

INTRODUCTION

Section 1.1.

The importance of matrix Lyapunov systems and their occurrence in a number of areas of applied mathematics such as dynamical programming, optimal filters, quantum mechanics and systems engineering etc., are well known. In this work we mainly focus our attention to the first order matrix Lyapunov systems of the form

$$X'(t) = A(t)X(t) + X(t)B(t), \quad (1.1.1)$$

where $A(t)$, $B(t)$ are square matrices of order ' n ' whose elements a_{ij} , b_{ij} are real or complex valued functions of ' t ' in the real interval $[a, b]$. For a given dynamic system with ' n ' degrees of freedom there may be available exactly ' n ' states observed at ' n ' different times. A mathematical description of such a system results in an n -point boundary value problem.

The study of existence and uniqueness of solutions to two point boundary value problems associated with non-homogeneous matrix Lyapunov systems was carried out by Murty, Prasad and Prasad [65]. Later, Murty, Howell and Sivasundaram [63] established existence and uniqueness of solutions to two and multi point boundary value problems associated with non-linear matrix Lyapunov systems. Recently, Murty and Rao [69] obtained existence and uniqueness of solutions to non-homogeneous matrix Lyapunov systems satisfying two point boundary conditions by using the technique of Kronecker product of matrices and with the help of Bartels-Stewarts algorithm and QR-algorithm to

evaluate the constant matrix involved in the general solution. The above type of boundary value problems occur in control engineering, dynamical systems and ecological systems.

A problem of great importance is that of determine the behaviour of a physical system in the neighborhood of an equilibrium state. If the system returns to this state after being subjected to small disturbances, it is called stable; otherwise it is called unstable. Consequently, when designing a system we would like to have a mathematical criterion for stability. Several classical results relating to stability, asymptotic stability for system of differential equations were quoted in the text books of Bellman [11], Coddington & Levinson [14], W.A.Coppel [18], M.R.M.Rao & S.Ahmad [81], and also the references there in.

The problem of solutions being Ψ -bounded and Ψ -stability for systems of ordinary differential equations on $R_+ = [0, \infty)$ has been studied by several authors; namely Akinyele [1], Avramescu [3], Constantin [16], and Diamandescu [[23]-[27]]. Akinyele [1] introduced the notion of Ψ -stability of degree k with respect to a function $\Psi \in C(R_+, R_+)$, which is increasing and differentiable on R_+ and is such that $\Psi(t) \geq 1$, for $t \geq 0$ and $\lim_{t \rightarrow \infty} \Psi(t) = b$, where $b \in [1, \infty)$. Later Constantin [16] introduced the notion of degree of stability and degree of boundedness of solutions of an ordinary differential equations, with respect to a continuous positive and non-decreasing function $\Psi : R_+ \rightarrow R_+$. Further, Morchalo [62] introduced the notion of Ψ -stability, Ψ -uniform stability, and Ψ -asymptotic stability of trivial solution of the non-linear system $x' = f(t, x)$, where Ψ is a scalar continuous function. He also obtained several new sufficient conditions for different types of Ψ -stability for the linear system $x' = A(t)x(t)$.

Moreover, sufficient conditions are also given for uniform Lipschitz stability of the system $x' = f(t, x) + g(t, x)$.

Recently, Diamandescu [[24], [25]] proved a necessary and sufficient condition for the existence of Ψ -bounded solution of non-homogeneous linear differential equation $x' = A(t)x + f(t)$. He also obtained [[23], [26], [27]] several sufficient conditions for various types of Ψ -stability of the trivial solution of the non-linear Volterra integro-differential system, under the assumption that the function Ψ is a continuous matrix valued function.

The study of dichotomy and conditioning of boundary value problems is an interesting area of current research, due to their invaluable use in the analysis of algorithms, in devising numerical schemes for solutions and also play an important role in estimating the global error due to small perturbations. In finding solutions to boundary value problems, the construction of Green's matrix is vital. It is well known that the construction of Green's matrix plays an important role in solving non-homogeneous as well as non-linear matrix systems.

In principle, dichotomy denotes a splitting of solution spaces into subspaces of solution with a markedly different growth behaviour, like increasing \leftrightarrow decreasing, increasing faster than a certain exponential rate \leftrightarrow increasing slower as compared with this rate. In fact, in many practical situations, we come across fundamental matrix solutions of system of first order differential equations of the form

$$Lx = x' - A(t)x \tag{1.1.2}$$

that have both increasing and decreasing components in the fundamental matrix, and it has become almost a convention that the solution space of $Lx = 0$

can be split up into a growing and decay part, i.e. there exists a dichotomy. An outstanding reference on dichotomy we refer to Coppel [19].

