CHAPTER-2

PRILIMINARIES

Section 2.1.

In this chapter we introduce the notation used in this thesis, present some standard results from books as well as from recent papers. Proofs are hinted for a few frequently used results.

Notation : Set theoretic symbols like $\in, \exists, \forall, \Rightarrow$ are freely used. The n-dimensional Euclidian space is denoted by $\mathbf{R}^n$. Let $\mathbf{R}^{m\times n}$ denotes the space of all $m \times n$ matrices whose elements are real numbers. Unless otherwise stated vectors in $\mathbf{R}^n$ are denoted by small Roman letters and the matrices in $\mathbf{R}^{m\times n}$ are denoted by capital Roman letters.

The transpose of a matrix 'A' is denoted by A^*. If 'A' is a non-singular matrix, A^{-1} stands for inverse of A. The unit matrix of order 'n' is denoted by I_n (whose order can be specified by the context in which it occurs) and the matrix all of whose elements are zero's is denoted by 0. Though the symbol 0 is used in several senses, no confusion arises, since the context in which it occurs clearly indicates what it stands for.

By an $m \times n$ matrix we mean a matrix with m-rows and n-columns and an $n \times n$ matrix merely said to be of order n (or square). A matrix function $A : t \to A(t)$ is denoted by $A(t)$ or $[a_{ij}(t)]$, where $a_{ij}(t)$ stands for the i-th row j-th column element which is a function of t. A matrix $A(t)$ is said to be continuous (differentiable) on an interval $J = [a, b]$ $(a \leq t \leq b)$ if and only if each component or element is continuous (differentiable) on the same interval. A derivative (integral) of a matrix A is the matrix obtained by differentiating (integrating) each component of A.

i.e. $$A'(t) = [a'_{ij}(t)] \quad \text{and} \quad \int_a^b A(t)dt = [\int_a^b a_{ij}(t)dt].$$

Higher derivatives are indicated by bracketed super scripts. $C^n[a, b]$ denotes the class of n-times continuously differentiable functions on $[a, b]$.

In the sequel, we use the following standard norm for vectors and matrices namely for $x \in \mathbf{R}^n$, $\|x\| = \sum_{i=1}^{n} |x_i|$ (or) $\|x\| = \max_i |x_i|$ (or) $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$ and for $A = [a_{ij}] \in \mathbf{R}^{m\times n}$, $\|A\| = \max_{i,j} |a_{ij}|$ (or) $\|A\| = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}$ (or) $\|A\| = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|$ respectively.

Kronecker product also known as a direct product or a tensor product is a concept having its origin in group theory and has important applications in particle physics. This technique has been successfully applied in various fields of matrix theory.

Definition 2.1.1. [39] Let $A = [a_{ij}] \in \mathbf{R}^{m\times n}$, we denote

$$\hat{A} = \text{Vec}A = \begin{bmatrix} A_{.1} \\ A_{.2} \\ \vdots \\ A_{.n} \end{bmatrix}, \text{ where } A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad (1 \le j \le n).$$

Definition 2.1.2. [39] Let $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{p\times q}$ then the Kronecker product of A and B written $A \otimes B$ is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}$$

is an $mp \times nq$ matrix and is in $\mathbf{R}^{mp\times nq}$.

The Kronecker product has the following properties and rules.

1. $(A \otimes B)^* = A^* \otimes B^*$
2. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ (provided A and B are invertible)
3. $\|A \otimes B\| = \|A\|\|B\| \quad (\|A\| = \max_{i,j} |a_{ij}|)$
4. The mixed product rule;

   $(A \otimes B)(C \otimes D)=(AC \otimes BD)$

   this rule holds, provided the dimension of the matrices are such that the various expressions exist.
5. If $A(t)$ and $B(t)$ are matrices, then

   $(A \otimes B)' = A' \otimes B + A \otimes B'$ $('= d/dt)$.
6. Vec $(AYB) = (B^* \otimes A)$ Vec Y.
7. If A and B are matrices both of order $n \times n$ then

   (i) Vec $(AX) = (I_n \otimes A)$ Vec X,

   (ii) Vec $(XA) = (A^* \otimes I_n)$ Vec X.
8. There exists a zero element $0_{mn} = 0_m \otimes 0_n$
9. There exists a unit element $I_{mn} = I_m \otimes I_n$.

Section 2.2.

