CHAPTER-6

# ON OBSERVABILITY OF FUZZY DYNAMICAL MATRIX LYAPUNOV SYSTEMS

## Section 6.1.

Controllability and observability are the two basic concepts that arise in the control of dynamical systems. In this chapter we continue our study of first order fuzzy dynamical matrix Lyapunov systems modeled by ;

$$X'(t) = A(t)X(t)+X(t)B(t)+F(t)U(t), \quad X(0) = X_0, \quad t > 0, \tag{6.1.1}$$

$$Y(t) = C(t)X(t) + D(t)U(t), \tag{6.1.2}$$

where $U(t)$ is a $n \times n$ fuzzy input matrix called fuzzy control and $Y(t)$ is a $n \times n$ fuzzy output matrix. Here $A(t), B(t), F(t), C(t)$, and $D(t)$ are matrices of order $n \times n$, whose elements are continuous functions of t on $\mathbf{J} = [0, T] \subset R$ $(T > 0)$, and discuss the concept of observability. Recently, the observability criteria for fuzzy dynamical control systems were studied by Ding and Kandel [31] and Ding, Maa and Kandel [32].

First, the input matrix $U(t)$ is taken as a product of $n$ fuzzy sets defined on $\mathbf{R}^n$. This enables us to incorporate a fuzzy rule base into the deterministic control system and to determine the new fuzzy rule base as we desire. In particular, we can use this method to study observability of the initial value. Given the input-and-output rule base, we can determine a fuzzy rule base for the initial value, and at the same time we give a formula to determine this rule base. Secondly, the input $U(t)$ chosen as a fuzzy matrix defined on $\mathbf{R}^{n\times n}$. Thus based on this input we investigate the concept of observability and introduce the notion of likely observability. Since the input is fuzzy, the state of the system is no more deterministic. For a given range of values for the initial state, we can determine which level of inputs and outputs observe the range of the initial state.

In section 6.2 we state some results relating to uniqueness of solutions to the corresponding Kronecker product system associated with (6.1.1), when $U(t)$ is a crisp continuous matrix. Further, we generate a deterministic control system with fuzzy inputs and outputs. Finally, we recall some definitions and results relating to fuzzy sets.

In section 6.3 we obtain sufficient conditions for the observability of fuzzy dynamical Lyapunov system by the use of fuzzy rule base, and highlight the main theorem with a suitable example.

In section 6.4 we introduce the notion of likely observability and present sufficient conditions of fuzzy dynamical system to be likely observable, and illustrate the main result with a suitable example.

This chapter extends some of the results of Ding and Kandel [31] and Ding, Maa and Kandel [32] to fuzzy dynamical matrix Lyapunov systems.

**Section 6.2.**

In this section we obtain a unique solution of the Kronecker product system corresponding to (6.1.1), when $U(t)$ is a crisp continuous matrix, and also generate a deterministic control system with fuzzy inputs and outputs, called a fuzzy dynamical Lyapunov system.

By applying the Vec operator to the matrix Lyapunov system (6.1.1) satisfying (6.1.2) and using the properties of Kronecker product, we have

$$\hat{X}'(t) = G(t)\hat{X}(t) + (I_n \otimes F(t))\hat{U}(t), \quad \hat{X}(0) = \hat{X}_0, \tag{6.2.1}$$

$$\hat{Y}(t) = (I_n \otimes C(t))\hat{X}(t) + (I_n \otimes D(t))\hat{U}(t), \tag{6.2.2}$$

where $G(t) = (B^* \otimes I_n) + (I_n \otimes A)$ is a $n^2 \times n^2$ matrix and $\hat{X}=$ Vec $X(t)$, $\hat{U}=$ Vec $U(t)$, $\hat{Y}=$ Vec $Y(t)$ are column matrices of order $n^2$.

The corresponding linear homogeneous system of (6.2.1) is

$$\hat{X}'(t) = G(t)\hat{X}(t), \quad \hat{X}(0) = \hat{X}_0. \tag{6.2.3}$$

**Lemma 6.2.1.** Let $\phi(t)$ and $\psi(t)$ be the fundamental matrices for the systems

$$X'(t) = A(t)X(t), \quad X(0) = I_n, \tag{6.2.4}$$

and

$$[X^*(t)]' = B^*(t)X^*(t), \quad X(0) = I_n \tag{6.2.5}$$

respectively. Then the matrix $\psi(t) \otimes \phi(t)$ is a fundamental matrix of (6.2.3) and the solution of (6.2.3) is $\hat{X}(t) = (\psi(t) \otimes \phi(t))\hat{X}_0$.

**Theorem 6.2.1.** Let $\phi(t)$ and $\psi(t)$ be the fundamental matrices for the systems (6.2.4) and (6.2.5), then the unique solution of (6.2.1) is

$$\hat{X}(t) = (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))\hat{U}(s)ds. \tag{6.2.6}$$

From the discussion of Section 5.3 of Chapter 5, we have the following result.

