CHAPTER-5

ON CONTROLLABILITY OF FUZZY DYNAMICAL MATRIX LYAPUNOV SYSTEMS

Section 5.1.

The importance of control theory in applied mathematics and its occurrence in several problems such as mechanics, electromagnetic theory, thermodynamics, artificial satellites etc., are well known. The purpose of this chapter is to provide sufficient conditions for controllability of first order fuzzy matrix Lyapunov systems modeled by ;

$$X'(t) = A(t)X(t) + X(t)B(t) + F(t)U(t), \ \ X(0) = X_0, \ \ t > 0, \tag{5.1.1}$$

$$Y(t) = C(t)X(t) + D(t)U(t), \tag{5.1.2}$$

where $U(t)$ is a $n \times n$ fuzzy input matrix called fuzzy control and $Y(t)$ is a $n \times n$ fuzzy output matrix. Here $A(t), B(t), F(t), C(t)$, and $D(t)$ are matrices of order $n \times n$, whose elements are continuous functions of t on $\mathbf{J} = [0, T] \subset R$ $(T > 0)$.

The problem of controllability and observability for system of ordinary differential equations has been studied by Barnett [6] and for matrix Lyapunov systems by Murty, Rao and Suresh Kumar [70]. Recently, the controllability criteria for fuzzy dynamical control systems were studied by Ding and Kandel [[29], [30]].

In section 5.2 we present some basic definitions and results relating to fuzzy sets and also obtain a unique solution to the corresponding Kronecker product system associated with (5.1.1), when $U(t)$ is a crisp continuous matrix.

In section 5.3 we provide a way to incorporate Kronecker product systems with fuzzy IF-THEN rules or fuzzy sets to form a fuzzy dynamical Lyapunov system, and also obtain its solution set.

In section 5.4 we present sufficient conditions for the controllability of the fuzzy dynamical Lyapunov system, with and with out using fuzzy rule base. The main results are illustrated with suitable examples.

This chapter generalizes some of the results of Ding and Kandel [[29], [30]] to matrix Lyapunov systems.

Section 5.2.

In this section we present some definitions and results relating to fuzzy sets and also obtain a unique solution of the Kronecker product system corresponding to (5.1.1), when $U(t)$ is a crisp continuous matrix.

Definition 5.2.1. A set valued function $F : \mathbf{J} \to P_k(\mathbf{R}^n)$ is said to be measurable, if it satisfies any one of the following equivalent conditions;

1. for all $u \in \mathbf{R}^n$, $t \to d_{F(t)}(u) = \inf_{v \in F(t)} \|u - v\|$ is measurable,
2. $\text{Gr}F = \{(t, u) \in \mathbf{J} \times \mathbf{R}^n : u \in F(t)\} \in \Sigma \times \beta(\mathbf{R}^n)$, where $\Sigma, \beta(\mathbf{R}^n)$ are Borel σ-field of $\mathbf{J}$ and $\mathbf{R}^n$ respectively (Graph measurability),
3. there exists a sequence $\{f_n(.)\}_{n\geq 1}$ of measurable functions such that $F(t) = \overline{\{f_n(.)\}_{n\geq 1}}$, for all $t \in \mathbf{J}$ (Castaing's representation).

We denote by S_F^1 the set of all selections of $F(.)$ that belong to the Lebesgue Bochner space $L^1_{\mathbf{R}^n}(\mathbf{J})$, i.e.

$$S_F^1 = \{f(.) \in L^1_{\mathbf{R}^n}(\mathbf{J}) : f(t) \in F(t) \ a.e.\}.$$

We present the Aumann's integral as follows;

$$(A)\int_{\mathbf{J}} F(t)dt = \{\int_{\mathbf{J}} f(t)dt, f(.) \in S_F^1\}.$$

Regarding fundamentals of differentiability and integrability of fuzzy functions, we refer to O.Kaleva [46], Lakshmikantham and Mohapatra [52].

In the sequel, we need the following representation theorem.

Theorem 5.2.1. [72] If $u \in E^n$, then

1. $[u]^\alpha \in P_k(\mathbf{R}^n)$, for all $0 \le \alpha \le 1$.

2. $[u]^{\alpha_2} \subset [u]^{\alpha_1}$, for all $0 \le \alpha_1 \le \alpha_2 \le 1$.

3. If $\{\alpha_k\}$ is a non-decreasing sequence converging to $\alpha > 0$, then $[u]^\alpha = \bigcap\limits_{k\ge1} [u]^{\alpha_k}$.

Conversely, if $\{A^\alpha : 0 \le \alpha \le 1\}$ is a family of subsets of $\mathbf{R}^n$ satisfying (1)-(3), then there exists a $u \in E^n$ such that $[u]^\alpha = A^\alpha$ for $0 < \alpha \le 1$ and $[u]^0 = \overline{\bigcup\limits_{0<\alpha\le1} A^\alpha} \subset A^0$.

