CHAPTER-7

INITIAL AND BOUNDARY VALUE PROBLEMS FOR FUZZY DIFFERENTIAL EQUATIONS

Section 7.1.

The study of initial and boundary value problems for non-linear fuzzy differential equations has attracted much attention in recent times. The initial value problems associated with first order fuzzy differential equations $y' = f(t, y)$ have been studied by several authors [[46], [47], [73]] on the metric space (E^n, D) of normal fuzzy convex sets with the distance given by the maximum of the Hausdorff distances between the corresponding level sets. In this direction, Kaleva [46] presents a uniqueness theorem when f is continuous and satisfies a Lipschitz condition. Further, Nieto [73] obtained a version of the classical existence theorem of Peano for initial value problems associated with first order fuzzy differential equations on the metric space (E^n, D) by using the classical Ascoli's theorem under the assumption of continuity and boundedness on f.

In 2001, Lakshmikantham, Murty and Turner [53] proved a set of sufficient conditions under which a two point boundary value problem associated with second order non-linear fuzzy differential equation has a unique solution. The main tools employed in the above paper are estimates on Green's function and a fixed point theorem of Banach. Further, O'Regan, Lakshmikantham and Nieto [75] obtained existence results of Wintner-type for fuzzy initial value problems and a super-linear type for fuzzy boundary value problems with the help of a generalized Schauder theorem in metric spaces. Recently, Georgiou, Nieto and Rosana Rodriguez-Lopez [38] proved existence and uniqueness of solutions for initial value problems associated with non-linear second order and n^{th}-order fuzzy differential equations satisfying a Lipschitz condition.

In this chapter, we prove existence and uniqueness theorems for first and second order non-linear fuzzy differential equations satisfying a modified Lipschitz condition. Further, we also obtain existence and uniqueness criteria to certain class of three point boundary value problems associated with third order non-linear fuzzy differential equations.

Section 7.2 deals with the existence and uniqueness theorem for initial value problems associated with first order non-linear fuzzy differential equations.

In section 7.3 we prove existence and uniqueness results for initial and boundary value problems for second order non-linear fuzzy differential equations with the help of Green's functions and contraction mapping theorem. Here, we use a modified Lipschitz condition that involves all the variables. The results obtained here, include more general class of problems than in [38], [46] and [53]] obtained for first and second order non-linear fuzzy differential equations.

In section 7.4 we present sufficient conditions for the existence and uniqueness of solutions of three point boundary value problems associated with a third order non-linear fuzzy differential equations by using appropriate Green's function for the associated boundary value problems and by defining a contraction mapping that yields an interval over which a unique solution exists. Here we choose a suitable three point boundary value problem, where the signs of the Green's function and its partial derivatives over the different intervals can be established. This section extend the results of V.Lakshmikantham etc. [53] developed for two point boundary value problems associated with second crder fuzzy differential equations to three point boundary value problems for third order fuzzy differential equations.

Section 7.2.

In this section we obtain an existence and uniqueness theorem for first order fuzzy differential equations.

Let $J = [a, b] \subset R$ and $f : J \times E^n \to E^n$ be continuous. A mapping $\phi : J \to E^n$ is a solution of the initial value problem

$$y' = f(t, y), \quad y(a) = k, \tag{7.2.1}$$

where k is a real constant, if and only if ϕ is a solution of the integral equation

$$y(t) = k + \int_a^t f(s, y(s))ds. \tag{7.2.2}$$

It is easily seen that $C(J, E^n)$, the set of all continuous mappings from J to E^n is complete with the distance

$$H(u, v) = \sup_{t \in J} \left\{ D(u(t), v(t)) e^{-\rho t} \right\},$$

where $u, v \in C(J, E^n)$ and $\rho(\geq 0) \in R$ is fixed.

Theorem 7.2.1. Let $f : J \times E^n \to E^n$ be a continuous map and suppose that there exists $M > 0$ such that

$$D(f(t, u_1), f(t, u_2)) \leq M D(u_1, u_2) e^{-\rho t}, \tag{7.2.3}$$

for all $t \in J, u_1, u_2 \in E^n$. Then the initial value problem (7.2.1) has a unique solution on the interval J.