**Result 6.2.1.** The system (6.2.1), (6.2.2) determines a fuzzy dynamical Lyapunov system in the following cases namely either (i) the input $\hat{U}(t)$ is take as the product of $n^2$ fuzzy sets defined on $\mathbf{R}^1$ or (ii) the input $\hat{U}(t)$ is a fuzzy set defined on $\mathbf{R}^{n^2}$. The fuzzy dynamical Lyapunov system can be written as

$$\hat{X}'(t) = G(t)\hat{X}(t) + (I_n \otimes F(t))\hat{U}(t), \quad \hat{X}(0) = \{\hat{X}_0\}, \tag{6.2.7}$$

$$\hat{Y}(t) = (I_n \otimes C(t))\hat{X}(t) + (I_n \otimes D(t))\hat{U}(t). \tag{6.2.8}$$

The solution set of the system is given by

$$\hat{X}(t) \in (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))\hat{U}(s)ds. \tag{6.2.9}$$

**Remark 6.2.1.** Consider a special case. If the input is in the form

$$\hat{U}(t) = \widetilde{u}_1(t) \times \widetilde{u}_2(t) \times \ldots \times u_i(t) \times \ldots \times \widetilde{u}_{n^2}(t),$$

where $\widetilde{u}_k(t) \in \mathbf{R}^1$, $k \neq i$ are crisp numbers, then the $i^{th}$ component of the solution set of (6.2.1) is a fuzzy set in $E^1$.

Now we recall some of the definitions and results from the previous chapter which are useful for studying observability properties of fuzzy dynamical Lyapunov systems.

**Definition 6.2.1.** Let $u, v \in E^1$, $k \in \mathbf{R}^1$ and $[u]^\alpha$ be the $\alpha$-level set of $u$. We define the sum of $u$ and $v$ by

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha = \{a+b : a \in [u]^\alpha, b \in [v]^\alpha\}, \tag{6.2.10}$$

the difference between $u$ and $v$ by

$$[u-v]^\alpha = [u]^\alpha - [v]^\alpha = \{a-b : a \in [u]^\alpha, b \in [v]^\alpha\}, \tag{6.2.11}$$

and the scalar product by

$$[ku]^\alpha = k[u]^\alpha = \{ka : a \in [u]^\alpha\}. \tag{6.2.12}$$

**Definition 6.2.2.** Let $x, y \in E^{n^2}$ and $x = x_1 \times x_2 \times \ldots \times x_{n^2}$, $y = y_1 \times y_2 \times \ldots \times y_{n^2}$, $x_i, y_i \in E^1$, $i = 1, 2, \ldots n^2$. If $y = z + x$, then $z = y - x$ which is defined by

$$[z]^\alpha = [y-x]^\alpha = [y]^\alpha - [x]^\alpha = \begin{bmatrix} [y_1]^\alpha - [x_1]^\alpha \\ \cdots \\ [y_{n^2}]^\alpha - [x_{n^2}]^\alpha \end{bmatrix}. \qquad (6.2.13)$$

If $y = w - x$, then $w = y + x$ which is defined by

$$[w]^\alpha = [y+x]^\alpha = [y]^\alpha + [x]^\alpha = \begin{bmatrix} [y_1]^\alpha + [x_1]^\alpha \\ \cdots \\ [y_{n^2}]^\alpha + [x_{n^2}]^\alpha \end{bmatrix}. \qquad (6.2.14)$$

**Definition 6.2.3.** Let

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n^2} \\ c_{21} & c_{22} & \cdots & c_{2n^2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n^21} & c_{n^22} & \cdots & c_{n^2n^2} \end{bmatrix}$$

be a $n^2 \times n^2$ matrix, $p = p_1 \times p_2 \times \ldots \times p_{n^2}$, $p_i \in E^1$, $i = 1, 2, \ldots n^2$, be a fuzzy set in $E^{n^2}$. $[p_i]^\alpha$ are $\alpha$-level sets of $p_i$. Define the product $Cp$ of $C$ and $p$ as

$$[Cp]^\alpha = C[p]^\alpha = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n^2} \\ c_{21} & c_{22} & \cdots & c_{2n^2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n^21} & c_{n^22} & \cdots & c_{n^2n^2} \end{bmatrix} \begin{bmatrix} [p_1]^\alpha \\ [p_2]^\alpha \\ \cdots \\ [p_{n^2}]^\alpha \end{bmatrix}$$

$$= \begin{bmatrix} c_{11}[p_1]^\alpha + \ldots + c_{1n^2}[p_{n^2}]^\alpha \\ c_{21}[p_1]^\alpha + \ldots + c_{2n^2}[p_{n^2}]^\alpha \\ \cdots \\ c_{n^21}[p_1]^\alpha + \ldots + c_{n^2n^2}[p_{n^2}]^\alpha \end{bmatrix}.$$

**Lemma 6.2.2.** *Cp* is a fuzzy set in $E^{n^2}$.

**Section 6.3.**

In this section we obtain sufficient conditions for observability of the fuzzy system (6.2.7), (6.2.8) by using fuzzy rule base.

Let $u_i(t) \in E^1$, $t \in \mathbf{J}$, $i = 1, 2, \ldots, n^2$ and define

$$\hat{U}(t) = (u_1(t), u_2(t), \ldots, u_{n^2}(t)) = u_1(t) \times u_2(t) \times \ldots \times u_{n^2}(t)$$

$$= \{(u_1^\alpha(t), u_2^\alpha(t), \ldots, u_{n^2}^\alpha(t)) : \alpha \in [0,1]\}$$

$$= \{(\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_{n^2}(t)) : \widetilde{u}_i(t) \in u_i^\alpha(t), \alpha \in [0,1]\},$$

where $u_i^\alpha(t)$ is the $\alpha$-level set of $u_i(t)$. From the above definition of $\hat{U}(t)$ and Theorem 5.2.1 of Chapter 5, it can be easily seen that $\hat{U}(t) \in E^{n^2}$.

**Definition 6.3.1.** The fuzzy system (6.2.7), (6.2.8) is said to be completely observable over the interval $\mathbf{J} = [0, T]$, if the knowledge of rule base of input $\hat{U}$ and output $\hat{Y}$ over $\mathbf{J}$ suffices to determine a rule base of initial state $\hat{X}_0$.