The problem

$$y' = A(t)y, \quad t \in J \tag{2.2.1}$$

$$y(t_0) = y_0, \tag{2.2.2}$$

where A is a $n \times n$ matrix, $t_0 \in J$ and y_0 is a given vector is called a matrix initial value problem.

Theorem 2.2.1. If a_{ij} are continuous on an interval J then there exists a unique solution to the initial value problem (2.2.1) and (2.2.2).

Any set of n-linearly independent solutions $y_1, y_2, \ldots, y_n$ of (2.2.1) is called a fundamental set of solutions and the matrix with $y_1, y_2, \ldots, y_n$ as its columns is called a fundamental matrix for the equation (2.2.1) and is denoted by Y. The fundamental matrix Y is non-singular.

Theorem 2.2.2. If Y is a fundamental matrix for the equation (2.2.1), then for any constant n-vector c, Yc is a solution of (2.2.1) and every solution of (2.2.1) is of this form.

The equation

$$y' = A(t)y + f, \tag{2.2.3}$$

where A and f are (usually) continuous on an interval J is termed as a non-homogeneous equation. If $f = 0$ on J, then it is called homogeneous.

Theorem 2.2.3. If $\widetilde{y}$ is any particular solution of the non-homogeneous equation (2.2.3) and Y is a fundamental matrix for the corresponding homogeneous equation (2.2.1), then y defined by

$$y = \widetilde{y} + Yc \tag{2.2.4}$$

is a solution of (2.2.3) for every constant n-vector 'c' and every solution of (2.2.3) is of this form, where

$$\widetilde{y}(t) = Y(t) \int_a^t Y^{-1}(s) f(s) ds \tag{2.2.5}$$

is a particular solution of (2.2.3).

Proof. y in (2.2.4) is evidently a solution of (2.2.3). If u is any other solution of (2.2.3), then $(u - \widetilde{y})' = A(u - \widetilde{y})$ so that $(u - \widetilde{y})$ is a solution of (2.2.1). Hence $(u - \widetilde{y}) = Yc$ or $u = \widetilde{y} + Yc$.

Consider the second order non-linear differential equation

$$y'' = f(t, y, y') \tag{2.2.6}$$

satisfying

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \tag{2.2.7}$$

where f is defined on a domain $D = J \times R \times R$.

The problem (2.2.6) and (2.2.7) is called one point boundary value problem or initial value problem.

Definition 2.2.1. A real valued function $y(t)$ defined on J is said to be a solution of the initial value problem (2.2.6) and (2.2.7) on J if

(i) $y'(t), y''(t)$ exists for $t \in J$

(ii) $y(t_0) = y_0, \quad y'(t_0) = y_1$, for $t_0 \in J$

(iii) the points $(t, y(t), y'(t)) \in D$ for $t \in J$

(iv) $y''(t) = f(t, y(t), y'(t))$, $\forall\ t \in J$.

Definition 2.2.2. The function $f(t, y(t), y'(t))$ is said to satisfy a Lipschitz condition on D if there exist two non-negative constants K and L such that

$$|f(t,y,y')-f(t,z,z')| \leq K|y-z|+L|y'-z'|, \forall\ (t,y,y'), (t,z,z') \in D. \tag{2.2.8}$$

Theorem 2.2.4. If $f(t, y, y')$ is continuous on a domain D which contains the point $(t_0, y(t_0), y'(t_0))$ and satisfies a Lipschitz condition (2.2.8), then there exists a twice continuously differentiable function $y(t)$, defined on J containing t_0, which satisfies (2.2.6) and (2.2.7) and has $(t, y(t), y'(t)) \in D$, $\forall\ t \in J$. Furthermore, this solution can be uniquely continued on J.

Theorem 2.2.5. If 'f' is continuous and satisfies a Lipschitz condition (2.2.8) in a domain D which contains the point $(t_0, y(t_0), y'(t_0))$ then the solution of (2.2.6) and (2.2.7) depends continuously on the initial conditions.

Let X be a non-empty set. If d is a metric for X, then the ordered pair (X, d) is called a metric space and d(x, y) is called the distance between x and y.

Definition 2.2.3. A metric space (X, d) is said to be complete if and only if every Cauchy sequence in X has a limit point in X.

Definition 2.2.4. A normed linear space is a vector space over the field of real numbers or complex numbers in which a real valued function $\|x\|$ is defined with the following properties:

(i) (a) $\|x\| \geq 0$ (b) $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\alpha x\| = |\alpha|\|x\|$

(iii) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$.

It may be noted that $d(x, y) = \|x - y\|$ is a metric for the space. It is called the natural metric.