Proof. For any $u \in C(J, E^n)$ define the operator $T : C(J, E^n) \to C(J, E^n)$ given by

$$[Tu](t) = k + \int_a^t f(s, u(s))ds, \quad \forall\ t \in J.$$

Consider

$$\begin{aligned}
H(Tu, Tv) &= \sup_{t\in J}\left\{D([Tu](t), [Tv](t))e^{-\rho t}\right\} \\
&= \sup_{t\in J}\left\{D(\int_a^t f(s, u(s))ds, \int_a^t f(s, v(s))ds)e^{-\rho t}\right\} \\
&\le \sup_{t\in J}\left\{\int_a^t D(f(s, u(s)), f(s, v(s)))ds\ \ e^{-\rho t}\right\} \\
&\le \sup_{t\in J}\left\{\int_a^t MD(u(s), v(s))dse^{-\rho s}\ \ e^{-\rho t}\right\} \\
&\le MH(u, v)\sup_{t\in J}\left\{(t-a)\ \ e^{-\rho t}\right\} \\
&\le MH(u, v)(b-a)e^{-\rho a}.
\end{aligned}$$

We can choose $\rho > 0$ such that $M(b-a)e^{-\rho a} < 1$ and T is a contraction mapping. By using contraction mapping theorem, T has a unique fixed point which is a unique solution of the initial value problem (7.2.1).

Example 7.2.1. Consider the initial value problem

$$y'(t) = qe^{-\rho t}y(t) + F(t), \quad y(1) = k, \quad t \in [1, 2] \tag{7.2.4}$$

where $F \in C([1, 2], E^n)$, and $q,\ k \in R$. Here

$$f(t, y) = qe^{-\rho t}y(t) + F(t).$$

Consider

$$D(f(t, y), f(t, z)) = D(qe^{-\rho t}y + F(t), qe^{-\rho t}z + F(t))$$

$$= |q|e^{-\rho t}D(y,z)$$

$$= MD(y,z)e^{-\rho t},$$

where $M = |q|$. Clearly, f satisfies (7.2.3). By taking $q = 1$, and from Theorem 7.2.1 the initial value problem (7.2.4) has a unique solution for all $\rho > 0$.

We denote by $C'(J, E^n)$ the set of all continuously differentiable mappings from J to E^n. For $u, v \in C'(J, E^n)$, we define the distance

$$H_1(u,v) = H(u,v) + H(u',v').$$

Lemma 7.2.1. (Lemma 1 of [38]) $(C'(J, E^n), H_1)$ is a complete metric space.

Section 7.3.

In this section we prove existence and uniqueness results for initial and boundary value problems associated with second order non-linear fuzzy differential equations. The results obtained in this section are illustrated with suitable examples.

Consider the non-linear fuzzy differential equation of second order

$$y'' + f(t, y, y') = 0 \tag{7.3.1}$$

satisfying

$$y(a) = m_1, \quad y'(a) = m_2, \tag{7.3.2}$$

where $f : J \times E^n \times E^n \to E^n$ is continuous, m_1 and m_2 are real constants.

If ϕ is a solution of (7.3.1) satisfying (7.3.2) if and only if ϕ is a solution of the integral equation

$$y(t) = m_1 + m_2(t-a) + \int_a^t (s-t)f(s, y(s), y'(s))ds. \tag{7.3.3}$$

Now we prove the existence and uniqueness theorem for initial value problem (7.3.1) satisfying (7.3.2) using the integral representation (7.3.3).

Theorem 7.3.1. Let $f : J \times E^n \times E^n \to E^n$ be a continuous map and suppose that there exist $M_1, M_2 > 0$ such that

$$D(f(t, u_1, u_2), f(t, v_1, v_2)) \leq [M_1 D(u_1, v_1) + M_2 D(u_2, v_2)]e^{-\rho t}, \qquad (7.3.4)$$

for all $t \in J, u_1, u_2, v_1, v_2 \in E^n$. Then the initial value problem (7.3.1) satisfying (7.3.2) has a unique solution on the interval J.

Proof. Consider the complete metric space $(C'(J, E^n), H_1)$. For any $u \in C'(J, E^n)$ define the operator $T : C'(J, E^n) \to C'(J, E^n)$ by

$$[Tu](t) = m_1 + m_2(t - a) + \int_a^t (s - t) f(s, u(s), u'(s))ds, \quad t \in J.$$

Using definitions of H_1, H, T, and (7.3.4) we have

$$H_1(Tu, Tv)$$

$$\begin{aligned}
&= H(Tu, Tv) + H([Tu]', [Tv]') \\
&= \sup_{t \in J} \left\{ D\left(\int_a^t (s - t) f(s, u(s), u'(s))ds, \int_a^t (s - t) f(s, v(s), v'(s))ds\right)e^{-\rho t} \right\} \\
&\qquad + \sup_{t \in J} \left\{ D\left(\int_a^t f(s, u(s), u'(s))ds, \int_a^t f(s, v(s), v'(s))ds\right)e^{-\rho t} \right\} \\
&\leq \sup_{t \in J} \left\{ \int_a^t |s - t| \; D(f(s, u(s), u'(s)), f(s, v(s), v'(s)))ds \; e^{-\rho t} \right\} \\
&\qquad + \sup_{t \in J} \left\{ \int_a^t D(f(s, u(s), u'(s)), f(s, v(s), v'(s)))ds \; e^{-\rho t} \right\} \\
&\leq \sup_{t \in J} \left\{ \int_a^t (t - s)[M_1 D(u(s), v(s)) + M_2 D(u'(s), v'(s))]e^{-\rho s} ds \; e^{-\rho t} \right\}
\end{aligned}$$