The condition numbers in principle indicate by how much any possible errors in the boundary conditions may be amplified; it turns out that they also play an important role in estimating the global error due to small perturbations. As per explicit ordinary differential equations, the conditioning of boundary value problems depends on bounds for quantities that contain the boundary conditions and associated Green's function. The conditioning of linear boundary value problems was studied by Mattheij [59]. He obtained a relationship between the conditioning and the rank of the boundary conditions. Further, discussed the stability of certain algorithms like multiple shooting is related to the conditioning of the problem and presented a stability analysis of this algorithm and established the fact that condition number is an important quantity in estimating the global error. The conditioning of boundary value problems in transferable differential-algebraic equations was studied by Lentini and Marz [55].

In 1987, De Hoog and Mattheij have shown in [22] that the concepts of dichotomy of solution space and well conditioning of associated boundary value problems are equivalent in some sense. Further, Murty and Lakshmi [64] have obtained results relating to dichotomy and conditioning for two point boundary value problems associated with system of first order square matrix differential equations satisfying the general boundary conditions. In 1996, Murty [66] studied conditioning for four point boundary value problems associated with general first order matrix differential equations involving square matrices. Later, Murty and Rao [67] obtained conditioning for three point boundary

value problems associated with system of first order differential equations involving rectangular matrices.

Generally, several systems are mostly related to uncertainty and inaccuracy. The problem of inaccuracy is considered in general an exact science and that of uncertainty is considered as vague or fuzzy and accidental. Since 1965, when Zadeh published his pioneering paper [87], hundreds of examples have been supplied where the nature of uncertainty in the behaviour of a system possesses fuzzy rather than stochastic nature. Non-stationary fuzzy systems described by fuzzy processes look as their natural extension into the time domain. From different view points they were carefully studied, for example, in [[49], [57]]. Solutions of fuzzy differential equations provide a noteworthy example of time dependent fuzzy sets.

The term “fuzzy differential equations” was coined in 1978 by the authors Kandel and Byatt [48]; much extended version of this short note was published two years later by [49]. Since then, the theory of fuzzy differential equations seems to have split into two independent branches, where the first one relies upon the notion of Hukuhara derivative [78] while the other does not.

In 1972, Chang and Zadeh [13] first introduced the concept of fuzzy derivative, followed by Dubois and Prade [35], who used the extension principle in their approach. In the mean time, Puri and Ralescu [77] used the notion of H-differentiability to extend the differential of set-valued functions to that of fuzzy functions. This led Seikkala [84] to introduce the notion of fuzzy derivative as an extension of the Hukuhara derivative and the fuzzy integral, which was the same as that proposed by Dubois and Prade [[33], [34]].

In recent years, control systems have assumed an increasingly important

role in the development and advancement of modern civilization and technology. Practically every aspect of our day-to-day activities is affected by some type of control systems. Control systems are found in abundance in all sectors of industry, such as quality control of manufactured products, automatic assembly line, machine-tool control, space technology and weapon systems, computer control, transportation systems, power systems, robotics and many others. Even the control of inventory and social and economic systems may be approach from the theory of automatic control.

In most branches of applied mathematics, the aim being is to analyze a given situation. Its main aim being to compel or control a system to behave in some desired fashion. Here 'system' is used to mean a collection of objects, which are related by interactions and produce various outputs in response to different inputs. The main interest is to control the system automatically, without direct human intervention.

The objective of the control is to transfer the state of the system to a desirable state from the initial state using the given input $u(t)$. However, the existence of such an input should be assured; this is the controllability condition. On the other hand it is sometimes necessary to know all state variables from measurement of the output $y(t)$, whose dimension is less than that of the state. The observability condition assures the construction of the state from the output. These properties are intrinsic for the systems and play an important role in the theory of linear systems.

Barnett and Cameron [6] studied controllability and observability for continuous first order systems, and also similar type of discrete systems. Results on control theory for discrete cases are given in [85] and [54]. Recently, Murty,

Rao and Suresh Kumar [70] obtained necessary and sufficient conditions for complete controllability, complete observability, and realizability associated with first order matrix Lyapunov systems under certain smoothness conditions.

