CHAPTER-8

EXTENSION OF LYAPUNOV THEORY TO FIVE POINT BOUNDARY VALUE PROBLEMS FOR THIRD ORDER DIFFERENTIAL EQUATIONS

Section 8.1.

The importance of boundary value problems and their occurrence in many physical problems are well known. In this chapter the problem of existence and uniqueness of solutions of five point boundary value problems for the third order differential equation

$$y''' = f(x, y, y', y'') \tag{8.1.1}$$

is studied. Here f is assumed to be continuous on $[a, c] \times R^3$ and solutions to initial value problems associated with (8.1.1) exist, are unique, and extend throughout $[a, c]$.

The technique of matching solutions, which is familiar in singular perturbation theory, is first utilized by Baily etc.[5] to study existence and uniqueness criteria of solutions of two point boundary value problems associated with second order non-linear differential equations. Barr and Sherman [9] extended the idea of matching solutions to three point boundary value problems associated with third order differential equations with a monotonicity restriction on f.

Several authors [[7], [9], [20], [43], [44], [60], [83]] used matching technique of solutions to obtain existence and uniqueness of solutions to three point boundary value problems associated with $n^{th}(n \geq 3)$ order non-linear differential equations. Moreover, existence and uniqueness criteria for boundary value problems are studied [82] by using Lyapunov functions, which are well known in stability theory.

The approach taken here is similar to that of Barr and Sherman [9] and Rao, Murty and Murty [82] and is based on the use of a solution matching technique and suitable 'Lyapunov-like' function defined later. Recently, Henderson and Tisdel [44] obtained existence and uniqueness of solutions of five point boundary value problems associated with (8.1.1), with the following monotonicity assumption on f :

For all $w \in R$,

$$f(x, v_1, v_2, w) > f(x, u_1, u_2, w),$$

(i) when $x \in (a, b]$, $u_1 \geq v_1$ and $v_2 > u_2$, or

(ii) when $x \in [b, c)$, $u_1 \leq v_1$ and $v_2 > u_2$.

In this chapter we replace this monotonicity condition by an appropriate 'Lyapunov-like' function and establish existence and uniqueness of solutions of five point boundary value problems.

Section 8.2 yields criteria under which solutions of (8.1.1) which satisfy boundary conditions at three points may be matched to obtain a unique solution of (8.1.1) satisfying boundary conditions at five points.

In section 8.3 with the help of a suitable 'Lyapunov-like' function we obtain at most one solution to the following three point boundary value problems (8.1.1) satisfying

$$y(a) - y(x_1) = y_1, \;\; y(b) = y_2, \;\; y^{(i)}(b) = m \;\; (i = 1, 2) \tag{8.1.2$_i$}$$

and

$$y(b) = y_2, \;\; y^{(i)}(b) = m, \;\; y(x_2) - y(c) = y_3 \;\; (i = 1, 2). \tag{8.1.3$_i$}$$

Further, with the added hypothesis that solutions exist for the problems (8.1.1) satisfying ($8.1.2_i$) and ($8.1.3_i$) a unique solution to the five point boundary value problem (8.1.1) satisfying

$$y(a) - y(x_1) = y_1, \;\; y(b) = y_2, \;\; y(x_2) - y(c) = y_3 \tag{8.1.4}$$

is constructed.

Section 8.2.

In this section the following theorem illustrates how the solutions of two three point boundary value problems are matched to obtain a unique solution to the five point boundary value problem.

Theorem 8.2.1. Let $y_1, y_2, y_3, b \in R$ with $a < x_1 < b < x_2 < c$ and suppose that

(i) for each $m \in R$ there exist solutions of (8.1.1) satisfying ($8.1.2_i$) or ($8.1.3_i$) ($i = 1, 2$),

(ii) for each $m \in R$ and each t there exists at most one solution of each of the following boundary value problems ;

$$y''' = f(x, y, y', y''),$$

$$y(a) - y(x_1) = y_1, \;\; y(b) = y_2, \;\; y''(t) = m, \text{ where } t \in (a, b],$$

$$y''' = f(x, y, y', y''),$$

$$y(b) = y_2, \;\; y''(t) = m, \;\; y(x_2) - y(c) = y_3, \text{ where } t \in [b, c).$$

Then there exists a unique solution to the boundary value problem (8.1.1) satisfying (8.1.4).

Proof. Let $y_1(x, m)$ denote a solution of (8.1.1) satisfying (8.1.2$_2$) with second derivative m at $x = b$.