$$+\sup_{t\in J}\left\{\int_a^t [M_1D(u(s),v(s))+M_2D(u'(s),v'(s))]e^{-\rho s}ds\ e^{-\rho t}\right\}$$

$$\leq \sup_{t\in J}\left\{\int_a^t (t-s)[M_1H(u,v)+M_2H(u',v')]ds\ e^{-\rho t}\right\}$$

$$+\sup_{t\in J}\left\{\int_a^t [M_1H(u,v)+M_2H(u',v')]ds\ e^{-\rho t}\right\}$$

$$= [M_1H(u,v)+M_2H(u',v')]\left(\sup_{t\in J}\left\{\int_a^t (t-s)ds\, e^{-\rho t}\right\}+\sup_{t\in J}\left\{\int_a^t ds\, e^{-\rho t}\right\}\right)$$

$$\leq \max\{M_1,M_2\}H_1(u,v)\left(\sup_{t\in J}\left\{\frac{(t-a)^2}{2}\ e^{-\rho t}\right\}+\sup_{t\in J}\left\{(t-a)\ e^{-\rho t}\right\}\right)$$

$$= H_1(u,v)\max\{M_1,M_2\}\left(\frac{(b-a)^2}{2}e^{-\rho a}+(b-a)e^{-\rho a}\right).$$

We can choose $\rho > 0$ such that

$$\max\{M_1,M_2\}\left(\frac{(b-a)^2}{2}e^{-\rho a}+(b-a)e^{-\rho a}\right) < 1.$$

It follows that T is a contraction mapping in the complete metric space $C'((J,E^n),H_1)$. By contraction mapping theorem, T has a unique fixed point u, which is a unique solution of the initial value problem (7.3.1) satisfying (7.3.2).

Now consider the non-linear fuzzy differential equation of second order

$$y''+f(t,y,y')=0 \tag{7.3.5}$$

satisfying the boundary conditions

$$y(a)=k_1, \quad y(b)=k_2, \tag{7.3.6}$$

where $f : J\times E^n\times E^n \to E^n$ is continuous, k_1 and k_2 are real constants. We know that $C'((J,E^n),H_1)$ is a complete metric space. For any $\phi \in$

$C'((J, E^n), H_1)$ define the operator $T\phi \in C'(J, E^n)$ by

$$[T\phi](t) = \int_a^b G(t,s) f(s, \phi(s), \phi'(s)) ds, \quad \forall \ t \in J$$

where $G(t, s)$ is the Green's function for the homogeneous boundary value problem. Hence, $\phi \in C'(J, E^n)$ is a solution of (7.3.5) satisfying (7.3.6) if and only if ϕ is a fixed point of T.

Theorem 7.3.2. Let $f : J \times E^n \times E^n \to E^n$ be continuous and suppose that there exist $M_1, M_2 > 0$ such that (7.3.4) is satisfied. Then the two point fuzzy boundary value problem (7.3.5) satisfying (7.3.6) has a unique solution on the interval J.

Proof. Consider the boundary value problem

$$y'' = 0 \tag{7.3.7}$$

satisfying

$$y(a) = 0, \quad y(b) = 0. \tag{7.3.8}$$

This problem has no non-trivial solution. Therefore, if h is any continuous function on J, the equation $y''(t) + h(t) = 0$ has a unique solution satisfying the boundary condition (7.3.8), given by

$$y(t) = \int_a^b G(t,s) h(s) ds,$$

where

$$G(t,s) = \begin{cases} \frac{(b-t)(s-a)}{b-a}, & a \le s \le t \le b, \\ \frac{(b-s)(t-a)}{b-a}, & a \le t \le s \le b. \end{cases}$$

Consider

$$\sup_{t \in J} \int_a^b |G(t,s)| ds = \sup_{t \in J} \left\{ \int_a^t |G(t,s)| ds + \int_t^b |G(t,s)| ds \right\}$$

$$= \sup_{t\in J}\left\{\frac{b-t}{b-a}\int_a^t (s-a)ds + \frac{t-a}{b-a}\int_t^b (b-s)ds\right\}$$
$$= \sup_{t\in J}\left\{\frac{(b-t)(t-a)}{2}\right\}.$$

This function attains its maximum value at $t = \frac{a+b}{2}$, and hence

$$\sup_{t\in J}\int_a^b |G(t,s)|ds \le \frac{(b-a)^2}{8}. \tag{7.3.9}$$

Again consider

$$\sup_{t\in J}\int_a^b |G_t(t,s)|ds = \sup_{t\in J}\left\{\frac{(t-a)^2+(t-b)^2}{2(b-a)}\right\}.$$