Definition 2.2.5. A normed linear space which is complete with respect to its natural metric is called a Banach space.

Theorem 2.2.6. The space $C(X)$ of all bounded real continuous functions defined on a metric space (X, d) forms a Banach space with the norm given by $\|f\| = \sup |f(t)|$.

One of the simplest and useful tools of the non-linear analysis is the principle of contraction mapping.

Definition 2.2.6. Let B be a Banach space. Let $T : B \rightarrow B$ be a mapping. If there exists an $\alpha \in (0.1)$ such that

$$||Tx_1 - Tx_2|| \leq \alpha||x_1 - x_2||, \forall\ x_1, x_2 \in B.$$

Then 'T' is called a contraction mapping and 'α' is called contraction constant of the mapping.

It is evident that every contraction mapping is continuous.

Definition 2.2.7. A point x in B is called a fixed point of the mapping T if

$$Tx = x.$$

Theorem 2.2.7. (Banach fixed point theorem or Contraction mapping theorem) Every contraction mapping defined on a Banach space has one and only one fixed point.

Consider the two point boundary value problem (2.2.6) satisfying

$$y(a) = y_1, \quad y(b) = y_2. \tag{2.2.9}$$

The following theorem provides a sufficient condition for existence and uniqueness of solutions to the boundary value problem (2.2.6) and (2.2.9).

Theorem 2.2.8. [5] Let $f(t, y, y')$ be continuous and satisfies Lipschitz condition (2.2.8) on D. If

$$K\frac{(b-a)^2}{8} + L\frac{(b-a)}{2} < 1,$$

then the boundary value problem (2.2.6) and (2.2.9) has one and only one solution.

Several results have been obtained relating estimates for length of interval for the existence and uniqueness of solutions to third order non-linear differential equations with boundary conditions at two points. Regarding two point boundary value problems for third order non-linear differential equations, Barr and Sherman [9] have demonstrated the following theorem.

Theorem 2.2.9. Let $[a, b] \in R$ and suppose $f(t, y, z, w)$ satisfies a Lipschitz condition with positive constants L_0, L_1 and L_2 and satisfies

$$L_2(b-a) + L_1\frac{(b-a)^2}{3} + L_0\frac{\sqrt{3}}{27}(b-a)^3 < 1,$$

then there exists a unique solution to the boundary value problem

$$y''' = f(t, y, y', y'') \tag{2.2.10}$$

satisfying

$$y(a) = y_1 \quad y'(a) = y_2, \quad y(b) = y_3, \tag{2.2.11}$$

where y_1, y_2, $y_3 \in R$.

Further, Moorti and Garner [60] also demonstrated a similar type of results for two point boundary value problems.

Theorem 2.2.10. Let $[a, b] \in R$ and suppose $f(t, y, z, w)$ satisfies a Lipschitz condition with positive constants L_0, L_1 and L_2 and satisfies

$$L_2\frac{2}{3}(b-a) + L_1\frac{(b-a)^2}{6} + L_0\frac{2}{81}(b-a)^3 < 1,$$

then there exists a unique solution to the boundary value problem (2.2.10) satisfying (2.2.11).

Section 2.3.

Let $P_k(\mathbf{R}^n)$ denotes the family of all non-empty compact convex subsets of $\mathbf{R}^n$. Define the addition and scalar multiplication in $P_k(\mathbf{R}^n)$ as usual. Then Radstrom [80] states that $P_k(\mathbf{R}^n)$ is a commutative semi-group under addition, which satisfies the cancellation law. Moreover, if $\alpha, \beta \in R$ and $A, B \in P_k(\mathbf{R}^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1.A = A$$

and if $\alpha, \beta \geq 0$, then $(\alpha+\beta)A = \alpha A + \beta A$. The distance between A and B is defined by the Hausdorff metric

$$d(A,B) = \inf\{\epsilon : A \subset N(B,\epsilon), B \subset N(A,\epsilon)\},$$

where

$$N(A,\epsilon) = \{x \in \mathbf{R}^n : ||x-y|| < \epsilon, \text{ for some } y \in A\}.$$

Let $J = [a,b] \subset R$ be a compact interval and denote

$$E^n = \{u : \mathbf{R}^n \to [0,1]/u \text{ satisfies (i)-(iv) below}\},$$

where

(i) u is normal, i.e. there exists an $x_0 \in \mathbf{R}^n$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, i.e. for $x, y \in \mathbf{R}^n$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)];$$

(iii) u is upper semi-continuous;

(iv) $[u]^0 = \overline{\{x \in \mathbf{R}^n/u(x) > 0\}}$ ($\overline{A}$= closure of set A) is compact.