First, we show that $y_1'(b, m)$ is an increasing function of m. By hypothesis, if $m_2 > m_1$, then

$$y_1''(b, m_2) > y_1''(b, m_1).$$

Define $w(x) = y_1(x, m_2) - y_1(x, m_1)$. By hypothesis

$$w(a) - w(x_1) = 0, \quad w(b) = 0 \quad \text{and} \quad w''(b) > 0.$$

Further, it is claimed that $w''(x) > 0$, for all $x \in (a, b]$. Suppose to the contrary there exists a point $p \in (a, b)$ such that $w''(p) \leq 0$. Since $w''(x)$ is continuous, there exists a point $q \in [p, b)$ such that $w''(q) = 0$. This implies that

$$y_1''(q, m_2) = y_1''(q, m_1) = k,$$

which is a contradiction to our hypothesis (ii). Since $w(a) = w(x_1)$, there exists a point $r \in (a, x_1)$ such that $w'(r) = 0$. This together with the above claim, implies

$$w'(x) > 0, \quad \text{for all } x \in (r, b].$$

In particular $w'(b) > 0$, hence $y_1'(b, m)$ is a strictly increasing function of m.

Let $y_2(x; m)$ denote a solution of (8.1.1) satisfying (8.1.3$_2$) with second derivative m at $x = b$. In a similar way it can be shown that $y_2'(b, m)$ is a strictly decreasing function of m.

Now it is claimed that $y_1'(b, m)$ has no jump discontinuities as a function of m. Suppose to the contrary $y_1'(b, m)$ has a jump discontinuity at $m = m_1$ such that

$$y_1'(b, m_1^-) = \alpha, \quad y_1'(b, m_1) = \beta \text{ and } y_1'(b, m_1^+) = \gamma.$$

By monotonicity of $y_1'(b,m)$ in m it follows that $\alpha \leq \beta \leq \gamma$, $\alpha < \gamma$. Let k be a real number different from β (i.e $k \neq \beta$) and $\theta(x)$ be a solution of (8.1.1) satisfying

$$y(a) - y(x_1) = y_1, \;\; y(b) = y_2, \;\; y'(b) = k.$$

It follows that $\theta''(b)$ exists and $\theta''(b) = p$. For $p = m_1$, $y_1(x,p) \equiv \theta(x)$ which leads to a contradiction. Similarly $y_2'(b,m)$ has no jump discontinuities.
Now we show that $y_1'(b,.) : R \overset{onto}{\rightarrow} R$.

Let $z_0 \in R$, the boundary value problem (8.1.1) satisfying

$$y(a) - y(x_1) = y_1, \;\; y(b) = y_2, \;\; y'(b) = z_0$$

has a solution ϕ . Let $\phi''(b) = q$, then $y_1(x,q) \equiv \phi(x)$ implies $y_1'(b,q) = \phi'(b) = z_0$. Similarly $y_2'(b,.)$ also maps from R onto R.

Thus both $y_1'(b,m)$ and $y_2'(b,m)$ are continuous strictly monotonic functions of m whose ranges are the set of all real numbers. By defining $Y'(b,m) = y_1'(b,m) - y_2'(b,m)$, we have

$$Y'(b,m) \rightarrow \infty \quad as \quad m \rightarrow +\infty,$$

$$Y'(b,m) \rightarrow -\infty \quad as \quad m \rightarrow -\infty.$$

Then there exists a $m_0 \in R$ such that $y_1'(b,m_0) = y_2'(b,m_0)$. By definition of $y_1(x,m_0)$ and $y_2(x,m_0)$, we have $y_1(b,m_0) = y_2(b,m_0)$ and $y_1''(b,m_0) = y_2''(b,m_0)$. Thus $y(x)$ defined by

$$y(x) = \begin{cases} y_1(x,m_0), & a \leq x \leq b \\ y_2(x,m_0), & b \leq x \leq c \end{cases}$$

is a solution of (8.1.1) satisfying (8.1.4).

To establish uniqueness, suppose that $y_1(x)$ and $y_2(x)$ are two distinct solutions of (8.1.1) satisfying (8.1.4). Let $w(x) = y_1(x) - y_2(x)$, then

$$w(a) - w(x_1) = 0, \;\; w(b) = 0, \;\; w(x_2) - w(c) = 0.$$

By application of Rolle's theorem, there exists a point $p_1 \in (a, x_1)$ and a point $p_2 \in (x_2, c)$ such that

$$w'(p_1) = 0, \;\; w'(p_2) = 0.$$

Again applying Rolle's theorem, there exists a point $p_3 \in (p_1, p_2)$ such that $w''(p_3) = 0$, i.e. $y_1''(p_3) = y_2''(p_3)$ which is a contradiction to hypothesis (ii).