The maximum of this function is attained at 'a' and 'b' which yields

$$\sup_{t\in J}\int_a^b |G_t(t,s)|ds \le \frac{b-a}{2}. \tag{7.3.10}$$

We know that $C'((J,E^n),H_1)$ is a complete metric space. For any $u \in C'(J,E^n)$ define the operator $T : C'(J,E^n) \to C'(J,E^n)$ by

$$[Tu](t) = \int_a^b G(t,s)f(s,u(s),u'(s))ds, \quad t\in J. \tag{7.3.11}$$

Using the bounds on G, G_t given by (7.3.9) and (7.3.10), definitions of H_1, H, T, and from (7.3.4), we have

$$H_1(Tu,Tv) = H(Tu,Tv)+H([Tu]',[Tv]')$$
$$= \sup_{t\in J}\left\{D(\int_a^b G(t,s)f(s,u(s),u'(s))ds, \int_a^b G(t,s)f(s,v(s),v'(s))ds)e^{-\rho t}\right\}$$
$$+\sup_{t\in J}\left\{D(\int_a^b G_t(t,s)f(s,u(s),u'(s))ds, \int_a^b G_t(t,s)f(s,v(s),v'(s))ds)e^{-\rho t}\right\}$$
$$\le \sup_{t\in J}\left\{\int_a^b |G(t,s)|D(f(s,u(s),u'(s)),f(s,v(s),v'(s)))ds\ e^{-\rho t}\right\}$$

$$+\sup_{t\in J}\left\{\int_a^b |G_t(t,s)|D(f(s,u(s),u'(s)),f(s,v(s),v'(s)))ds\ e^{-\rho t}\right\}$$

$$\leq \sup_{t\in J}\left\{\int_a^b |G(t,s)|[M_1D(u(s),v(s))+M_2D(u'(s),v'(s))]e^{-\rho s}ds\ e^{-\rho t}\right\}$$

$$+\sup_{t\in J}\left\{\int_a^b |G_t(t,s)|[M_1D(u(s),v(s))+M_2D(u'(s),v'(s))]e^{-\rho s}ds\ e^{-\rho t}\right\}$$

$$\leq \sup_{t\in J}\left\{\int_a^b |G(t,s)|[M_1H(u,v)+M_2H(u',v')]ds\ e^{-\rho t}\right\}$$

$$+\sup_{t\in J}\left\{\int_a^b |G_t(t,s)|[M_1H(u,v)+M_2H(u',v')]ds\ e^{-\rho t}\right\}$$

$$= [M_1H(u,v)+M_2H(u',v')]$$

$$\left(\sup_{t\in J}\left\{\int_a^b |G(t,s)|ds\, e^{-\rho t}\right\}+\sup_{t\in J}\left\{\int_a^b |G_t(t,s)|ds\, e^{-\rho t}\right\}\right)$$

$$\leq \max\{M_1,M_2\}H_1(u,v)\left(\sup_{t\in J}\left\{\frac{(b-a)^2}{8}\ e^{-\rho t}\right\}+\sup_{t\in J}\left\{\frac{(b-a)}{2}\ e^{-\rho t}\right\}\right)$$

$$= \max\{M_1,M_2\}H_1(u,v)\left(\frac{(b-a)^2}{8}\ e^{-\rho a}+\frac{(b-a)}{2}\ e^{-\rho a}\right).$$

We can choose $\rho > 0$ such that

$$\max\{M_1,M_2\}\left(\frac{(b-a)^2}{8}\ e^{-\rho a}+\frac{(b-a)}{2}\ e^{-\rho a}\right)<1,$$

and T is contraction mapping. By contraction mapping theorem, T has a unique fixed point, which is a unique solution of the boundary value problem (7.3.5) satisfying (7.3.8).

By applying the above procedure to the boundary value problem

$$y''+f(t,y(t)+p(t),y'(t)+p'(t))=0$$

$$y(a)=0,\quad y(b)=0,$$

where p is a polynomial of first degree such that $p(a) = k_1$, $p(b) = k_2$ a unique solution $y_1(t)$ is constructed. Let $y(t) = y_1(t) + p(t)$. Then it is easily seen that y is a solution of the boundary value problem (7.3.5) satisfying (7.3.6).

Theorem 7.3.3. Let $f : J \times E^n \times E^n \to E^n$ be continuous and suppose that there exist $M_1, M_2 > 0$ such that (7.3.4) is satisfied. Then the two point fuzzy boundary value problems of the second kind, (7.3.5) satisfying

$$y'(a) = k_1, \quad y(b) = k_2, \tag{7.3.12}$$

and (7.3.5) satisfying

$$y(a) = k_1, \quad y'(b) = k_2, \tag{7.3.13}$$

have unique solutions on the interval J.