Let $u_i^\ell$, $y_i^\ell$, $i = 1, 2, \ldots, n^2$, $\ell = 1, 2, \ldots, m$ be fuzzy sets in $E^1$. We assume that the rule base for the input and output are

$$\begin{aligned} R^\ell &: \text{IF } \widetilde{u}_1(t) \text{ is in } u_1^\ell(t) \text{ and } \ldots \text{ and } \widetilde{u}_{n^2}(t) \text{ is in } u_{n^2}^\ell(t), \\ &\quad \text{THEN } \widetilde{y}_1(t) \text{ is in } y_1^\ell(t) \text{ and } \ldots \text{ and } \widetilde{y}_{n^2}(t) \text{ is in } y_{n^2}^\ell(t), \\ &\qquad \ell = 1, 2, \ldots, m, \end{aligned} \tag{6.3.1}$$

and the relation between input and output is

$$\hat{Y}(t) = (I_n \otimes C(t))\hat{X}(t) + (I_n \otimes D(t))\hat{U}(t). \tag{6.3.2}$$

**Theorem 6.3.1.** Assume the fuzzy rule base (6.3.1) holds, then the system (6.2.7), (6.2.8) is completely observable over the interval $\mathbf{J}$, if $(I_n \otimes C(T))$

$(\psi(T) \otimes \phi(T))$ is non-singular. Furthermore, if $\hat{X}_0 = \left(\tilde{x}_0^1, \tilde{x}_0^2, \ldots, \tilde{x}_0^{n^2}\right)$, then we have the following rule base for the initial value $\hat{X}_0$ ;

$$\begin{aligned} R^\ell : &\text{IF } \tilde{u}_1(T) \text{ is in } u_1^\ell(T) \text{ and } \ldots \text{ and } \tilde{u}_{n^2}(T) \text{ is in } u_{n^2}^\ell(T), \\ &\text{IF } \tilde{y}_1(T) \text{ is in } y_1^\ell(T) \text{ and } \ldots \text{ and } \tilde{y}_{n^2}(T) \text{ is in } y_{n^2}^\ell(T), \\ &\text{THEN } \tilde{x}_0^1 \text{ is in } x_0^\ell(1) \text{ and } \ldots \text{ and } \tilde{x}_0^{n^2} \text{ is in } x_0^\ell(n^2), \\ &\qquad \ell = 1, 2, \ldots, m, \end{aligned} \tag{6.3.3}$$

where

$$x_0^\ell(i) = [(I_n \otimes C(T))(\psi(T) \otimes \phi(T))]^{-1} \{V_i^\ell(T) - (I_n \otimes D(T))\hat{U}(T)$$

$$-(I_n \otimes C(T)) \int_0^T (\psi(T-s) \otimes \phi(T-s))(I_n \otimes F(s))H_i^\ell(s)ds\}, \tag{6.3.4}$$

$$\hat{X}_0 = [(I_n \otimes C(T))(\psi(T) \otimes \phi(T))]^{-1} \{\tilde{y}(T) - (I_n \otimes D(T))\tilde{u}(T)$$

$$-(I_n \otimes C(T)) \int_0^T (\psi(T-s) \otimes \phi(T-s))(I_n \otimes F(s))\tilde{u}(s)ds\}, \tag{6.3.5}$$

and

$$\begin{aligned} H_i^\ell(t) &= \tilde{u}_1(t) \times \ldots \times u_i^\ell(t) \times \ldots \times \tilde{u}_{n^2}(t), \\ V_i^\ell(t) &= \tilde{y}_1(t) \times \ldots \times y_i^\ell(t) \times \ldots \times \tilde{y}_{n^2}(t), \end{aligned} \tag{6.3.6}$$

$i = 1, 2, \ldots, n^2$, $\ell = 1, 2, \ldots, m$.

**Proof.** Without loss of generality, we prove this theorem by considering $\ell = 1$. Let

$$\tilde{u}(t) = (\tilde{u}_1(t), \tilde{u}_2(t), \ldots, \tilde{u}_{n^2}(t)), \quad \tilde{y}(t) = (\tilde{y}_1(t), \tilde{y}_2(t), \ldots, \tilde{y}_{n^2}(t)),$$

$\mu_{u_i^1(t)}(\tilde{u}_i(t))$ be the grade of the membership of $\tilde{u}_i(t)$ in $u_i^1(t)$, $\mu_{y_i^1(t)}(\tilde{y}_i(t))$ be the grade of the membership of $\tilde{y}_i(t)$ in $y_i^1(t)$. Since $(I_n \otimes C(T))(\psi(T) \otimes \phi(T))$ is non-singular and from (6.2.6), we have

$$\hat{X}_0 = [(I_n \otimes C(T))(\psi(T) \otimes \phi(T))]^{-1} \{\tilde{y}(T) - (I_n \otimes D(T))\tilde{u}(T)$$

$$-(I_n \otimes C(T)) \int_0^T (\psi(T-s) \otimes \phi(T-s))(I_n \otimes F(s))\tilde{u}(s)ds\}. \qquad (6.3.7)$$