A fuzzy set valued mapping $F : \mathbf{J} \to E^n$ is called fuzzy integrably bounded if $F_0(t)$ is integrably bounded.

Definition 5.2.2. Let $F : \mathbf{J} \to E^n$ be a fuzzy integrably bounded mapping. The fuzzy integral of F over $\mathbf{J}$ denoted by $\int_{\mathbf{J}} F(t)dt$, is defined level-set-wise by

$$\left[\int_{\mathbf{J}} F(t)dt\right]^\alpha = (A)\int_{\mathbf{J}} F_\alpha(t)dt, \ \ 0 < \alpha \le 1.$$

Let $F : \mathbf{J} \times E^n \to E^n$, consider the fuzzy initial value problem

$$u' = F(t,u), \ \ u(0) = u_0. \tag{5.2.1}$$

Definition 5.2.3. A mapping $u : \mathbf{J} \to E^n$ is a fuzzy weak solution to (5.2.1) if it is continuous and satisfies the integral equation

$$u(t) = u_0 + \int_0^t F(s,u(s))ds, \ \ \forall \ t \in \mathbf{J}.$$

If F is continuous, then this weak solution also satisfies (5.2.1) and we call it fuzzy strong solution to (5.2.1).

Now by applying the Vec operator to the matrix Lyapunov system (5.1.1) satisfying (5.1.2) and using the properties of Kronecker products, we have

$$\hat{X}'(t) = G(t)\hat{X}(t) + (I_n \otimes F(t))\hat{U}(t), \quad \hat{X}(0) = \hat{X}_0, \tag{5.2.2}$$

$$\hat{Y}(t) = (I_n \otimes C(t))\hat{X}(t) + (I_n \otimes D(t))\hat{U}(t), \tag{5.2.3}$$

where $G(t) = (B^* \otimes I_n) + (I_n \otimes A)$ is a $n^2 \times n^2$ matrix and $\hat{X}=$ Vec $X(t)$, $\hat{U}=$ Vec $U(t)$, $\hat{Y}=$ Vec $Y(t)$ are column matrices of order n^2.

The corresponding linear homogeneous system of (5.2.2) is

$$\hat{X}'(t) = G(t)\hat{X}(t), \quad \hat{X}(0) = \hat{X}_0. \tag{5.2.4}$$

Lemma 5.2.1. Let $\phi(t)$ and $\psi(t)$ be the fundamental matrices for the systems

$$X'(t) = A(t)X(t), \quad X(0) = I_n, \tag{5.2.5}$$

and

$$[X^*(t)]' = B^*(t)X^*(t), \quad X(0) = I_n \tag{5.2.6}$$

respectively. Then the matrix $\psi(t) \otimes \phi(t)$ is a fundamental matrix of (5.2.4) and the solution of (5.2.4) is $\hat{X}(t) = (\psi(t) \otimes \phi(t))\hat{X}_0$.

Theorem 5.2.2. Let $\phi(t)$ and $\psi(t)$ be the fundamental matrices for the systems (5.2.5) and (5.2.6), then the unique solution of (5.2.2) is

$$\hat{X}(t) = (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes \hat{F}(s))\hat{U}(s)ds. \tag{5.2.7}$$

Proof. The proof follows along similar lines to that of Theorem 3.2.1 of Chapter 3.

Section 5.3.

In this section we generate a fuzzy dynamical Lyapunov system associated with (5.2.2), (5.2.3), and also obtained its solution set.

Let $u_i(t) \in E^1$, $t \in \mathbf{J}$, $i = 1, 2, \ldots, n^2$ and define

$$\hat{U}(t) = (u_1(t), u_2(t), \ldots, u_{n^2}(t)) = u_1(t) \times u_2(t) \times \ldots \times u_{n^2}(t)$$

$$= \{(u_1^\alpha(t), u_2^\alpha(t), \ldots, u_{n^2}^\alpha(t)) : \alpha \in [0,1]\}$$

$$= \{(\tilde{u}_1(t), \tilde{u}_2(t), \ldots, \tilde{u}_{n^2}(t)) : \tilde{u}_i(t) \in u_i^\alpha(t), \alpha \in [0,1]\},$$

where $u_i^\alpha(t)$ is the α-level set of $u_i(t)$. From the above definition of $\hat{U}(t)$ and Theorem 5.2.1, it can be easily seen that $\hat{U}(t) \in E^{n^2}$.

Now we show that the following system

$$\hat{X}'(t) = G(t)\hat{X}(t) + (I_n \otimes F(t))\hat{U}(t), \hat{X}(0) = \hat{X}_0 \tag{5.3.1}$$

$$\hat{Y}(t) = (I_n \otimes C(t))\hat{X}(t) + (I_n \otimes D(t))\hat{U}(t), \tag{5.3.2}$$

determines a fuzzy system, when fuzzy control $\hat{U}(t)$ as defined above.