Proof. First, we consider the boundary value problem

$$y'' = 0 \tag{7.3.14}$$

satisfying

$$y(a) = 0, \qquad y'(b) = 0. \tag{7.3.15}$$

This problem has no non-trivial solution. Therefore, if g is any continuous function on J, the equation

$$y''(t) + g(t) = 0 \tag{7.3.16}$$

satisfying the boundary condition (7.3.15) has a unique solution, given by

$$y(t) = \int_a^b K(t,s)g(s)ds,$$

where

$$K(t,s) = \begin{cases} s-a, & a \leq s \leq t \leq b, \\ t-a, & a \leq t \leq s \leq b. \end{cases}$$

Consider

$$\sup_{t\in J} \int_a^b |K(t,s)|ds = \sup_{t\in J} \left\{ \int_a^t (s-a)ds + (t-a) \int_t^b ds \right\}$$

$$= \sup_{t\in J} \left\{ \frac{(t-a)^2}{2} + (t-a)(b-t) \right\}.$$

This function attains its maximum value at $t = b$, and hence

$$\sup_{t\in J} \int_a^b |K(t,s)|ds \leq \frac{(b-a)^2}{2}. \tag{7.3.17}$$

Again consider

$$\sup_{t\in J} \int_a^b |K_t(t,s)|ds = \sup_{t\in J} \{b-t\} \leq b-a. \tag{7.3.18}$$

For any $v \in C'(J, E^n)$ define the operator $T : C'(J, E^n) \to C'(J, E^n)$ by

$$[Tv](t) = \int_a^t K(t,s) f(s, v(s), v'(s))ds, \quad t \in J. \tag{7.3.19}$$

Similarly using the bounds on K, K_t given by (7.3.17) and (7.3.18), definitions of H_1, H, T, (7.3.4), and following the procedure as in Theorem 7.3.2, we have

$$H_1(Tv, Tw)$$

$$\leq [M_1 H(v,w) + M_2 H(v', w')] \left(\sup_{t\in J} \left\{ \int_a^b |K(t,s)|ds \, e^{-\rho t} \right\} + \sup_{t\in J} \left\{ \int_a^b |K_t(t,s)|ds \, e^{-\rho t} \right\} \right)$$

$$\leq \max\{M_1, M_2\} H_1(v,w) \left(\sup_{t\in J} \left\{ \frac{(b-a)^2}{2} \, e^{-\rho t} \right\} + \sup_{t\in J} \left\{ (b-a) \, e^{-\rho t} \right\} \right)$$

$$= \max\{M_1, M_2\} H_1(v,w) \left(\frac{(b-a)^2}{2} \, e^{-\rho a} + (b-a) \, e^{-\rho a} \right).$$

We can choose $\rho > 0$ such that

$$\max\{M_1, M_2\}\left(\frac{(b-a)^2}{2} + (b-a)\right) \quad e^{-\rho a} < 1,$$

and T is a contraction mapping. By contraction mapping theorem, T has a unique fixed point, which is a unique solution of the boundary value problem (7.3.5) satisfying (7.3.15).

By applying the above procedure to the boundary value problem

$$y'' + f(t, y(t) + q(t), y'(t) + q'(t)) = 0,$$

$$y(a) = 0, \quad y'(b) = 0,$$

where q is a polynomial of first degree such that $q(a) = k_1$, $q'(b) = k_2$ a unique solution $\tilde{y}(t)$ is constructed. Let $y(t) = \tilde{y}(t) + q(t)$. Then it is easily seen that $y(t)$ is a solution of the boundary value problem (7.3.5) satisfying (7.3.13).

Similarly we can prove that the boundary value problem (7.3.5) satisfying (7.3.12) has a unique solution on J.

Example 7.3.1. Consider the two point boundary value problem

$$y''(t) = q_1 e^{-\rho t} y(t) + q_2 e^{-\rho t} y'(t) + \phi(t), \quad t \in [1, 2] \tag{7.3.20}$$

$$y(1) = k_1, \qquad y(2) = k_2, \tag{7.3.21}$$

where $\phi \in C([1,2], E^n)$, q_1, q_2, k_1, $k_2 \in R$ and $\rho(\geq 0) \in R$ is fixed. Here

$$f(t, y_1, y_2) = q_1 e^{-\rho t} y_1(t) + q_2 e^{-\rho t} y_2(t) + \phi(t) \tag{7.3.22}$$

Consider

$$D(f(t, y_1, y_2), f(t, z_1, z_2))$$

$$= D(q_1 e^{-\rho t} y_1(t) + q_2 e^{-\rho t} y_2(t) + \phi(t), q_1 e^{-\rho t} z_1(t) + q_2 e^{-\rho t} z_2(t) + \phi(t))$$

$$= D(q_1e^{-\rho t}y_1(t) + q_2e^{-\rho t}y_2(t), q_1e^{-\rho t}z_1(t) + q_2e^{-\rho t}z_2(t))$$

$$\leq |q_1|e^{-\rho t}D(y_1, z_1) + |q_2|e^{-\rho t}D(y_2, z_2)$$

$$= [M_1D(y_1, z_1) + M_1D(y_1, z_1)]e^{-\rho t},$$

where $M_1 = |q_1|$ and $M_2 = |q_2|$. Therefore f satisfies the modified Lipschitz condition.