From the relation (6.3.2), when the input and output are both fuzzy sets it follows from the Definition 6.2.2,

$$(I_n \otimes C(t))\hat{X}(t) = \hat{Y}(t) - (I_n \otimes D(t))\hat{U}(t) \qquad (6.3.8)$$

is a fuzzy set. Substituting the solution of (6.2.7) in (6.3.8), we have

$$(I_n \otimes C(t)) \left( (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))\hat{U}(s)ds \right)$$

$$= \hat{Y}(t) - (I_n \otimes D(t))\hat{U}(t).$$

Using Definitions 6.2.2 and 6.2.3,

$$(I_n \otimes C(t))(\psi(t) \otimes \phi(t))\hat{X}_0 = \hat{Y}(t) - (I_n \otimes D(t))\hat{U}(t)$$

$$-(I_n \otimes C(t)) \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))\hat{U}(s)ds.$$

Since $(I_n \otimes C(T))(\psi(T) \otimes \phi(T))$ is non-singular, we have

$$\hat{X}_0 = [(I_n \otimes C(T))(\psi(T) \otimes \phi(T))]^{-1} \{\hat{Y}(T) - (I_n \otimes D(T))\hat{U}(T)$$

$$-(I_n \otimes C(T)) \int_0^T (\psi(T-s) \otimes \phi(T-s))(I_n \otimes F(s))\hat{U}(s)ds\}.$$

Now, the initial value $\hat{X}_0$ is no more a crisp value, but should be a fuzzy set. In order to determine each component of $\hat{X}_0$, let us assume

$$H_i^1(t) = \tilde{u}_1(t) \times \ldots \times u_i^1(t) \times \ldots \times \tilde{u}_{n^2}(t),$$

$$V_i^1(t) = \tilde{y}_1(t) \times \ldots \times y_i^1(t) \times \ldots \times \tilde{y}_{n^2}(t), \ \ i = 1, 2, \ldots, n^2.$$

From Remark 6.2.1, we know that the $i^{th}$ component of the set

$$(\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))H_i^1(s)ds$$

is a fuzzy set in $E^1$. From Lemma 6.2.2, we know that the product

$$(I_n \otimes C(t)) \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))H_i^1(s)ds$$

is a fuzzy set in $E^{n^2}$. Hence $\hat{X}_0$ is a fuzzy set in $E^{n^2}$, and the $i^{th}$ component of it denoted as $x_0^1(i)$ is a fuzzy set in $E^1$. The grade of membership of $\widetilde{x}_0^i$ in $x_0^1(i)$ is defined by

$$\mu_{x_0^1(i)}\left(\widetilde{x}_0^i\right) = \min\left\{\mu_{u_i^1(t)}\left(\widetilde{u}_i(t)\right), \mu_{y_i^1(t)}\left(\widetilde{y}_i(t)\right)\right\}. \tag{6.3.9}$$

Now, we are in a position to determine the rule base for the initial value and is given by (6.3.3), (6.3.4), (6.3.5) and (6.3.6).

In general it is difficult to compute $x_0^\ell(i)$, but to solve the real problems, we choose the following method of approximation. Now we take the point $\left(\widetilde{x}_0^i, \mu_{x_0^\ell(i)}\left(\widetilde{x}_0^i\right)\right)$ and the zero-level set $[x_0^\ell(i)]^0$ to determine a triangle as the new fuzzy set $x_0^\ell(i)$.

We can use the center average defuzzifier

$$\widetilde{x}_0^i = \frac{\sum_{\ell=1}^{m} \left(\widetilde{x}_0^i\right)^\ell \mu_{x_0^\ell(i)}\left(\left(\widetilde{x}_0^i\right)^\ell\right)}{\sum_{\ell=1}^{m} \mu_{x_0^\ell(i)}\left(\left(\widetilde{x}_0^i\right)^\ell\right)}$$

to determine the initial value $\hat{X}_0 = \left(\widetilde{x}_0^1, \widetilde{x}_0^2, \ldots, \widetilde{x}_0^{n^2}\right)$. To obtain more accurate value for the initial state, more rule bases may be provided.

The advantage of this method is that given a deterministic system with fuzzy input and output rule base we can determine the rule base for initial value with out solving the equation.

**Example 6.3.1.** Consider the fuzzy matrix Lyapunov system

$$X'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X(t) + X(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} U(t), \ 0 \le t \le \frac{\pi}{2}$$

$$Y(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X(t).$$

The $\alpha$-level sets of fuzzy input $\hat{U}(t)$ and fuzzy output $\hat{Y}(t)$ by rule base 1 and rule base 2 are given below ;

$$\text{Rule 1. } [\hat{U}^{(1)}]^{\alpha} = \begin{bmatrix} [0, -0.75(\alpha - 1)] \\ [0.75(\alpha - 1) + 1, 1] \\ [0, -0.5(\alpha - 1)] \\ [0.5(\alpha - 1) + 1, 1] \end{bmatrix}, \quad [\hat{Y}^{(1)}]^{\alpha} = \begin{bmatrix} [0, -2(\alpha + 1)] \\ [0.5\alpha + 2.5, 3] \\ [0, -1.5(\alpha - 1)] \\ [0.5(\alpha - 1) + 3, 3] \end{bmatrix}.$$

$$\text{Rule 2. } [\hat{U}^{(2)}]^{\alpha} = \begin{bmatrix} [0, -0.8(\alpha - 1)] \\ [0.8\alpha + 0.2, 1] \\ [0, -0.5(\alpha - 1)] \\ [0.5\alpha + 0.5, 1] \end{bmatrix}, \quad [\hat{Y}^{(2)}]^{\alpha} = \begin{bmatrix} [0, -1.5(\alpha - 1)] \\ [\alpha + 1, 2] \\ [0, -2.5(\alpha - 1)] \\ [2\alpha + 1, 3] \end{bmatrix}.$$

From rule base 1, we select

$$\tilde{u}^1 = [\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4]^* = [0.5, 0.85, 0.4, 0.75]^*$$

the grades of the membership of $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$, and $\tilde{u}_4$ are $\frac{1}{3}$, 0.8, 0.2, and $\frac{1}{2}$ respectively. Also

$$\tilde{y}^1 = [\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4]^* = [1, 2.8, 0.5, 2.9]^*$$

the grades of the membership of $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$, and $\tilde{y}_4$ are $\frac{1}{2}$, 0.6, $\frac{2}{3}$, and 0.8 respectively.

