.36.** The $E_1E_2 \cdots E_n$ -reachable set $R_{1,2,\ldots,n}(1)$ of the vertex 1 is the set $[S]$. 

56
Proof. By definition,

\[ R(1) = \{ x : \text{there exists an } E_1E_2\cdots E_n \text{-path from 1 to } x \} \]

Observe that

\[ x \in R_{1,2,\ldots,n}(1) \iff \text{there exists an } E_1E_2\cdots E_n \text{-path from 1 to } x, \text{ say} \]
\[ (1, x_1, x_2, \ldots, x_n, x) \]
\[ \iff x \in [S]. \]

Therefore, \( R_{1,2,3,\ldots,n}(1) = [S] \). \( \square \)

**Corollary 3.1.37.** The \( E_i \)-reachable set \( R_i(1) \) of the vertex 1 is the set \(<S_i>\).

**Corollary 3.1.38.** The \( E_1E_2\cdots E_n \)-antecedent set \( Q_{1,2,\ldots,n}(1) \) of the vertex 1 is the set \([S]_\ell\).

**Proof.** Observe that

\[ x \in Q_{1,2,\ldots,n}(1) \iff \text{there exists an } E_1E_2\cdots E_n \text{-path from } x \text{ to 1, say} \]
\[ (x, x_1, x_2, \ldots, x_n, 1) \]
\[ \iff x \in [S]_\ell. \]

Therefore,

\[ Q_{1,2,\ldots,n}(1) = [S]_\ell. \]

This completes the proof. \( \square \)

**Corollary 3.1.39.** The \( E_i \)-antecedent set \( Q_i(1) \) of the vertex 1 is the set \(<S_i>_{\ell}\).