In particular, if we take $q_1 = 2, q_2 = 4$, then the two point boundary value problem (7.3.20) satisfying (7.3.21) has a unique solution for all values of $\rho \geq 1$.

Example 7.3.2. Consider Example 7.3.1 with boundary conditions (7.3.21) replaced by

$$y(1) = k_1, \qquad y'(2) = k_2. \tag{7.3.23}$$

By taking $q_1 = 1$ and $q_2 = \frac{3}{2}$, and from Theorem 7.3.3 the boundary value problem (7.3.20) satisfying (7.3.23) has a unique solution for all values of $\rho \geq 1$.

Section 7.4.

In this section we develop existence and uniqueness criteria to certain class of three point boundary value problems associated with third order non-linear fuzzy differential equations, with the help of Green's functions and contraction mapping principle. First, we state some theorems which are useful for later discussion.

Theorem 7.4.1. [46] Let $F : J \to E^n$ be continuous. Then for all $t \in J$ the integral $G(t) = \int_a^t F(t)dt$ is differentiable and $G'(t) = F(t)$.

Theorem 7.4.2. [46] Let $F : J \to E^n$ be continuously differentiable on J. Then

$$D(F(b), F(a)) \leq (b - a) \sup\{D(F'(t), \tilde{0})/t \in I\},$$

where $\tilde{0}(x) = 1$, if $x = 0$ and $\tilde{0}(x) = 0$, if $x \neq 0$.

Now we recall a well known criterion for compactness in a space of continuous functions between metric spaces, i.e., Ascoli's theorem.

Theorem 7.4.3. Let X be a compact metric space and Y any metric space. A subset Φ of the space $C(X, Y)$ of continuous mappings of X into Y is totally bounded in the metric of uniform convergence if and only if Φ is equicontinuous on X and $\Phi(x) = \{\phi(x)/\phi \in \Phi\}$ is a totally bounded subset of Y for each $x \in X$.

Let $J_1 = [0, c]$ be a closed subinterval of R and assume that $f : J_1 \times E^n \times E^n \times E^n \to E^n$ is continuous. We consider the non-linear fuzzy differential equation of third order

$$y''' = f(t, y, y', y'') \tag{7.4.1}$$

satisfying three point boundary conditions

$$y'(0) = y_1, \quad y(b) = y_2, \quad y''(c) = y_3. \tag{7.4.2}$$

Denote by $C^2(J_1, E^n)$ the set of all twice continuously differentiable mappings from J_1 to E^n. We define for any $\phi, \psi \in C^2(J_1, E^n)$ by

$$\mathbf{H}(\phi, \psi) = K \max_{t \in J_1} D(\phi(t), \psi(t)) + L \max_{t \in J_1} D(\phi'(t), \psi'(t)) + M \max_{t \in J_1} D(\phi''(t), \psi''(t)).$$

Then $(C^2(J_1, E^n), \mathbf{H})$ is a complete metric space. For any $\phi \in C^2(J_1, E^n)$ define $T\phi \in C^2(J_1, E^n)$ by

$$[T\phi](t) = \int_0^c G(t, s) f(s, \phi(s), \phi'(s), \phi''(s)) ds, \ \forall \ t \in J_1$$

where $G(t, s)$ is the Green's function for the homogeneous boundary value problem. Hence $\phi \in C^2(J_1, E^n)$ is a solution of (7.4.1) satisfying (7.4.2) if and only if ϕ is a fixed point of T.

We know that, if $\phi \in C^2(J_1, E^n)$ is a solution of the initial value problem

$$y''' = f(t, y, y', y'') \qquad (7.4.3)$$

$$y(0) = y_0, \quad y'(0) = m_1, \quad y''(0) = m_2 \qquad (7.4.4)$$

if and only if ϕ is a solution of the integral equation

$$y(t) = y_0 + m_1 t + \frac{m_2 t^2}{2} + \int_0^t \frac{(t-s)^2}{2} f(s, y(s), y'(s), y''(s))ds. \qquad (7.4.5)$$

On the other hand, if we set $y'' = z$ then (7.4.3) and (7.4.4) becomes

$$z' = f(t, y, y', z) \qquad (7.4.6)$$

$$y(0) = y_0, \quad y'(0) = m_1, \quad z(0) = m_2. \qquad (7.4.7)$$

If ψ is a solution of (7.4.6) satisfying (7.4.7), then ψ is a solution of

$$z(t) = m_2 + \int_0^t f(s, y(s), y'(s), z(s))ds. \qquad (7.4.8)$$

For any $\psi \in C^2(J_1, E^n)$ define $F\psi$ as

$$[F\psi](t) = m_2 + \int_0^t f(s, y(s), y'(s), \psi(s))ds. \qquad (7.4.9)$$

Lemma 7.4.1. Suppose that there exists $N \geq 0$ such that

$$D(f(t, y, y', y''), \tilde{0}) \leq N, \ \forall \ t \in J_1, \quad y, y', y'' \in E^n. \qquad (7.4.10)$$

Then F is compact, i.e. F transforms bounded sets into relatively compact sets.