Most popular fuzzy logic systems discussed in the literature may be classified into three types: pure fuzzy systems, Takagi and Sugeno's fuzzy systems, and fuzzy logic systems using fuzzifiers and defuzzifiers. We know that fuzzy logic systems are very useful in helping to establish an intelligent control theory. There are two main approaches to intelligent control: one that combines differential equations with expert systems in artificial intelligence, the so called expert control, and the other that combines differential equations with discrete event systems or Markov Chain. The first approach is practically useful, but it is difficult to analyze because the formulations of differential equations are based on mathematical formulas and the other is based on symbolic artificial intelligence. The second approach is mathematically well developed, but the theory is complex and applications to real practical problems are not easy.

Recently, Ding and Kandel [[29]-[31]], Ding, Ming Ma and Kandel [32] provided a way to incorporate differential equations with fuzzy sets or fuzzy IF-THEN rules to form a new fuzzy logic system, called fuzzy dynamical system, which can be regarded as an approach to intelligent control. In [29] and [32] they studied the controllability and observability properties of linear system modeled by

$$\begin{aligned}x'(t) &= A(t)x(t) + B(t)u(t), \quad x(0) = x_0 \\y(t) &= C(t)x(t) + D(t)u(t),\end{aligned}\tag{1.1.3}$$

where A, B, C, D are matrices, whose elements are continuous functions, and

$u(t)$ is a fuzzy set. Further, in [[30], [31]] they continued their study on controllability and observability of fuzzy dynamical systems (1.1.3) with the input $u(t)$ in another form namely as a product of ' n ' fuzzy sets defined in real intervals.

The initial value problem for first order fuzzy differential equation

$$y' = f(t, y), \quad y(t_0) = y_0 \quad (1.1.4)$$

has been studied by several authors [[28], [46], [47], [51], [84]] on the metric space of (E^n, D) given by the maximum of the Hausdörff distance between the corresponding level sets. In [46], the author presents a existence and uniqueness theorem when f is continuous and satisfies Lipschitz condition. Further, Kaleva [47] obtained a Peano-like theorem by restricting to continuous mapping f defined on some locally compact subsets of (E^n, D) . In [73], Nieto proved a version of the classical Peano's theorem for the fuzzy initial value problem (1.1.4), where f is continuous and bounded, on the metric space (E^n, D) , by using the classical version of Ascoli's theorem. The result obtained in this paper compliments the existence and uniqueness result of Kaleva [46].

Recently, Georgiou and Kougias [37] proved the existence of a unique solution for the Cauchy problem of second order fuzzy differential equations given by

$$\begin{aligned} y'' &= f(t, y, y') \\ y(t_0) &= k_1, \quad y'(t_0) = k_2 \end{aligned} \quad (1.1.5)$$

under the assumption of continuity on ' f ' and the Lipschitz character of ' f ' with respect to the second variable. Further, Georgiou, Nieto and Rosana Rodriguez-Lopez [38] obtained existence and uniqueness results for initial

value problems associated with second order and n^{th} order non-linear fuzzy differential equations, provided 'f' is continuous and satisfies a Lipschitz condition that involves all the variables, but the temporal one.

Relating to two point boundary value problems for second order non-linear fuzzy differential equations Lakshmikantham and Mohapatra [52] and Lakshmikantham, Murty and Turner [53] are studied, providing results on existence and uniqueness of solutions. Further, O'Regan, Lakshmikantham and Nieto [75] proved a Wintner-type result for fuzzy initial value problems and a superlinear-type result for fuzzy boundary value problems.

The technique of matching solutions which is well known in singular perturbation theory is utilized in boundary value problems by Bailey, etc., [5] to study existence and uniqueness of solutions to two point boundary value problems. Barr and Sherman [9] extend the idea of matching solutions to three point boundary value problems associated with third order differential equations with the following monotonicity condition on f .

The function $f(t, u, v, w)$ is said to satisfy monotonicity condition 'A' at a point 'b' on (a, c) , if there exist 'a' and 'b' such that $a < b < c$;

$$\begin{aligned} u_1 \geq u_2, v_1 < v_2 &\Rightarrow f(t, u_1, v_1, w) < f(t, u_2, v_2, w), \quad \forall t \in (a, b) \\ u_1 \leq u_2, v_1 < v_2 &\Rightarrow f(t, u_1, v_1, w) < f(t, u_2, v_2, w), \quad \forall t \in [b, c). \end{aligned} \tag{1.1.6}$$

In 1978, Moorti and Garner [60] obtained existence and uniqueness of solutions to three point boundary value problem associated with n^{th} order differential equations satisfying n boundary conditions, with the generalized monotonicity assumption at 'b' on (a, c) . Further, Das and Lalli [20] have weakened the hypothesis of Barr and Sherman [9] and obtained existence and uniqueness of solutions to third order three point boundary value problems.