From rule base 2, we select

$$\tilde{u}^2 = [\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4]^* = [0.5, 0.8, 0.25, 0.75]^*$$

the grades of the membership of $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$, and $\tilde{u}_4$ are $\frac{3}{8}$, $\frac{3}{4}$, $\frac{1}{2}$, and $\frac{1}{2}$ respectively.

Also

$$\tilde{y}^2 = [\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4]^* = [1, 1.75, 2, 1.5]^*,$$

the grades of the membership of $\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3$, and $\widetilde{y}_4$ are $\frac{1}{3}$, $\frac{3}{4}$, 0.2, and 0.25 respectively.

For rule base 1, by formula (6.3.4), (6.3.5), we have

$$\hat{X}_0 = e^{-\frac{\pi}{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \left( \begin{bmatrix} 1 \\ 2.8 \\ 0.5 \\ 2.9 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \int_0^{\frac{\pi}{2}} e^{\frac{\pi}{2}-s} \right.$$

$$\cdot \begin{bmatrix} \cos(\frac{\pi}{2}-s) & -\sin(\frac{\pi}{2}-s) & 0 & 0 \\ \sin(\frac{\pi}{2}-s) & \cos(\frac{\pi}{2}-s) & 0 & 0 \\ 0 & 0 & \cos(\frac{\pi}{2}-s) & -\sin(\frac{\pi}{2}-s) \\ 0 & 0 & \sin(\frac{\pi}{2}-s) & \cos(\frac{\pi}{2}-s) \end{bmatrix}$$

$$\left. e^s \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.85 \\ 0.4 \\ 0.75 \end{bmatrix} ds \right) = \begin{bmatrix} -1.142 \\ -0.9324 \\ -1.046 \\ -0.9532 \end{bmatrix},$$

and

$$x_0^1(1) = e^{-\frac{\pi}{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \left( \begin{bmatrix} [0, -2(\alpha-1)] \\ 2.8 \\ 0.5 \\ 2.9 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \int_0^{\frac{\pi}{2}} e^{\frac{\pi}{2}-s} \right.$$

$$\begin{bmatrix} \cos(\frac{\pi}{2}-s) & -\sin(\frac{\pi}{2}-s) & 0 & 0 \\ \sin(\frac{\pi}{2}-s) & \cos(\frac{\pi}{2}-s) & 0 & 0 \\ 0 & 0 & \cos(\frac{\pi}{2}-s) & -\sin(\frac{\pi}{2}-s) \\ 0 & 0 & \sin(\frac{\pi}{2}-s) & \cos(\frac{\pi}{2}-s) \end{bmatrix} e^s \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} [0,-0.75(\alpha-1)] \\ 0.85 \\ 0.4 \\ 0.75 \end{bmatrix} ds \Bigg) = \begin{bmatrix} [-1.6+0.75\alpha, -0.434-0.416\alpha] \\ [-1.4324, -0.6824-0.75\alpha] \\ -1.046 \\ -0.9532 \end{bmatrix},$$

when $\alpha = 0$, we get the biggest interval [-1.6, -0.434] and $\widetilde{x}_0^1$=-1.142 is located in this interval. We choose its membership grade in $x_0^1(1)$ as

$$\mu_{x_0^1(1)}\left(\widetilde{x}_0^1\right) = \min\{\frac{1}{3}, \frac{1}{2}\} = \frac{1}{3} = 0.333.$$

$$x_0^1(2) = e^{-\frac{\pi}{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \left( \begin{bmatrix} 1 \\ [0.5\alpha+2.5, 3] \\ 0.5 \\ 2.9 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \int_0^{\frac{\pi}{2}} e^{\frac{\pi}{2}-s} \right.$$

$$\begin{bmatrix} \cos(\frac{\pi}{2}-s) & -\sin(\frac{\pi}{2}-s) & 0 & 0 \\ \sin(\frac{\pi}{2}-s) & \cos(\frac{\pi}{2}-s) & 0 & 0 \\ 0 & 0 & \cos(\frac{\pi}{2}-s) & -\sin(\frac{\pi}{2}-s) \\ 0 & 0 & \sin(\frac{\pi}{2}-s) & \cos(\frac{\pi}{2}-s) \end{bmatrix} e^s \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left. \begin{bmatrix} 0.5 \\ [0.75(\alpha-1)+1, 1] \\ 0.4 \\ 0.75 \end{bmatrix} ds \right) = \begin{bmatrix} [-1.292, -0.542-0.75\alpha] \\ [-1.124, -0.27-0.854\alpha] \\ -1.046 \\ -0.9532 \end{bmatrix},$$

when $\alpha = 0$, we get the biggest interval [-1.124, -0.27] and $\widetilde{x}_0^2$=-0.9324 is located in this interval. We choose its membership grade in $x_0^1(2)$ as

$$\mu_{x_0^1(2)}\left(\widetilde{x}_0^2\right) = \min\{0.8, 0.6\} = 0.6.$$

$$x_0^1(3) = e^{-\frac{\pi}{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \left( \begin{bmatrix} 1 \\ 2.8 \\ [0, -1.5(\alpha-1)] \\ 2.9 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \int_0^{\frac{\pi}{2}} e^{\frac{\pi}{2}-s} \right.$$