For $0 < \alpha \leq 1$, the α-level set is denoted and defined by

$[u]^\alpha = \{x \in \mathbf{R}^n/u(x) \geq \alpha\}$. Obviously, $[u]^\alpha \in P_k(\mathbf{R}^n)$ for all $0 \leq \alpha \leq 1$.

The real numbers can be embedded to E^1 by the correspondence

$$c \to \tilde{c}(t) = \begin{cases} 1 & \text{if } t = c, \\ 0 & \text{elsewhere.} \end{cases}$$

It is well known that

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [cu]^\alpha = c[u]^\alpha,$$

for all $u, v \in E^n$, $c \in R$, $0 \leq \alpha \leq 1$.

In the sequel, we need the following representation theorem.

Theorem 2.3.1. [72] If $u \in E^n$, then

1. $[u]^\alpha \in P_k(\mathbf{R}^n)$, for all $0 \leq \alpha \leq 1$,

2. $[u]^{\alpha_2} \subset [u]^{\alpha_1}$, for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$,

3. If $\{\alpha_k\}$ is a non-decreasing sequence converging to $\alpha > 0$, then $[u]^\alpha = \bigcap_{k \geq 1} [u]^{\alpha_k}$.

Conversely, if $\{A^\alpha : 0 \leq \alpha \leq 1\}$ is a family of subsets of $\mathbf{R}^n$ satisfying (1)-(3), then there exists a $u \in E^n$ such that $[u]^\alpha = A^\alpha$ for $0 < \alpha \leq 1$ and $[u]^0 = \overline{\bigcup_{0<\alpha\leq 1} A^\alpha} \subset A^0$.

Define $D : E^n \times E^n \rightarrow [0, \infty)$ by the equation

$$D(u,v) = \sup\{d([u]^\alpha, [v]^\alpha)/\alpha \in [0,1]\},$$

where d is the Hausdorff metric defined in $P_k(\mathbf{R}^n)$. Then it is easy to show that D is a metric in E^n and using results of [[21], [79]], we see that (E^n, D) is a complete metric space, but not locally compact. Moreover, the distance D verifies that

$$D(u+w, v+w) = D(u,v), \quad u, v, w \in E^n,$$
$$D(\lambda u, \lambda v) = |\lambda| D(u,v), \quad u, v \in E^n, \lambda \in R,$$
$$D(u+w, v+z) \leq D(u,v) + D(w,z), \quad u, v, w, z \in E^n.$$

We note that (E^n, D) is not a vector space. But it can be embedded isomorphically as a cone in a Banach space [80].

Definition 2.3.1. We say that a mapping $F : J \longrightarrow E^n$ is strongly measurable if for all $\alpha \in [0,1]$ the set-valued mapping $F_\alpha : J \rightarrow P_k(\mathbf{R}^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is (Lebesgue) measurable, when $P_k(\mathbf{R}^n)$ endowed with the topology generated by the Hausdorff metric d.

Definition 2.3.2. A mapping $F : J \rightarrow E^n$ is called integrably bounded if there exists an integrable function h such that $\|x\| \leq h(t)$, for all $x \in F_0(t)$.

Definition 2.3.3. Let $F : J \longrightarrow E^n$. The integral of F over J, denoted $\int_J F(t)dt$ or $\int_a^b F(t)dt$, is defined level-wise by the equation

$$\begin{aligned}\left[\int_J F(t)dt\right]^\alpha &= \int_J F_\alpha(t)dt \\ &= \left\{\int_J f(t)dt / f : J \longrightarrow \mathbf{R}^n \text{ is a measurable selection for } F_\alpha\right\},\end{aligned}$$

for all $0 < \alpha \leq 1$.

Theorem 2.3.2. If $F : J \longrightarrow E^n$ is strongly measurable and integrably bounded then F is integrable.

Corollary 2.3.1. If $F : J \longrightarrow E^n$ is continuous then it is integrable.

Theorem 2.3.3. [46] Let $F : J \longrightarrow E^n$ be integrable and $c \in J$. Then

$$\int_a^b F = \int_a^c F + \int_c^b F.$$

Theorem 2.3.4. [46] Let $F, G : J \longrightarrow E^n$ be integrable and $\lambda \in R$. Then

(i) $\int F + G = \int F + \int G$,

(ii) $\int \lambda F = \lambda \int F$,

(iii) $D(F, G)$ is integrable,

(iv) $D(\int F, \int G) \leq \int D(F, G)$.