Assume that $\hat{U}(t)$ is continuous in E^{n^2}. The set $\hat{U}^\alpha(t) = u_1^\alpha \times u_2^\alpha \times \ldots \times u_{n^2}^\alpha$ is a convex and compact set in $\mathbf{R}^{n^2}$. For any positive number T, consider the following differential inclusions;

$$\hat{X}'(t) \in G(t)\hat{X}(t) + (I_n \otimes F(t))\hat{U}^\alpha(t), \ \ t \in \mathbf{J} = [0, T] \tag{5.3.3}$$

$$\hat{X}(0) = \hat{X}_0. \tag{5.3.4}$$

Let $\hat{X}^\alpha$ be the solution set of inclusions (5.3.3), (5.3.4).

Claim (i). $[\hat{X}(t)]^\alpha \in P_k(\mathbf{R}^{n^2})$, for every $0 \leq \alpha \leq 1$, $t \in \mathbf{J}$.

First, we prove that $\hat{X}^\alpha$ is non-empty, compact and convex in $C(\mathbf{J}, \mathbf{R}^{n^2})$. Since $\hat{U}^\alpha(t)$ has measurable selection, we have $\hat{X}^\alpha$ is non-empty.

Let $K = \max_{t\in \mathbf{J}} \|\phi(t)\|$, $L = \max_{t\in \mathbf{J}} \|\psi(t)\|$, $M = \max_{t\in \mathbf{J}}\{\|\tilde{u}(t)\| : \tilde{u}(t) \in \hat{U}^\alpha(t)\}$, $N = \max_{t\in \mathbf{J}} \|F(t)\|$.

If for any $\hat{X} \in \hat{X}^\alpha$, then there is a selection $\tilde{u}(t) \in \hat{U}^\alpha(t)$ such that

$$\hat{X}(t) = (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))\tilde{u}(s)ds.$$

Then

$$\begin{aligned}
\|\hat{X}(t)\| &\leq \|(\psi(t) \otimes \phi(t))\hat{X}_0\| + \int_0^t \|(\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))\tilde{u}(s)\|ds \\
&\leq \|\psi(t)\| \ \|\phi(t)\| \ \|\hat{X}_0\| + \int_0^t \|\psi(t-s)\| \ \|\phi(t-s)\| \ \|F(s)\| \ \|\tilde{u}(s)\|ds \\
&\leq KL\|\hat{X}_0\| + KLNMT.
\end{aligned}$$

Thus $\hat{X}^\alpha$ is bounded.

For any $t_1, t_2 \in \mathbf{J}$

$$\begin{aligned}
\hat{X}(t_1) - \hat{X}(t_2) = {} & (\psi(t_1) \otimes \phi(t_1))\hat{X}_0 + \int_0^{t_1} (\psi(t_1-s) \otimes \phi(t_1-s))(I_n \otimes F(s))\tilde{u}(s)ds \\
& -(\psi(t_2) \otimes \phi(t_2))\hat{X}_0 - \int_0^{t_2} (\psi(t_2-s) \otimes \phi(t_2-s))(I_n \otimes F(s))\tilde{u}(s)ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\hat{X}(t_1) - \hat{X}(t_2)\| \leq {} & \|(\psi(t_1) \otimes \phi(t_1)) - (\psi(t_2) \otimes \phi(t_2))\| \ \|\hat{X}_0\| \\
& + \int_{t_2}^{t_1} \|(\psi(t_1-s) \otimes \phi(t_1-s))(I_n \otimes F(s))\tilde{u}(s)\|ds \\
& + \int_0^{t_2} \|[(\psi(t_1-s) \otimes \phi(t_1-s)) - (\psi(t_2-s) \otimes \phi(t_2-s))] \\
& \qquad (I_n \otimes F(s))\tilde{u}(s)\|ds \\
\leq {} & \|(\psi(t_1) \otimes \phi(t_1)) - (\psi(t_2) \otimes \phi(t_2))\| \ \|\hat{X}_0\| + KLNM|t_1 - t_2| \\
& + MN \int_0^T \|(\psi(t_1-s) \otimes \phi(t_1-s)) - (\psi(t_2-s) \otimes \phi(t_2-s))\|ds.
\end{aligned}$$

Since $\phi(t)$ and $\psi(t)$ are both uniformly continuous on $\mathbf{J}$, $\hat{X}$ is equi-continuous. Thus $\hat{X}^\alpha$ is relatively compact. If $\hat{X}^\alpha$ is closed, then it is compact.

Let $\hat{X}_k \in \hat{X}^\alpha$ and $\hat{X}_k \to \hat{X}$. For each $\hat{X}_k$, there is a $\tilde{u}_k \in \hat{U}^\alpha(t)$ such that

$$\hat{X}_k(t) = (\psi(t)\otimes\phi(t))\hat{X}_0 + \int_0^t (\psi(t-s)\otimes\phi(t-s))(I_n\otimes F(s))\tilde{u}_k(s)ds. \qquad (5.3.5)$$

Since $\tilde{u}_k \in \hat{U}^\alpha(t)$, it is closed, then there exists a subsequence $\{\tilde{u}_{k_j}\}$ of $\{\tilde{u}_k\}$ converging weakly to $\tilde{u} \in \hat{U}^\alpha(t)$. From Mazur's Theorem [17], there exists a sequence of numbers $\lambda_j > 0$, $\sum_j \lambda_j = 1$ such that $\sum_j \lambda_j \tilde{u}_{k_j}$ converges strongly to $\tilde{u}$.