$$\begin{bmatrix} \cos(\frac{\pi}{2}-s) & -\sin(\frac{\pi}{2}-s) & 0 & 0 \\ \sin(\frac{\pi}{2}-s) & \cos(\frac{\pi}{2}-s) & 0 & 0 \\ 0 & 0 & \cos(\frac{\pi}{2}-s) & -\sin(\frac{\pi}{2}-s) \\ 0 & 0 & \sin(\frac{\pi}{2}-s) & \cos(\frac{\pi}{2}-s) \end{bmatrix} e^s \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left. \begin{bmatrix} 0.5 \\ 0.85 \\ [0, -0.5(\alpha-1)] \\ 0.75 \end{bmatrix} ds \right) = \begin{bmatrix} -1.142 \\ -0.9324 \\ [-1.25+0.5\alpha, -0.438-0.312\alpha] \\ [-1.3532, -0.8532-0.5\alpha] \end{bmatrix},$$

when $\alpha = 0$, we get the biggest interval [-1.25, -0.438] and $\widetilde{x}_0^3$=-1.046 is located in this interval. We choose its membership grade in $x_0^1(3)$ as

$$\mu_{x_0^1(3)}\left(\widetilde{x}_0^3\right) = \min\{0.2, \frac{2}{3}\} = 0.2.$$

$$x_0^1(4) = e^{-\frac{\pi}{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \left( \begin{bmatrix} 1 \\ 2.8 \\ 0.5 \\ [0.5(\alpha-1)+3, 3] \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \int_0^{\frac{\pi}{2}} e^{\frac{\pi}{2}-s} \right.$$

$$\begin{bmatrix} \cos(\frac{\pi}{2}-s) & -\sin(\frac{\pi}{2}-s) & 0 & 0 \\ \sin(\frac{\pi}{2}-s) & \cos(\frac{\pi}{2}-s) & 0 & 0 \\ 0 & 0 & \cos(\frac{\pi}{2}-s) & -\sin(\frac{\pi}{2}-s) \\ 0 & 0 & \sin(\frac{\pi}{2}-s) & \cos(\frac{\pi}{2}-s) \end{bmatrix} e^s \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 \\ 0.85 \\ 0.4 \\ [0.5(\alpha-1)+1,1] \end{bmatrix} ds \Bigg) = \begin{bmatrix} -1.142 \\ -0.9324 \\ [-1.296, -0.796-0.5\alpha] \\ [-1.224, -0.62-0.604\alpha] \end{bmatrix},$$

when $\alpha = 0$, we get the biggest interval [-1.224, -0.62] and $\tilde{x}_0^4$=-0.9532 is located in this interval. We choose its membership grade in $x_0^1(4)$ as

$$\mu_{x_0^1(4)}\left(\tilde{x}_0^4\right) = \min\{\frac{1}{2}, 0.8\} = \frac{1}{2} = 0.5.$$

Similarly by using rule base 2, we obtain the values of $\hat{X}_0$, $x_0^2(i)$, $i = 1, 2, 3, 4$ and are given below ;

$$\hat{X}_0 = \begin{bmatrix} -1.092 \\ -0.664 \\ -0.584 \\ -0.812 \end{bmatrix}, \quad x_0^2(1) = \begin{bmatrix} [-1.6+0.8\alpha, -0.488-0.312\alpha] \\ [-1.164, -0.364-0.8\alpha] \\ -0.584 \\ -0.812 \end{bmatrix},$$

$$x_0^2(2) = \begin{bmatrix} [-1.292, -0.492-0.8\alpha] \\ [-0.916, 0.092-1.008\alpha] \\ -0.584 \\ -0.812 \end{bmatrix}, x_0^2(3) = \begin{bmatrix} -1.092 \\ -0.664 \\ [0.5\alpha-1.25, -0.23-0.52\alpha] \\ [-1.062, -0.562-0.5\alpha] \end{bmatrix},$$

and

$$x_0^2(4) = \begin{bmatrix} -1.092 \\ -0.664 \\ [-0.834, -0.334-0.5\alpha] \\ [-1.374, -0.458-0.916\alpha] \end{bmatrix}.$$

Also the grades of the membership of $\tilde{x}_0^1 = -1.092, \tilde{x}_0^2 = -0.664, \tilde{x}_0^3 = -0.584$, $\tilde{x}_0^4 = -0.812$ in $x_0^2(1)$, $x_0^2(2)$, $x_0^2(3)$, $x_0^2(4)$ are 0.333, 0.75, 0.2, 0.25 respectively.

We can use the center average defuzzifier to determine $\hat{X}_0 = (\tilde{x}_0^1, \tilde{x}_0^2, \tilde{x}_0^3, \tilde{x}_0^4)$, where

$$\tilde{x}_0^1 = \frac{\sum_{\ell=1}^{2} \tilde{x}_0^\ell \ \mu_{x_0^\ell(1)}\left(\tilde{x}_0^\ell\right)}{\sum_{\ell=1}^{2} \mu_{x_0^\ell(1)}\left(\tilde{x}_0^\ell\right)} = \frac{-1.142 \times 0.333 + (-1.092) \times 0.333}{0.333 + 0.333} = -1.117,$$