Let $u, v \in E^n$. If there exists a $w \in E^n$ such that $u = v + w$ then we call w the H-difference of u and v, denoted $u - v$.

Definition 2.3.4. A mapping $F : J \to E^n$ is differentiable at $t_0 \in J$ if there exists a $F'(t_0) \in E^n$ such that the limits

$$\lim_{h \to 0+} \frac{F(t_0 + h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0+} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist and equal to $F'(t_0)$.

Here the limit is taken in the metric space (E^n, D). At the end points of J we consider only the one-sided derivatives.

Theorem 2.3.5. Let $F : J \to E^1$ be differentiable. Denote $F_\alpha = [f_\alpha, g_\alpha]$, $\alpha \in [0, 1]$. Then f_α and g_α are differentiable and

$$[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)].$$

Theorem 2.3.6. If $F : J \to E^n$ is differentiable then it is continuous.

Theorem 2.3.7. If $F, G : J \to E^n$ are differentiable and $\lambda \in R$ then

$$(F + G)'(t) = F'(t) + G'(t) \quad \text{and} \quad (\lambda F)'(t) = \lambda F'(t).$$

Theorem 2.3.8. [46] Let $F : J \to E^n$ be continuous. Then for all $t \in J$ the integral $G(t) = \int_a^t F(t)dt$ is differentiable and $G'(t) = F(t)$.

Theorem 2.3.9. Let $F : J \to E^n$ be differentiable and assume that the derivative F' is integrable over J. Then for each $s \in J$, we have

$$F(s) = F(a) + \int_a^s F'.$$

Theorem 2.3.10. [46] Let $F : J \to E^n$ be continuously differentiable on J. Then

$$D(F(b), F(a)) \leq (b - a) \sup\{D(F'(t), \tilde{0})/t \in I\}.$$

Let $a > 0$ and assume that $f : J \to E^n$ is continuous. Consider the initial value problem

$$y'(t) = f(t, y(t)), \ \ y(a) = y_0. \tag{2.3.1}$$

Lemma 2.3.1. A mapping $y : J \to E^n$ is a solution to the problem (2.3.1) if and only if it is continuous and satisfies the integral equation

$$y(t) = y_0 + \int_a^t f(s, y(s))ds,$$

for all $t \in J$.

Theorem 2.3.11. [46] Let $f : J \times E^n \to E^n$ be continuous and assume that there exists a $K > 0$ such that

$$D(f(t,x)), f(t,y)) \leq KD(x,y),$$

for all $t \in J$, $x, y \in E^n$. Then the problem (2.3.1) has a unique solution on J.

Theorem 2.3.12. [73] Suppose that $f : J \times E^n \to E^n$ is continuous and bounded i.e. there exists $r \geq 0$ such that

$$D(f(t,y), \tilde{0}) \leq r, \quad t \in J, \quad y \in E^n.$$

Then the initial value problem (2.3.1) possesses at least one solution on the interval J.

Consider the non-linear fuzzy differential equation of second order

$$y'' = f(t, y, y') \tag{2.3.2}$$

satisfying

$$y(a) = y_1, \quad y'(a) = m, \tag{2.3.3}$$

where $f : J \times E^n \times E^n \to E^n$ is continuous.

If ϕ is a solution of (2.3.2) satisfying (2.3.3) if and only if ϕ is a solution of the integral equation

$$y(t) = y_1 + m(t-a) + \int_a^t (t-s) f(s, y(s), y'(s)) ds.$$

Theorem 2.3.13. [53] Suppose that $f : J \times E^n \times E^n \to E^n$ is continuous and suppose there exists an $M > 0$ such that $D(f(t,y,y'), \tilde{0}) \leq M$. Then the initial value problem (2.3.2) satisfying (2.3.3) possesses at least one solution on the interval J.

Theorem 2.3.14. [53] Let $f \in C(J \times E^n \times E^n, E^n)$ and satisfy

$$D[f(t,u,u'), f(t,v,v')] \leq KD(u,v) + LD(u',v')$$

and assume that

$$\alpha = K\frac{(b-a)^2}{8} + L\frac{(b-a)}{2} < 1.$$

Then the two point fuzzy boundary value problem

$$y'' = f(t,y,y'); \;\; y(a) = y_1, \quad y(b) = y_2,$$

has one and only one solution.