Here the matching of solutions in hypothesis (i) was accomplished by depending on hypothesis (ii) which is about uniqueness of solutions of four point boundary value problems. Hence it is preferable to match solutions of three point boundary value problems without the help of a hypothesis involving four point boundary value problems. This was achieved in the next section with the use of 'Lyapunov-like' function.

Section 8.3.

In this section we define the Lyapunov function and establish Lemmas which are useful for proving our main theorem regarding existence and uniqueness of solutions of five point boundary value problems.

Suppose y_1 and y_2 are solutions of (8.1.1) satisfying ($8.1.2_i$) or ($8.1.3_i$) (i=1, 2). Write $y = y_1 - y_2$. Then

$$y''' = F(x, y, y', y'') = f(x, y+y_2, y'+y_2', y''+y_2'') - f(x, y_2, y_2', y_2'') \qquad (8.3.1)$$

and

$$F(x, 0, 0, 0) = 0. \qquad (8.3.2)$$

Then the boundary conditions ($8.1.2_i$) and ($8.1.3_i$) (i=1, 2) respectively become

$$y(a) - y(x_1) = 0, \ \ y(b) = 0, \ \ y^{(i)}(b) = 0 \ \ (i = 1, 2) \qquad (8.3.3_i)$$

and

$$y(b) = 0, \ \ y^{(i)}(b) = 0, \ \ y(x_2) - y(c) = 0 \ \ (i = 1, 2). \qquad (8.3.4_i)$$

Hence $y(x) \equiv 0$ is a solution of (8.3.1) satisfying ($8.3.3_i$) or ($8.3.4_i$) ($i = 1, 2$). Thus we have proved the following:

Lemma 8.3.1. The problem (8.1.1) satisfying ($8.1.2_i$) or ($8.1.3_i$) ($i = 1, 2$) has a unique solution if and only if $y(x) \equiv 0$ is the only solution of (8.3.1) satisfying ($8.3.3_i$) or ($8.3.4_i$) ($i = 1, 2$).

Definition 8.3.1. A Lyapunov function $V(x, y, y', y'')$ is a continuous locally Lipschitzian real valued function with respect to (y, y', y''). Corresponding to $V(x, y, y', y'')$ we define

$$V_f'(x, y, y', y'') = \lim_{h \to 0^+} \inf \frac{1}{h}[V(x + h, y + hy', y' + hy'', y'' + hf) - V(x, y, y', y'')]$$

$$V'(x, y, y', y'') = \lim_{h \to 0^+} \inf \frac{1}{h}[V(x+h, y(x+h), y'(x+h), y''(x+h)) - V(x, y, y', y'')]$$

where f is a function defined and continuous on a domain $M = [a, c] \times N$, where $[a, c]$ is a interval on the real line and $N \subset R^3$. Choose $M = M_1 \cup M_2$, where $M_1 = [a, b] \times N$ and $M_2 = [b, c] \times N$.

Lemma 8.3.2. If $V(x, y, y', y'')$ is a Lyapunov function and $y(x)$ is a solution of (8.1.1) then $V'(x, y, y', y'') = V_f'(x, y, y', y'')$ and $V(x, y, y', y'')$ is non-increasing (non-decreasing) if and only if $V_f'(x, y, y', y'') \leq 0$ ($V_f'(x, y, y', y'') \geq 0$).

Proof. Analogous to the proof of Yoshizawa [p.4 of [86]].

Lemma 8.3.3. For F defined in (8.3.1), if there exists a Lyapunov function $V(x, y, y', y'')$ defined on M_1 such that

(i) $V(x, y, y', y'') = 0$ if $y = 0$,

(ii) $V(x, y, y', y'') > 0$ if $y \neq 0$,

(iii) $V'_F(x, y, y', y'') \geq 0$ in the interior of M_1.

Then for each $m \in R$, there exists at most one solution to the three point boundary value problems (8.1.1) satisfying ($8.1.2_i$) ($i = 1, 2$).