Proof. Let B be a bounded set in $C^2(J_1, E^n)$. The set $FB = \{F\psi/\psi \in B\}$ is totally bounded if and only if it is equi-continuous and for every $t \in J_1$ the set

$$[FB](t) = \{[F\psi](t)/t \in J_1\}$$

is bounded subset of E^n. For any t_0, $t_1 \in J_1$ with $t_0 \leq t_1$ and $\psi \in B$ we have

$$D([F\psi](t_0), [F\psi](t_1)) \leq |t_1 - t_0| \sup\{D(f(t, y(t), y'(t), \psi(t)), \tilde{0})/t \in J_1\}$$

$$= |t_1 - t_0| \sup\{D(f(t, y(t), y'(t), y''(t)), \tilde{0})/t \in J_1\}$$

$$\leq |t_1 - t_0| N.$$

Thus FB is equi-continuous. Now for any fixed t we have that

$$D([F\psi](t), [F\psi](t')) \leq |t - t'| \sup\{D(f(t, y(t), y'(t), y''(t)), \tilde{0})/t \in J_1\}$$

$$\leq |t - t'| N,$$

for every $t' \in J_1$, $y \in B$.

Hence we see that $\{[F\psi](t)/t \in J_1, \psi \in B\}$ is bounded in E^n. By Ascoli's theorem we conclude that FB is a relatively compact subset of $C^2(J_1, E^n)$. This completes the proof of the lemma.

Theorem 7.4.4. Suppose that $f : J_1 \times E^n \times E^n \times E^n \to E^n$ is continuous and bounded, i.e. f satisfies (7.4.10). Then the initial value problem (7.4.3) satisfying (7.4.4) possesses at least one solution on the interval J_1.

Proof. Consider the ball $B = \{\psi \in C^2(J_1, E^n)/\mathbf{H}(\psi, \tilde{0}) \leq N_1\}$, where $N_1 = cN$ in the metric space $(C^2(J_1, E^n), \mathbf{H})$. If $x \in FB, FB = \{F\phi/\phi \in B\}$, then $x = F\phi$ for some $\phi \in B$. Consider

$$D([F\phi](t), [F\phi](0)) \leq |t - 0| \sup\{D(f(t, \phi(t), \phi'(t), \phi''(t), \tilde{0})/t \in J_1\}$$

$$\leq tN \leq cN = N_1.$$

By defining $\overline{0} : J_1 \to E^n$, $\overline{0}(t) = \tilde{0}$, $t \in J_1$, we have

$$\mathbf{H}(F\phi, F\tilde{0}) = \sup\{D([\,F\phi\,](t), [\,F\overline{0}\,](t))/t \in J\} \leq N_1.$$

Therefore $x = F\phi \in B$ and thus $FB \subset B$. By Lemma 7.4.1, F is compact and in consequence, it has a fixed point $\phi \in B$, which is a solution of (7.4.6) satisfying (7.4.7).

Theorem 7.4.5. Let $f \in C[J_1 \times E^n \times E^n \times E^n, E^n]$ and satisfy

$$D\left[f(t,u,u',u''), f(t,v,v',v'')\right] \leq KD(u,v)+LD(u',v')+MD(u'',v'') \quad (7.4.11)$$

and assume that

$$\beta = K\frac{b^2(3c-b)}{6} + L\frac{c^2}{2} + Mc < 1. \quad (7.4.12)$$

Then the three point fuzzy boundary value problem

$$y''' = f(t,y,y',y'') \quad (7.4.13)$$

$$y'(0) = y_1, \quad y(b) = y_2, \quad y''(c) = y_3. \quad (7.4.14)$$

has one and only one solution.

Proof. Clearly the homogeneous boundary value problem has only the trivial solution. Hence the three point boundary value problem (7.4.13) satisfying the boundary conditions

$$y'(0) = 0, \quad y(b) = 0, \quad y''(c) = 0, \quad (7.4.15)$$

has a unique solution given by

$$y(t) = \int_0^c G(t,s)f(s,y(s),y'(s),y''(s))ds,$$

where

$$\underset{s\in[0,b]}{G(t,s)} = \begin{cases} \frac{s(2b-s)-t^2}{2}, & \text{if } t < s, \\ s(b-t), & \text{if } s < t, \end{cases}$$

$$G(t,s)_{s\in[b,c]} = \begin{cases} \frac{b^2-t^2}{2}, & \text{if } t < s, \\ \frac{b^2+s^2}{2} - ts, & \text{if } s < t. \end{cases}$$

The signs of the Green's function and its partial derivatives are given below over different intervals.