Thus from (5.3.5), we have

$$\sum_j \lambda_j \hat{X}_{k_j}(t) = \sum_j \lambda_j (\psi(t)\otimes\phi(t))\hat{X}_0$$
$$+ \int_0^t (\psi(t-s)\otimes\phi(t-s))(I_n\otimes F(s)) \sum_j \lambda_j \tilde{u}_{k_j}(s)ds. \qquad (5.3.6)$$

From Fatou's lemma, taking the limit on (5.3.6), we obtain

$$\hat{X}(t) = (\psi(t)\otimes\phi(t))\hat{X}_0 + \int_0^t (\psi(t-s)\otimes\phi(t-s))(I_n\otimes F(s))\tilde{u}(s)ds. \qquad (5.3.7)$$

Thus $\hat{X}(t) \in \hat{X}^\alpha$, and hence $\hat{X}^\alpha$ is closed.

Let $\hat{X}_1$, $\hat{X}_2 \in \hat{X}^\alpha$, then there exist $\tilde{u}_1, \tilde{u}_2 \in \hat{U}^\alpha(t)$ such that

$$\hat{X}_1'(t) = G(t)\hat{X}_1(t) + (I_n\otimes F(t))\tilde{u}_1(t)$$

and

$$\hat{X}_2'(t) = G(t)\hat{X}_2(t) + (I_n\otimes F(t))\tilde{u}_2(t).$$

Let $\hat{X} = \lambda\hat{X}_1(t) + (1-\lambda)\hat{X}_2(t)$, $0 \le \lambda \le 1$, then

$$\hat{X}' = \lambda\hat{X}_1'(t) + (1-\lambda)\hat{X}_2'(t)$$

$$= \lambda \left(G(t)\hat{X}_1(t) + (I_n \otimes F(t))\tilde{u}_1(t)\right) + (1-\lambda)\left(G(t)\hat{X}_2(t) + (I_n \otimes F(t))\tilde{u}_2(t)\right)$$

$$= G(t)\left[\lambda\hat{X}_1(t) + (1-\lambda)\hat{X}_2(t)\right] + (I_n \otimes F(t))\left[\lambda\tilde{u}_1(t) + (1-\lambda)\tilde{u}_2(t)\right].$$

Since $\hat{U}^\alpha(t)$ is convex, $\lambda\tilde{u}_1(t) + (1-\lambda)\tilde{u}_2(t) \in \hat{U}^\alpha(t)$, we have

$$\hat{X}'(t) \in G(t)\hat{X}(t) + (I_n \otimes F(t))\hat{U}^\alpha(t),$$

i.e. $\hat{X} \in \hat{X}^\alpha$. Thus $\hat{X}^\alpha$ is convex. Therefore $\hat{X}^\alpha$ is nonempty, compact and convex in $C(\mathbf{J}, \mathbf{R}^{n^2})$. Thus, from Arzela-Ascoli theorem, we know that $[\hat{X}(t)]^\alpha$ is compact in $\mathbf{R}^{n^2}$ for every $t \in \mathbf{J}$. Also it is obvious that $[\hat{X}(t)]^\alpha$ is convex in $\mathbf{R}^{n^2}$. Thus we have $[\hat{X}(t)]^\alpha \in P_k(\mathbf{R}^{n^2})$, for every $t \in \mathbf{J}$. Hence the claim.

Claim (ii). $[\hat{X}(t)]^{\alpha_2} \subset [\hat{X}(t)]^{\alpha_1}$, for all $0 \le \alpha_1 \le \alpha_2 \le 1$.

Let $0 \le \alpha_1 \le \alpha_2 \le 1$. Since $u_i^{\alpha_2}(t) \subset u_i^{\alpha_1}(t)$, we have

$$\hat{U}^{\alpha_2}(t) = u_1^{\alpha_2}(t) \times u_2^{\alpha_2}(t) \times \ldots \times u_{n^2}^{\alpha_2}(t)$$

$$\subset u_1^{\alpha_1}(t) \times u_2^{\alpha_1}(t) \times \ldots \times u_{n^2}^{\alpha_1}(t) = \hat{U}^{\alpha_1}(t),$$