$$\tilde{x}_0^2 = \frac{\sum_{\ell=1}^{2} \tilde{x}_0^\ell \ \mu_{x_0^\ell(2)}\left(\tilde{x}_0^\ell\right)}{\sum_{\ell=1}^{2} \mu_{x_0^\ell(2)}\left(\tilde{x}_0^\ell\right)} = \frac{-0.9324 \times 0.6 + (-0.664) \times 0.75}{0.6 + 0.75} = -0.7833,$$

$$\tilde{x}_0^3 = \frac{\sum_{\ell=1}^{2} \tilde{x}_0^\ell \ \mu_{x_0^\ell(3)}\left(\tilde{x}_0^\ell\right)}{\sum_{\ell=1}^{2} \mu_{x_0^\ell(3)}\left(\tilde{x}_0^\ell\right)} = \frac{-1.046 \times 0.2 + (-0.584) \times 0.2}{0.2 + 0.2} = -0.815,$$

$$\tilde{x}_0^4 = \frac{\sum_{\ell=1}^{2} \tilde{x}_0^\ell \ \mu_{x_0^\ell(4)}\left(\tilde{x}_0^\ell\right)}{\sum_{\ell=1}^{2} \mu_{x_0^\ell(4)}\left(\tilde{x}_0^\ell\right)} = \frac{-0.9532 \times 0.5 + (-0.812) \times 0.25}{0.5 + 0.25} = -0.9061.$$

**Section 6.4.**

In this section we introduce the notion of likely observability and also obtain sufficient conditions for fuzzy dynamical system (6.2.7), (6.2.8) to be likely observable.

**Definition 6.4.1.** The fuzzy system (6.2.7), (6.2.8) is said to be likely observable at $\alpha$-level over the interval $[0, T]$, if the knowledge of the $\alpha$-level input $\hat{U}(t)$ and the $\alpha$-level output $\hat{Y}(t)$ over $[0, T]$ suffices to determine the range of the initial state $\hat{X}_0$.

The norm of a matrix $A(t) = [a_{ij}(t)]$ is defined by $\|A(t)\| = \max_{ij} |a_{ij}(t)|$ and maximum norm defined by $\|A\| = \max_{t\in[0,T]} \|A(t)\|$.

**Theorem 6.4.1.** The fuzzy system (6.2.7), (6.2.8) is likely observable on level $\alpha$ over the interval $[0,T]$, if $(I_n \otimes C(T))(\psi(T)\otimes\phi(T))$ is non-singular. Furthermore, let $u_0(t)$ and $y_0(t)$ be the center points of $\hat{U}(t)$ and $\hat{Y}(t)$ respectively, and let $\widetilde{X}^\alpha$ be the possible initial point on $\alpha$-level, then the range estimate for the initial value on $\alpha$-level is given by

$$\begin{aligned}||\widetilde{X}^\alpha - \hat{X}_0|| \le\ & ||[(I_n \otimes C(T))(\psi(T)\otimes\phi(T))]^{-1}|| \\ & \Bigg(\max_{y_\alpha(T)\in\hat{Y}^\alpha(T)} ||y_\alpha(T) - y_0(T)|| \\ & +||D(T)|| \max_{u_\alpha(T)\in\hat{U}^\alpha(T)} ||u_\alpha(T) - u_0(T)|| \\ & +||C(T)||.||\psi||\ ||\phi||\ ||F|| \\ & \int_0^T \max_{u_\alpha(t)\in\hat{U}^\alpha(t)} ||u_\alpha(t) - u_0(t)||dt\Bigg).\end{aligned} \tag{6.4.1}$$

**Proof.** Let $\hat{X}^\alpha$ be the solution set of fuzzy system (6.2.7), (6.2.8), then from (6.2.9), we have

$$\hat{X}^\alpha(T) \in (\psi(T)\otimes\phi(T))\hat{X}_0 + \int_0^T (\psi(T-s)\otimes\phi(T-s))(I_n\otimes F(s))\hat{U}^\alpha(s)ds.$$

It follows that

$$\begin{aligned}\hat{Y}^\alpha(T) \in\ & (I_n\otimes C(T))\Big((\psi(T)\otimes\phi(T))\hat{X}_0 \\ & + \int_0^T (\psi(T-s)\otimes\phi(T-s))(I_n\otimes F(s))\hat{U}^\alpha(s)ds\Big) + (I_n\otimes D(T))\hat{U}^\alpha(T),\end{aligned}$$

where $\hat{Y}^\alpha(T)$ is the $\alpha$-level set of $\hat{Y}(T)$, which implies that

$$\begin{aligned}(I_n\otimes C(T))(\psi(T)\otimes\phi(T))\hat{X}_0 \in\ & \hat{Y}^\alpha(T) - (I_n\otimes D(T))\hat{U}^\alpha(T) \\ & -(I_n\otimes C(T))\int_0^T (\psi(T-s)\otimes\phi(T-s))(I_n\otimes F(s))\hat{U}^\alpha(s)ds.\end{aligned}$$

Let $\widetilde{X}^\alpha$ be the possible initial value, then we can write the above equation as

$$(I_n\otimes C(T))(\psi(T)\otimes\phi(T))\widetilde{X}^\alpha \in \hat{Y}^\alpha(T) - (I_n\otimes D(T))\hat{U}^\alpha(T)$$

$$-(I_n \otimes C(T)) \int_0^T (\psi(T-s) \otimes \phi(T-s))(I_n \otimes F(s))\hat{U}^\alpha(s)ds. \qquad (6.4.2)$$