Multipoint boundary value problems, for which the number of boundary points is greater than the order of the differential equations were considered by many authors [[4], [40], [41], [58]]. They studied the questions of existence of solutions of multipoint boundary value problems associated with second and higher order differential equations. Recently, Henderson and Tisdell [44] extended the matching technique of solutions to obtain existence and uniqueness of solutions of five point boundary value problems associated with third order differential equations, with the use of monotonicity restriction on 'f'.

A theoretical technique that has been proved extremely useful in initial value theory [86], but does not seem to have its due in boundary value theory is the direct method of Lyapunov. In this direction Yoshizawa [86] and Okamura [74] have demonstrated that uniqueness of solutions to initial value problems is equivalent to the existence of a Lyapunov function. Kato and Strauss [50], Bernfeld [12] provided necessary and sufficient conditions for the existence of solutions to initial value problem on $[t_0, \infty)$, with the use of Lyapunov functions. With regard to two point boundary value problems for second order differential equations. George and Sutton [36] formulated a Lyapunov theory by the addition of three conditions to the usual definition of a Lyapunov function as given in [86], which provides sufficient conditions for the existence and uniqueness of solutions. Further, Barr and Miletta [8] provided necessary and sufficient conditions for the existence and uniqueness of solutions to two point boundary value problems associated with second order differential equations. Later, Rao, Murty and Murty [82] extend the Lyapunov theory for the existence and uniqueness of solutions to three point boundary value problems associated with third order non-linear differential equations. In this paper,

they replaced the monotonicity condition of Barr and Sherman [9] by using an appropriate extended Lyapunov function and the technique of matching solutions.

Section 1.2.

In this thesis we mainly deal with the study of qualitative properties of first order matrix Lyapunov systems. This is done in Chapters 3 to Chapter 6. The remaining two Chapters, namely 7 and 8 deals with existence and uniqueness of solutions to boundary value problems associated with non-linear differential equations.

A brief sketch of the work done in each chapter is mentioned here under.

In Chapter 2 we introduce the notation used in this thesis and present some properties of Kronecker product of matrices and also present some standard results on differential equations, boundary value problems, fuzzy differential equations etc., which are useful for later discussion.

The concepts of Ψ -boundedness and Ψ -stability for matrix Lyapunov systems are not studied so far in the literature. One of the chapters of this thesis, namely Chapter 3 is devoted to study the concepts of Ψ -boundedness, Ψ -stability, and Ψ -asymptotic stability of matrix Lyapunov systems. First, we establish existence and uniqueness of solutions of initial value problems associated with non-homogeneous matrix Lyapunov systems of the form

$$X'(t) = A(t)X(t) + X(t)B(t) + F(t), \quad (1.2.1)$$

where $F(t)$ is a $n \times n$ real valued continuous matrix on R_+ , by converting the problem into a Kronecker product initial value problem. We also obtain a

necessary and sufficient condition for the existence of at least one Ψ -bounded solution for the Kronecker product problem, under the assumption that $\hat{F}(t)$ ($\text{Vec } F$) is a Lebesgue Ψ -integral function. Further, we establish some sufficient conditions for Ψ -stability, Ψ -uniform stability, and Ψ -asymptotic stability of trivial solutions of homogeneous ($\hat{F}(t) = 0$) as well as non-homogeneous Kronecker product system associated with (1.2.1). The results obtained in this chapter are illustrated with suitable examples. This chapter includes some of the results of Diamandescu [[23], [24], [26]] as a particular case when $B(t) = 0$, X and F (in [24] and [26] $F = 0$) are column vectors.

The study of dichotomy and conditioning of two point boundary value problems has aroused great interest during the last two decades. In Chapter 4 we develop dichotomy and conditioning of the general first order matrix Lyapunov system of the form

$$LX = X'(t) - (A(t)X(t) + X(t)B(t)) = F(t), \quad (1.2.2)$$

satisfying two point boundary conditions

$$MX(a)N + RX(b)S = Q. \quad (1.2.3)$$

We first obtain existence and uniqueness criteria by converting the problem into a Kronecker product boundary value problem, and the solution is expressed in terms of Green's matrix for the corresponding homogeneous boundary value problem. Further, we also obtain the close relationships between the stability bounds of the Kronecker product boundary value problem on the one hand, and the growth behaviour of the fundamental matrix solution on the other hand. Finally, we conclude that the condition number is the right

criterion to indicate possible error amplification of the perturbed boundary conditions.