Thus, we have the selection inclusion $S^1_{\hat{U}^{\alpha_2}(t)} \subset S^1_{\hat{U}^{\alpha_1}(t)}$ and the following differential inclusion

$$\hat{X}'(t) \in G(t)\hat{X} + (I_n \otimes F(t))\hat{U}^{\alpha_2}(t)$$

$$\subset G(t)\hat{X} + (I_n \otimes F(t))\hat{U}^{\alpha_1}(t).$$

Consider the differential inclusions

$$\hat{X}'(t) \in G(t)\hat{X} + (I_n \otimes F(t))\hat{U}^{\alpha_2}(t), \quad t \in \mathbf{J} \tag{5.3.8}$$

and

$$\hat{X}'(t) \in G(t)\hat{X} + (I_n \otimes F(t))\hat{U}^{\alpha_1}(t), \quad t \in \mathbf{J}. \tag{5.3.9}$$

Let $\hat{X}^{\alpha_2}$ and $\hat{X}^{\alpha_1}$ be the solution sets of (5.3.8) and (5.3.9) respectively. Clearly, the solution of (5.3.8) satisfies the following inclusion

$$\hat{X}(t) \in (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s)) S^1_{\hat{U}^{\alpha_2}(s)} ds$$
$$\subset (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s)) S^1_{\hat{U}^{\alpha_1}(s)} ds.$$

Thus $\hat{X}^{\alpha_2} \subset \hat{X}^{\alpha_1}$, and hence $\hat{X}^{\alpha_2}(t) \subset \hat{X}^{\alpha_1}(t)$. Hence the claim.

Claim (iii). If $\{\alpha_k\}$ is a non-decreasing sequence converging to $\alpha > 0$, then $\hat{X}^{\alpha}(t) = \bigcap_{k\geq 1} \hat{X}^{\alpha_k}(t)$.

Let $\hat{U}^{\alpha_k}(t) = u_1^{\alpha_k} \times u_2^{\alpha_k} \times \ldots \times u_{n^2}^{\alpha_k}$; $\hat{U}^{\alpha}(t) = u_1^{\alpha} \times u_2^{\alpha} \times \ldots \times u_{n^2}^{\alpha}$ and consider the inclusions;

$$\hat{X}'(t) \in G(t)\hat{X} + (I_n \otimes F(t))\hat{U}^{\alpha_k}(t) \tag{5.3.10}$$

and

$$\hat{X}'(t) \in G(t)\hat{X} + (I_n \otimes F(t))\hat{U}^{\alpha}(t). \tag{5.3.11}$$

Let $\hat{X}^{\alpha_k}$ and $\hat{X}^{\alpha}$ be the solution sets of (5.3.10) and (5.3.11) respectively. Since $u_i(t)$ is a fuzzy set and from Theorem 5.2.1, we have

$$u_i^{\alpha} = \bigcap_{k\geq 1} u_i^{\alpha_k}.$$

Consider

$$\hat{U}^{\alpha}(t) = u_1^{\alpha} \times u_2^{\alpha} \times \ldots \times u_{n^2}^{\alpha} = \bigcap_{k\geq 1} u_1^{\alpha_k} \times \bigcap_{k\geq 1} u_2^{\alpha_k} \times \ldots \times \bigcap_{k\geq 1} u_{n^2}^{\alpha_k}$$
$$= \bigcap_{k\geq 1} (u_1^{\alpha_k} \times u_2^{\alpha_k} \times \ldots \times u_{n^2}^{\alpha_k}) = \bigcap_{k\geq 1} \hat{U}^{\alpha_k}(t)$$

and then $S^1_{\hat{U}^{\alpha}(t)} = S^1_{\bigcap_{k\geq 1} \hat{U}^{\alpha_k}(t)}$. Therefore

$$\hat{X}'(t) \in G(t)\hat{X} + (I_n \otimes F(t))\hat{U}^{\alpha}(t)$$

$$= G(t)\hat{X} + (I_n \otimes F(t)) \bigcap_{k\geq 1} \hat{U}^{\alpha_k}(t)$$

$$\subset G(t)\hat{X} + (I_n \otimes F(t))\hat{U}^{\alpha_k}(t) \ \ (k = 1, 2, \ldots).$$

Thus, we have $\hat{X}^{\alpha} \subset \hat{X}^{\alpha_k}$, $k = 1, 2, \ldots$, which implies that

$$\hat{X}^{\alpha} \subset \bigcap_{k\geq 1} \hat{X}^{\alpha_k}. \tag{5.3.12}$$

Let $\hat{X}$ be the solution set to the inclusions

$$\hat{X}'(t) \in G(t)\hat{X} + (I_n \otimes F(t))\hat{U}^{\alpha_k}(t), k \geq 1.$$

Then

$$\hat{X}(t) \in (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))S^1_{\hat{U}^{\alpha_k}(t)}ds;$$

it follows that

$$\hat{X}(t) \in (\psi(t) \otimes \phi(t))\hat{X}_0 + \bigcap_{k\geq 1} \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))S^1_{\hat{U}^{\alpha_k}}ds$$

$$\subset (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))S^1_{\bigcap_{k\geq 1} \hat{U}^{\alpha_k}}ds$$

$$= (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))S^1_{\hat{U}^{\alpha}}ds.$$