Proof. The proof of the problem (8.1.1) satisfying ($8.1.2_2$) will be given. A similar proof holds for the other boundary value problem. Suppose $y_1(x)$ and $y_2(x)$ are two distinct solutions of (8.1.1) satisfying ($8.1.2_2$). Write $w(x) = y_1(x) - y_2(x)$. Then

$$w''' = F(x, w, w', w''), \tag{8.3.5}$$

$$w(a) - w(x_1) = 0, \quad w(b) = 0, \quad w''(b) = 0, \tag{8.3.6}$$

where $F(x, 0, 0, 0) = 0$. From Lemma 8.3.1 it suffices to show that $w(x) \equiv 0$ is the only solution of the boundary value problem (8.3.5) satisfying (8.3.6). Suppose $\phi(x)$ is a non-trivial solution of the problem (8.3.5) satisfying (8.3.6), then there exists a $\eta \in (a, b)$ such that $\phi(\eta) \neq 0$. Hence

$$V(\eta, \phi(\eta), \phi'(\eta), \phi''(\eta)) > 0. \tag{8.3.7}$$

Since $V'_F(x, y, y', y'') \geq 0$ in the interior of M_1 and from Lemma 8.3.2 it follows that $V(x, y, y', y'')$ is non-decreasing along the solution $\phi(x)$. Thus $\eta < b$ implies

$$V(\eta, \phi(\eta), \phi'(\eta), \phi''(\eta)) \leq V(b, \phi(b), \phi'(b), \phi''(b)) = 0. \tag{8.3.8}$$

Thus (8.3.7) and (8.3.8) contradict each other and hence $y_1(x) \equiv y_2(x)$.

Lemma 8.3.4. For F defined in (8.3.1), if there exists a Lyapunov function $V(x, y, y', y'')$ defined on M_2 such that

(i) $V(x, y, y', y'') = 0$ if $y = 0$,

(ii) $V(x, y, y', y'') > 0$ if $y \neq 0$,

(iii) $V_F'(x, y, y', y'') \leq 0$ in the interior of M_2.

Then for each $m \in R$, there exists at most one solution to the three point boundary value problems (8.1.1) satisfying ($8.1.3_i$) $(i = 1, 2)$.

Proof. Analogous to the proof of Lemma 8.3.3.

Theorem 8.3.1. Let $y_1, y_2, y_3, b \in R$ with $a < x_1 < b < x_2 < c$. Suppose that

(i) for each $m \in R$ there exist solutions of (8.1.1) satisfying ($8.1.2_i$) or ($8.1.3_i$) $(i = 1, 2)$,

(ii) $V(x, y, y', y'')$ is a Lyapunov function as in Lemmas 8.3.3 and 8.3.4.

Then there exists at most one solution to the boundary value problem (8.1.1) satisfying (8.1.4).

Proof. From Lemmas 8.3.3 and 8.3.4 the solutions of (8.1.1) satisfying ($8.1.2_i$) or ($8.1.3_i$) $(i = 1, 2)$ are unique.

Let $y_1(x, m)$ denote the solution of (8.1.1) satisfying ($8.1.2_2$) with second derivative m at $x = b$. By hypothesis, if $m_2 > m_1$, then

$$y_1''(b, m_2) > y_1''(b, m_1).$$

Let $w(x) = y_1(x, m_2) - y_1(x, m_1)$, since $w''(x)$ is continuous and $w''(b) > 0$, either

(1) $w''(x) > 0$, for all $x \in [a, b]$,

(2) $w''(a) = 0$ and $w''(x) > 0$, for all $x \in (a, b]$

or

(3) there exists a point $q \in (a, b)$ such that $w''(q) = 0$ and $w''(x) > 0$, for all $x \in (q, b]$ holds.

First, we show that in all the above three cases there exists a point $p \in (a, b)$ such that $w'(x) > 0$, for all $x \in (p, b]$.

Suppose (1) or (2) holds. Since $w(a) = w(x_1)$ there exists a point $p \in (a, x_1)$ such that

$$w'(p) = 0. \tag{8.3.9}$$

Here w satisfies $w''' = F(x, w, w', w'')$. Since (1) or (2) holds and (8.3.9) implies

$$w'(x) > 0, \quad \text{for all } x \in (p, b].$$

Suppose (3) holds.

Claim. There exists a point $p \in [q, b)$ such that $w'(p) > 0$ and from (3) it follows that

$$w'(x) > 0, \quad \text{for all } x \in (p, b].$$

Suppose to the contrary

$$w'(x) < 0, \quad \text{for all } x \in [q, b). \tag{8.3.10}$$

Since $w(b) = 0$ implies $w(q) > 0$. Hence

$$V(q, w(q), w'(q), w''(q)) > 0. \tag{8.3.11}$$

Since $V_F'(x, y, y', y'') \geq 0$ in the interior of M_1 and from Lemma 8.3.2 it follows that $V(x, y, y', y'')$ is non-decreasing along $w(x)$. Since $q < b$ implies

$$V(q, w(q), w'(q), w''(q)) \leq V(b, w(b), w'(b), w''(b)) = 0. \tag{8.3.12}$$

Thus (8.3.11) and (8.3.12) contradict each other. Hence

$$w'(x) > 0, \quad \text{for all } x \in (p, b].$$

In particular, in all the three cases $w'(b) > 0$, therefore $y_1'(b, m)$ is a strictly increasing function of m.