Control theory is an interdisciplinary area of research, where many mathematical concepts and methods work together to produce an impressive body of applied mathematics. Controllability and observability are the two basic concepts that arise in the control of dynamical systems.

Chapter 5 is devoted to the study of controllability of first order fuzzy dynamical matrix Lyapunov systems modeled by;

$$\begin{aligned} X'(t) &= A(t)X(t) + X(t)B(t) + F(t)U(t), \quad X(0) = X_0 \\ Y(t) &= C(t)X(t) + D(t)U(t), \end{aligned} \tag{1.2.4}$$

where $U(t)$ is a $n \times n$ fuzzy input matrix called fuzzy control and $Y(t)$ is a $n \times n$ fuzzy output matrix. Here $A(t), B(t), F(t), C(t)$, and $D(t)$ are matrices of order $n \times n$, whose elements are continuous functions of t on $J = [0, T] \subset \mathbb{R}$ ($T > 0$). First, we generate a deterministic control system with fuzzy inputs and outputs called a fuzzy dynamical Lyapunov system. Here, the fuzzy input is taken in the form either (i) as a product of ' n ' fuzzy sets defined on \mathbb{R}^n or (ii) a fuzzy set defined on $\mathbb{R}^{n \times n}$. Further, we present a set of sufficient conditions for the controllability of the fuzzy dynamical Lyapunov system, with and without using the fuzzy rule base. The results of this chapter are highlighted with suitable examples. This chapter generalizes some of the results of Ding and Kandel [[29], [30]].

In Chapter 6 we continue to study the first order fuzzy dynamical matrix Lyapunov system given by (1.2.4) and discuss the concept of observability. First, we formulate a fuzzy dynamical Lyapunov system and also obtain its solution set. Next, we study observability property of fuzzy dynamical Lya-

Lyapunov system, and give sufficient conditions for the system to be completely observable. Further, we introduce the notion of likely observability and provide sufficient conditions for the system to be likely observable. The main results of this chapter are illustrated with suitable examples. This chapter extends some of the results of Ding and Kandel [31] and Ding, Maa and Kandel [32] to fuzzy dynamical matrix Lyapunov systems.

Chapter 7 deals with the study of existence and uniqueness of solutions of initial and boundary value problems associated with first, second, and third order non-linear fuzzy differential equations. First, we obtain existence and uniqueness theorem for initial value problem associated with first order non-linear fuzzy differential equations. Further, we prove existence and uniqueness theorem for initial and boundary value problems for second order non-linear fuzzy differential equations, with the help of Green's function and contraction mapping theorem. Here, we use a modified Lipschitz condition that involves all the variables. Later, we prove sufficient conditions for the existence and uniqueness of solutions of a certain class of third order non-linear fuzzy differential equations. The results obtained in this chapter not only include more general class of problems than in [[37], [38], [46]] but also extend some of the results of Lakshmikantham etc., [53] developed for non-linear fuzzy differential equations.

The questions of existence and uniqueness of solutions to five point boundary value problems associated with third order non-linear differential equations are studied in Chapter 8. First, we show how the solutions of two three point boundary value problems are matched to obtain a unique solution of five point boundary value problem. In this chapter we extend the Lyapunov

theory for the existence and uniqueness of solutions to five point boundary value problems. Here, we replace the monotonicity condition of [44] by using an appropriate ‘Lyapunov like’ function and the technique of matching solutions. We may also note that even though the monotonicity condition in [44] fail, a suitable ‘Lyapunov like’ function may exist to ensure the existence and uniqueness of solutions to five point boundary value problems. An example to this effect is also given in this chapter.

Section 1.3.

The following numbering system is used in this thesis. [1], [2], ..., [87] stand for references given at the end of the eighth chapter. This thesis is divided into eight chapters. In each chapter Definitions, Results, Theorems, Lemmas etc., are numbered in the decimal notation. The first integer indicates the number of the chapter, the second integer the number of the section and the third one indicates the relative order of the result etc., in that section. All these are marked on the left side of the page.

In addition to these, certain equations and expressions are numbered on the right side of the page as (a.b.c), where ‘c’ denotes the relative order of the equation in bth section of the Chapter ‘a’.