This implies that $\hat{X} \in \hat{X}^{\alpha}$. Therefore

$$\bigcap_{k\geq 1} \hat{X}^{\alpha_k} \subset \hat{X}^{\alpha}. \tag{5.3.13}$$

From (5.3.12) and (5.3.13), we have

$$\hat{X}^{\alpha} = \bigcap_{k\geq 1} \hat{X}^{\alpha_k},$$

and hence

$$\hat{X}^{\alpha}(t) = \bigcap_{k\geq 1} \hat{X}^{\alpha_k}(t).$$

From Claims (i)-(iii) and applying Theorem 5.2.1, there exists $\hat{X}(t) \in E^{n^2}$ on **J** such that $\hat{X}^{\alpha}(t)$ is a solution set to the differential inclusions (5.3.3) and (5.3.4). Hence the system (5.3.1), (5.3.2) is a fuzzy dynamical Lyapunov system, and it can be expressed as

$$\hat{X}'(t) = G(t)\hat{X}(t) + (I_n \otimes F(t))\hat{U}(t), \quad \hat{X}(0) = \{\hat{X}_0\} \tag{5.3.14}$$

$$\hat{Y}(t) = (I_n \otimes C(t))\hat{X}(t) + (I_n \otimes D(t))\hat{U}(t). \tag{5.3.15}$$

Remark 5.3.1. Consider a special case. If the input is in the form

$$\hat{U}(t) = \widetilde{u}_1(t) \times \widetilde{u}_2(t) \times \ldots \times u_i(t) \times \ldots \times \widetilde{u}_{n^2}(t),$$

where $\widetilde{u}_k(t) \in \mathbf{R}^1$, $k \neq i$ are crisp numbers, then the i^{th} component of the solution set of (5.3.1) is a fuzzy set in E^1.

Remark 5.3.2. It is easily seen from the above discussion that the system (5.3.1), (5.3.2) determines a fuzzy dynamical Lyapunov system, when the input $\hat{U}(t)$ is a fuzzy set defined on $\mathbf{R}^{n^2}$. The solution set of the system is given by

$$\hat{X}(t) \in (\psi(t) \otimes \phi(t))\hat{X}_0 + \int_0^t (\psi(t-s) \otimes \phi(t-s))(I_n \otimes F(s))\hat{U}(s)ds. \tag{5.3.16}$$

Section 5.4.

In this section we obtain sufficient conditions for the controllability of the fuzzy dynamical Lyapunov system (5.3.14), (5.3.15), with and without using fuzzy rule base.

Definition 5.4.1. The fuzzy system (5.3.14), (5.3.15) is said to be completely controllable if for any initial state $\hat{X}(0) = \hat{X}_0$ and any given final state $\hat{X}_f$ there exists a finite time $t_1 > 0$ and a control $\hat{U}(t)$, $0 \leq t \leq t_1$, such that $\hat{X}(t_1) = \hat{X}_f$.

Lemma 5.4.1. If F is a fuzzy set, then $\int_0^T F dt = TF$.

Proof. Let $[F]^\alpha$ be the α-level set of F. Since

$$\left[\int_0^T F dt\right]^\alpha = \int_0^T [F]^\alpha dt = T[F]^\alpha.$$

From the definition of fuzzy set, we have $\int_0^T F dt = TF$.

Lemma 5.4.2. Let P, Q be two fuzzy sets and $h(t)$ be a non zero continuous function on $\mathbf{J} = [0, T]$, satisfying

$$\int_0^T h(t) P dt = \int_0^T h(t) Q dt$$

then $P = Q$.

Proof. For each α-level, we have

$$\int_0^T h(t)[P]^\alpha dt = \left[\int_0^T h(t) P dt\right]^\alpha = \left[\int_0^T h(t) Q dt\right]^\alpha = \int_0^T h(t)[Q]^\alpha dt. \quad (5.4.1)$$

Suppose that $P \neq Q$, then for some $\alpha \in [0,1]$, we have $[P]^\alpha \neq [Q]^\alpha$. Without loss of generality, we assume that $P, Q \in E^1$. Let $P^\alpha = [P_{\min}(\alpha), P_{\max}(\alpha)]$ and $Q^\alpha = [Q_{\min}(\alpha), Q_{\max}(\alpha)]$. Then we have either (i) $P_{\min}(\alpha) \neq Q_{\min}(\alpha)$ or (ii) $P_{\max}(\alpha) \neq Q_{\max}(\alpha)$ holds.

If (i) holds, then

$$\int_0^T h(t) P_{\min}(\alpha) dt \neq \int_0^T h(t) Q_{\min}(\alpha) dt.$$

If (ii) holds, then

$$\int_0^T h(t) P_{\max}(\alpha) dt \neq \int_0^T h(t) Q_{\max}(\alpha) dt.$$

Thus in both cases (i) and (ii), we have

$$\int_0^T h(t)[P_{\min}(\alpha), P_{\max}(\alpha)] dt \neq \int_0^T h(t)[Q_{\min}(\alpha), Q_{\max}(\alpha)] dt.$$

It implies that

$$\int_0^T h(t)[P]^\alpha dt \neq \int_0^T h(t)[Q]^\alpha dt,$$

which is a contradiction to (5.4.1). Hence $P = Q$.