Function	$t < s \le b$,	$t \le b < s$,	$b < t < s$,	$s < t \le b$,	$s \le b < t$,	$b < s < t$
$G(t,s)$	nn	nn	$-$	nn	np	$-$
$G_t(t,s)$	np	np	$-$	np	np	$-$
$G_{tt}(t,s)$	$-$	$-$	$-$	0	0	0

where nn : non-negative, np : non-positive.

It can be easily shown by elementary methods that

$$\max_{0\le t\le c} \int_0^c |G(t,s)|ds \le \frac{b^2(3c-b)}{6}, \tag{7.4.16}$$

$$\max_{0\le t\le c} \int_0^c |G_t(t,s)|ds \le \frac{c^2}{2}, \tag{7.4.17}$$

and

$$\max_{0\le t\le c} \int_0^c |G_{tt}(t,s)|ds \le c. \tag{7.4.18}$$

Now we define $T : C^2(J_1, E^n) \to C^2(J_1, E^n)$ by

$$[Tu](t) = \int_0^c G(t,s)f(s,u(s),u'(s),u''(s))ds, \tag{7.4.19}$$

for all $u \in C^2(J_1, E^n)$, $t \in J_1 = [0,c]$.

Using the bounds on G, G_t and G_{tt} given by (7.4.16), (7.4.17) and (7.4.18), from the definition of $\mathbf{H}$ and (7.4.19), we have

$$D([Tu](t),[Tv](t)) = D\left(\int_0^c G(t,s)f(s,u(s),u'(s),u''(s))ds,\right.$$

$$\left.\int_0^c G(t,s)f(s,v(s),v'(s),v''(s))ds,\right)$$

$$\leq \int_0^c |G(t,s)|D\,(f(s,u(s),u'(s),u''(s)),$$

$$f(s,v(s),v'(s),v''(s)))\,ds$$

$$\leq \int_0^c |G(t,s)|[KD(u(s),v(s)) + LD(u'(s),v'(s))$$

$$+MD(u''(s),v''(s))]ds$$

$$\leq \mathbf{H}(u,v)\int_0^c |G(t,s)|ds$$

$$\leq \frac{b^2(3c-b)}{6}\mathbf{H}(u,v).$$

Therefore

$$D([Tu](t),[Tv](t)) \leq \frac{b^2(3c-b)}{6}\mathbf{H}(u,v). \tag{7.4.20}$$

Similarly

$$D([Tu]'(t),[Tv]'(t)) \leq \int_0^c |G_t(t,s)|[KD(u(s),v(s)) + LD(u'(s),v'(s))$$

$$+MD(u''(s),v''(s))]ds \tag{7.4.21}$$

$$\leq \mathbf{H}(u,v)\int_0^c |G_t(t,s)|ds \leq \frac{c^2}{2}\mathbf{H}(u,v).$$

and

$$D([Tu]''(t),[Tv]''(t)) \leq \int_0^c |G_{tt}(t,s)|[KD(u(s),v(s)) + LD(u'(s),v'(s))$$

$$+MD(u''(s),v''(s))]ds \tag{7.4.22}$$

$$\leq \mathbf{H}(u,v) \int_0^c |G_{tt}(t,s)| ds \leq c\mathbf{H}(u,v).$$

Thus from the inequalities (7.4.20)-(7.4.22), we have

$$\mathbf{H}(Tu,Tv) \leq [K\frac{b^2(3c-b)}{6} + L\frac{c^2}{2} + Mc]\mathbf{H}(u,v)$$

$$\leq \beta\mathbf{H}(u,v).$$

Since $\beta = K\frac{b^2(3c-b)}{6} + L\frac{c^2}{2} + Mc < 1$, it follows that T is a contraction mapping in the complete metric space $C^2((J_1, E^n), \mathbf{H})$. By contraction mapping theorem T has a unique fixed point u, which is a unique solution of the boundary value problem (7.4.13) satisfying (7.4.15).

By applying the above procedure to the boundary value problem

$$y''' = f(t, y(t) + p(t), y'(t) + p'(t), y''(t) + p''(t))$$

$$y'(0) = 0, \quad y(b) = 0, \quad y''(c) = 0,$$

where p is a polynomial of second degree such that $p'(a) = y_1$, $p(b) = y_2$, $p''(c) = y_3$, a unique solution $y_1(t)$ is constructed. Let $y(t) = y_1(t) + p(t)$, then it is easily seen that y is a solution of the boundary value problem (7.4.13), (7.4.14).