Since $u_0$, $y_0$ are the center points of $\hat{U}$, $\hat{Y}$ respectively, we have

$$(I_n \otimes C(T))(\psi(T) \otimes \phi(T))\hat{X}_0 = y_0(T) - (I_n \otimes D(T))u_0(T)$$

$$-(I_n \otimes C(T)) \int_0^T (\psi(T-s) \otimes \phi(T-s))(I_n \otimes F(s))u_0(s)ds. \qquad (6.4.3)$$

From (6.4.2), (6.4.3) and the fact that $(I_n \otimes C(T))(\psi(T) \otimes \phi(T))$ is non-singular, we can estimate the distance between $\widetilde{X^\alpha}$ and $\hat{X}_0$ as follows;

$$\|[(I_n \otimes C(T))(\psi(T) \otimes \phi(T))]^{-1} \left(\widetilde{X^\alpha} - \hat{X}_0\right) \|$$

$$\leq \max d \Big(\hat{Y}^\alpha(T) - (I_n \otimes D(T))\hat{U}^\alpha(T)$$

$$-(I_n \otimes C(T)) \int_0^T (\psi(T-s) \otimes \phi(T-s))(I_n \otimes F(s))\hat{U}^\alpha(s)ds,$$

$$y_0(T) - (I_n \otimes D(T))u_0(T)$$

$$-(I_n \otimes C(T)) \int_0^T (\psi(T-s) \otimes \phi(T-s))(I_n \otimes F(s))u_0(s)ds\Big)$$

$$\leq \max_{y_\alpha(T) \in \hat{Y}^\alpha(T)} ||y_\alpha(T) - y_0(T)|| + ||D(T)|| \max_{u_\alpha(T) \in \hat{U}^\alpha(T)} ||u_\alpha(T) - u_0(T)||$$

$$+||C(T)|| \; ||\psi|| \; ||\phi|| \; ||F|| \int_0^T \max_{u_\alpha(t) \in \hat{U}^\alpha(t)} ||u_\alpha(t) - u_0(t)||dt.$$

Hence

$$||\widetilde{X^\alpha} - \hat{X}_0|| \leq ||[(I_n \otimes C(T))(\psi(T) \otimes \phi(T))]^{-1}||$$

$$\Big( \max_{y_\alpha(T) \in \hat{Y}^\alpha(T)} ||y_\alpha(T) - y_0(T)||$$

$$+||D(T)|| \max_{u_\alpha(T) \in \hat{U}^\alpha(T)} ||u_\alpha(T) - u_0(T)||$$

$$+||C(T)|| \; ||\psi|| \; ||\phi|| \; ||F||$$

$$\int_0^T \max_{u_\alpha(t) \in \hat{U}^\alpha(t)} ||u_\alpha(t) - u_0(t)||dt \Big).$$

**Remark 6.4.1.** The condition of the Theorem 6.4.1 is only sufficient but not necessary since one level is not enough to determine the non-singularity of $(I_n \otimes C(T))(\psi(T) \otimes \phi(T))$.

**Note 6.4.1.** Given the input and output we can observe the initial state in some range, i.e. the system is observable in the fuzzy sense.

**Example 6.4.1** Consider the fuzzy dynamical matrix Lyapunov system (6.1.1) satisfying (6.1.2) with

$$A(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; B(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; F(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix},$$

$$C(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; D(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } X_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Also assume that $\alpha$-level sets of input $\hat{U}(t)$ and output $\hat{Y}(t)$ are

$$\hat{U}^\alpha = \begin{bmatrix} [\alpha - 1, 1 - \alpha] \\ [\alpha - 1, 1 - \alpha] \\ [0.1(\alpha - 1), 0.1(1 - \alpha)] \\ [0.1(\alpha - 1), 0.1(1 - \alpha)] \end{bmatrix}, \; \hat{Y}^\alpha = \begin{bmatrix} [-\sqrt{1-\alpha}, \sqrt{1-\alpha}] \\ [-\sqrt{1-\alpha}, \sqrt{1-\alpha}] \\ [\frac{1}{2} - \sqrt{1-\alpha}, \frac{1}{2} + \sqrt{1-\alpha}] \\ [\frac{1}{2} - \sqrt{1-\alpha}, \frac{1}{2} + \sqrt{1-\alpha}] \end{bmatrix}.$$

Then the fundamental matrices of (6.2.4), (6.2.5) are

$$\phi(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \; \psi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}.$$

Clearly the center points of $\hat{U}(t)$ and $\hat{Y}(t)$ are

$$u_0 = [0, 0, 0, 0]^*, \; y_0 = \left[0, 0, \frac{1}{2}, \frac{1}{2}\right]^*.$$

And also

$$||\phi|| = \max_{0 \le t \le T} ||\phi(t)|| = 1, \; ||\psi|| = \max_{0 \le t \le T} ||\psi(t)|| = e^T, ||y_\alpha - y_0|| = \sqrt{1-\alpha},$$

$$\|[(I_n \otimes C(T))(\psi(T) \otimes \phi(T))]^{-1}\| = e^{-T}, \|u_\alpha - u_0\| = 1 - \alpha,$$

$$\|F\| = e^T, \|C\| = 1, \|D\| = 0.$$

Substituting these values in (6.4.1), we have

$$\|\widetilde{X}^\alpha - \hat{X}_0\| \leq e^{-T}\sqrt{1-\alpha} + T(1-\alpha)e^T.$$

From the above inequality, it is easily observed that as $T$ becomes larger, our estimated area becomes larger. This means that as $T$ increases it is difficult to determine the initial value. As $\alpha \to 1$, then $\|\widetilde{X}^\alpha - \hat{X}_0\| \to 0$, i.e. $\widetilde{X}^\alpha$ approaches the initial value $\hat{X}_0$.