Definition 5.4.2. Let $u, v \in E^1$, $k \in \mathbf{R}^1$ and $[u]^\alpha$ be the α-level set of u. We define the sum of u and v by

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha = \{a+b : a \in [u]^\alpha, b \in [v]^\alpha\}, \tag{5.4.2}$$

the difference between u and v by

$$[u-v]^\alpha = [u]^\alpha - [v]^\alpha = \{a-b : a \in [u]^\alpha, b \in [v]^\alpha\}, \tag{5.4.3}$$

and the scalar product by

$$[ku]^\alpha = k[u]^\alpha = \{ka : a \in [u]^\alpha\}. \tag{5.4.4}$$

Definition 5.4.3. Let $x, y \in E^{n^2}$ and $x = x_1 \times x_2 \times \ldots \times x_{n^2}$, $y = y_1 \times y_2 \times \ldots \times y_{n^2}$, $x_i, y_i \in E^1$, $i = 1, 2, \ldots n^2$. If $y = z + x$, then $z = y - x$ which is defined by

$$[z]^\alpha = [y-x]^\alpha = [y]^\alpha - [x]^\alpha = \begin{bmatrix} [y_1]^\alpha - [x_1]^\alpha \\ \ldots \\ [y_{n^2}]^\alpha - [x_{n^2}]^\alpha \end{bmatrix}. \tag{5.4.5}$$

If $y = w - x$, then $w = y + x$ which is defined by

$$[w]^\alpha = [y+x]^\alpha = [y]^\alpha + [x]^\alpha = \begin{bmatrix} [y_1]^\alpha + [x_1]^\alpha \\ \ldots \\ [y_{n^2}]^\alpha + [x_{n^2}]^\alpha \end{bmatrix}. \tag{5.4.6}$$

Definition 5.4.4. Let

$$C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n^2} \\ c_{21} & c_{22} & \dots & c_{2n^2} \\ \dots & \dots & \dots & \dots \\ c_{n^21} & c_{n^22} & \dots & c_{n^2n^2} \end{bmatrix}$$

be a $n^2 \times n^2$ matrix, $p = p_1 \times p_2 \times \dots \times p_{n^2}$, $p_i \in E^1$, $i = 1, 2, \dots n^2$, be a fuzzy set in E^{n^2}, $[p_i]^\alpha$ are α-level sets of p_i. Define the product Cp of C and p as

$$[Cp]^\alpha = C[p]^\alpha = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n^2} \\ c_{21} & c_{22} & \dots & c_{2n^2} \\ \dots & \dots & \dots & \dots \\ c_{n^21} & c_{n^22} & \dots & c_{n^2n^2} \end{bmatrix} \begin{bmatrix} [p_1]^\alpha \\ [p_2]^\alpha \\ \dots \\ [p_{n^2}]^\alpha \end{bmatrix}$$

$$= \begin{bmatrix} c_{11}[p_1]^\alpha + \dots + c_{1n^2}[p_{n^2}]^\alpha \\ c_{21}[p_1]^\alpha + \dots + c_{2n^2}[p_{n^2}]^\alpha \\ \dots \\ c_{n^21}[p_1]^\alpha + \dots + c_{n^2n^2}[p_{n^2}]^\alpha \end{bmatrix}.$$

All these definitions yield the following Lemma.

Lemma 5.4.3. Cp is a fuzzy set in E^{n^2}.

Proof. The proof is similar to the proof of Lemma 3.1 [31].

Theorem 5.4.1. The fuzzy system (5.3.14), (5.3.15) is completely controllable if the $n^2 \times n^2$ symmetric controllability matrix

$$W(0,T) = \int_0^T (\psi(T-t) \otimes \phi(T-t))(I_n \otimes F(t))(I_n \otimes F(t))^* (\psi(T-t) \otimes \phi(T-t))^* dt \qquad (5.4.7)$$

is non-singular. Furthermore, the fuzzy control $\hat{U}(t)$ transfer the state of the system from $\hat{X}(0) = \hat{X}_0$ to a fuzzy state $\hat{X}(T) = \hat{X}_f = \left(x_{f_1}, x_{f_2}, \dots, x_{f_{n^2}}\right)$ can be determined by the following fuzzy rule base

$$\begin{aligned} \text{R}: \text{IF } \widetilde{x}_1 \text{ is in } x_{f_1} \text{ and } \ldots \text{ and } \widetilde{x}_{n^2} \text{ is in } x_{f_{n^2}}, \\ \text{THEN } \widetilde{u}_1 \text{ is in } u_1 \text{ and } \ldots \text{ and } \widetilde{u}_{n^2} \text{ is in } u_{n^2}, \end{aligned} \tag{5.4.8}$$

where

$$\begin{aligned} (\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, u_i(t), \ldots, \widetilde{u}_{n^2}(t)) &= \frac{1}{T}(I_n \otimes F(t))^{-1} \\ &(\psi(T-t)\otimes\phi(T-t))^{-1} (\widetilde{x}_1(T), \widetilde{x}_2(T), \ldots, x_{f_i}, \ldots, \widetilde{x}_{n^2}(T)) \\ &-(I_n\otimes F(t))^*(\psi(T-t)\otimes\phi(T-t))^* W^{-1}(0,T)(\psi(T)\otimes\phi(T))\hat{X}_0, \end{aligned} \tag{5.4.9}$$

$i = 1, 2, \ldots, n^2$.