Section 2.4.

Now we present some existence and uniqueness theorems for two and three point boundary value problems by the use of technique of matching solutions [[5], [9], [20]]. Further, we also present some sufficient conditions for existence and uniqueness criteria for boundary value problems with the help of Lyapunov functions [[8], [36], [82]].

Theorem 2.4.1. [5] Let $a < c < b$. Suppose that

(i) $f(t, y, y')$ is continuous on $D = [a, b] \times \mathbf{R}^2$

(ii) all initial value problems have unique solutions on $[a, b]$

(iii) all boundary value problems of the form (2.2.6) satisfying

$$y(a) = y_1, \;\; y'(c') = m, \;\; \text{whenever} \;\; c' \in (a, c]$$

$$y'(c') = m, \;\; y(b) = y_2, \text{whenever} \;\; c' \in [c, b)$$

$$y(a) = y_1, \;\; y(c) = y_3$$

and

$$y(c) = y_3, \;\; y(b) = y_2$$

have unique solutions.

Then there exists a unique solution to the boundary value problem (2.2.6) and (2.2.9).

Theorem 2.4.2. [9] Let $y_1, y_2, y_3 \in R$, $b \in R$ with $a < b < c$, and suppose that

(i) for each $m \in R$ there exist solutions of (2.2.10) satisfying

$$y(a) = y_1, \quad y(b) = y_2, \quad y^{(i)}(b) = m \quad (i = 1, 2) \tag{2.4.1$_i$}$$

and

(2.2.10) satisfying

$$y(b) = y_2, \quad y^{(i)}(b) = m, \quad y(c) = y_3, \quad (i = 1, 2), \tag{2.4.2$_i$}$$

(ii) f satisfies condition 'A' (1.1.6) at 'b' on (a, c).

Then there exists a unique solution to the three point boundary value problem

$$y''' = f(t, y, y', y'') \tag{2.4.3}$$

satisfying

$$y(a) = y_1, \quad y(b) = y_2, \quad y(c) = y_3. \tag{2.4.4}$$

Das and Lalli [20] have weakened the hypothesis of Barr and Sherman [9] and obtained the following theorem regarding existence and uniqueness of solutions to three point boundary value problems.

Theorem 2.4.3 [20] Suppose that

(i) for each $m \in R$ there exist solutions of (2.4.3) satisfying ($2.4.1_1$) and (2.4.3) satisfying ($2.4.2_1$)

(ii) f satisfies condition 'A' (1.1.6) at b on (a, c).

Then the boundary value problem (2.4.3) and (2.4.4) has a unique solution.

George and Sutton [36] recently developed a Lyapunov theory for two point boundary value problems which gives sufficient condition for uniqueness.

Definition 2.4.1. A Lyapunov function $V(t, y, y')$ is a continuous real valued function, locally Lipschitzian with respect to (y, y'). Corresponding to $V(t, y, y')$ define

$$V_f(t, y, y') = \lim_{h \to 0^+} \inf \tfrac{1}{h}[V(t + h, y + hy', y' + hf(t, y, y')) - V(t, y, y')],$$

$$V'(t, y, y') = \lim_{h \to 0^+} \inf \tfrac{1}{h}[V(t + h, y(t + h), y'(t + h)) - V(t, y, y')].$$

Let $u(t)$ be a solution of the boundary value problem (2.2.6) and (2.2.9). Suppose $v(t)$ is another solution of (2.2.6) and (2.2.9). Write $y = u - v$, then y satisfies

$$y'' = f(t, u, u') - f(t, u - y, u' - y') = F(t, y, y'), \tag{2.4.5}$$

$$y(a) = 0, \;\; y(b) = 0. \tag{2.4.6}$$

Theorem 2.4.4 For F defined in (2.4.5) if there exists a Lyapunov function $V(t, y, y')$ defined on $D = [a, b] \times D_1$, where $D_1 \subset \mathbf{R}^2$ such that

(i) $V(t, y, y') = 0$ if $y = 0$

(ii) $V(t, y, y') > 0$ if $y \neq 0$

(iii) $V'_F(t, y, y') \geq 0$ in the interior of D,

then there is at most one solution of (2.2.6) and (2.2.9).

Recently, Rao, Murty and Murty [82] obtained existence and uniqueness theorem for three point boundary value problem (2.4.3) and (2.4.4) by replacing monotonicity condition 'A' (1.1.6) of Theorem 2.4.2 by an appropriate Lyapunov function.