Let $y_2(x, m)$ denote a solution of (8.1.1) satisfying $(8.1.3_2)$. A similar proof to the above shows that $y_2'(b, m)$ is a strictly decreasing function of m.

Claim. $y_1'(b, .) : R \overset{onto}{\rightarrow} R$.

Let $u_0 \in R$. The boundary value problem

$$y''' = f(x, y, y', y''),$$

$$y(a) - y(x_1) = y_1, \quad y(b) = y_2, \quad y'(b) = u_0,$$

has a unique solution ψ. Let $\psi''(b) = m$. Hence ψ is a solution of (8.1.1) satisfying $(8.1.2_2)$. But this problem has a unique solution. Therefore $\psi = y_1(., m)$ and so $y_1'(b, m) = u_0$. Hence the claim. Similarly $y_2'(b, .) : R \overset{onto}{\rightarrow} R$.

Thus both $y_1'(b, m)$ and $y_2'(b, m)$ are continuous strictly monotonic functions of m whose ranges are the set of all real numbers. Denote

$$Y'(b, m) = y_1'(b, m) - y_2'(b, m),$$

then

$$Y'(b, m) \to \infty \text{ as } m \to +\infty \quad \text{and} \quad Y'(b, m) \to -\infty \text{ as } m \to -\infty.$$

Thus there exists a $m_0 \in R$ such that

$$y_1'(b, m_0) = y_2'(b, m_0).$$

According to hypothesis,

$$y_1(b, m_0) = y_2(b, m_0) \text{ and } y_1''(b, m_0) = y_2''(b, m_0).$$

Thus

$$y(x) = \begin{cases} y_1(x, m_0), & a \le x \le b, \\ y_2(x, m_0), & b \le x \le c, \end{cases}$$

is a solution of (8.1.1) satisfying (8.1.4).

Suppose $y_1(x)$ and $y_2(x)$ are two distinct solutions of (8.1.1) satisfying (8.1.4). Write $y(x) = y_1(x) - y_2(x)$, then

$$y'''(x) = F(x, y, y', y''), \tag{8.3.13}$$

$$y(a) - y(x_1) = 0, y(b) = 0, y(x_2) - y(c) = 0, \tag{8.3.14}$$

where $F(x, 0, 0, 0) = 0$.

Claim. $y(x) \equiv 0$, for all $x \in [a, c]$.

Suppose $y_0(x)$ is a non-trivial solution of the boundary value problem (8.3.13) satisfying (8.3.14). Then there exists a point $p \in [a, c]$ such that $y_0(p) \neq 0$. Since $y_0(b) = 0$, we have either $p \in [a, b)$ or $p \in (b, c]$ and $y_0(p) \neq 0$. For $p \in [a, b)$, we have

$$V(p, y_0(p), y_0'(p), y_0''(p)) > 0. \tag{8.3.15}$$

Since $V_F'(x, y, y', y'') \geq 0$ in the interior of M_1 and from Lemma 8.3.2 it follows that $V(x, y, y', y'')$ is non-decreasing along $y_0(x)$. Since $p < b$ implies

$$V(p, y_0(p), y_0'(p), y_0''(p)) \leq V(b, y_0(b), y_0'(b), y_0''(b)) = 0. \tag{8.3.16}$$

Thus (8.3.15) and (8.3.16) contradict each other. Similarly the other case follows. Hence

$$y_1(x) = y_2(x), \ \forall \ x \in [a, c].$$

Remark 8.3.1. It may be noted that even when the monotonicity condition is not satisfied by f, a Lyapunov function satisfying all the requirements as in Theorem 8.3.1 may exist to ensure the existence and uniqueness of solutions to five point boundary value problems as seen from the following example;

$$y'''+y' = 0, \tag{8.3.17}$$

$$y(0) - y(\frac{\pi}{4}) = \frac{-1}{\sqrt{2}},\ y(\frac{\pi}{2}) = 1,\ y(\frac{3\pi}{4}) - y(\pi) = \frac{1}{\sqrt{2}}. \tag{8.3.18}$$

Here $V(x, y, y', y'') = y^2$ is a Lyapunov function for (8.3.17) on $M = [0, \pi] \times N = M_1 \cup M_2$, where $M_1 = [0, \frac{\pi}{2}] \times N$, $M_2 = [\frac{\pi}{2}, \pi] \times N$ and $N \subset R^3$.