Proof. Suppose that the symmetric controllability matrix $W(0,T)$ is non-singular, so $W^{-1}(0,T)$ exists. Multiplying $W^{-1}(0,T)(\psi(T)\otimes\phi(T))\hat{X}_0$ on both sides of (5.4.7), we have

$$\begin{aligned} (\psi(T)\otimes\phi(T))\hat{X}_0 = \int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))(I_n\otimes F(t))^* \\ (\psi(T-t)\otimes\phi(T-t))^* W^{-1}(0,T)(\psi(T)\otimes\phi(T))\hat{X}_0 dt. \end{aligned} \tag{5.4.10}$$

Now the problem is to find the control $\hat{U}(t)$ such that

$$\begin{aligned} \hat{X}(T) = \hat{X}_f = (\psi(T)\otimes\phi(T))\hat{X}_0 \\ + \int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))\hat{U}(t)dt. \end{aligned} \tag{5.4.11}$$

Since $\hat{X}$ is fuzzy and from Lemma 5.4.3, $\hat{U}(t)$ must be fuzzy, otherwise the fuzzy left side of (5.4.11) is not equal to the crisp right side. By Lemma 5.4.1 $\hat{X}_f$ can be written as

$$\hat{X}_f = \frac{1}{T}\int_0^T \hat{X}_f dt = \frac{1}{T}\int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))$$

$$(I_n \otimes F(t))^{-1}(\psi(T-t)\otimes\phi(T-t))^{-1}\hat{X}_f dt. \qquad (5.4.12)$$

From (5.4.11) and (5.4.12), we have

$$\frac{1}{T}\int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))(I_n\otimes F(t))^{-1}(\psi(T-t)\otimes\phi(T-t))^{-1}\hat{X}_f dt$$

$$= (\psi(T)\otimes\phi(T))\hat{X}_0 + \int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))\hat{U}(t)dt. \qquad (5.4.13)$$

Combining (5.4.10) and (5.4.13), we have

$$\frac{1}{T}\int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))(I_n\otimes F(t))^{-1}(\psi(T-t)\otimes\phi(T-t))^{-1}\hat{X}_f dt$$

$$= \int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))(I_n\otimes F(t))^*(\psi(T-t)\otimes\phi(T-t))^* W^{-1}(0,T)$$

$$(\psi(T)\otimes\phi(T))\hat{X}_0 dt + \int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))\hat{U}(t)dt.$$

It follows that

$$\int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))\hat{U}(t)dt$$

$$= \int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))\{\frac{1}{T}(I_n\otimes F(t))^{-1}(\psi(T-t)\otimes\phi(T-t))^{-1}$$

$$\hat{X}_f - (I_n\otimes F(t))^*(\psi(T-t)\otimes\phi(T-t))^* W^{-1}(0,T)(\psi(T)\otimes\phi(T))\hat{X}_0\}dt.$$

By using Lemma 5.4.2, we get

$$\hat{U}(t) = \frac{1}{T}(I_n\otimes F(t))^{-1}(\psi(T-t)\otimes\phi(T-t))^{-1}\hat{X}_f$$

$$-(I_n\otimes F(t))^*(\psi(T-t)\otimes\phi(T-t))^* W^{-1}(0,T)(\psi(T)\otimes\phi(T))\hat{X}_0. \qquad (5.4.14)$$

Now we have two special cases for (5.4.14). First, let $\hat{X}(T) = \hat{X}_f = (\widetilde{x}_1(T), \widetilde{x}_2(T), \ldots, \widetilde{x}_{n^2}(T))$ be a crisp point, then we will get a corresponding control $\hat{U}(t) = (\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_{n^2})$, satisfying (5.4.14).

Second, let $\hat{X}(T) = (\widetilde{x}_1(T), \widetilde{x}_2(T), \ldots, x_{f_i}, \ldots, \widetilde{x}_{n^2}(T))$, then the corresponding control $\hat{U}(t)$ will take the form $\hat{U}(t) = (\widetilde{u}_1, \widetilde{u}_2, \ldots, u_i, \ldots, \widetilde{u}_{n^2})$ in which the i^{th} component of $\hat{U}(t)$ is a fuzzy set in E^1. Obviously, $\widetilde{u}_i(t)$ is in $u_i(t)$, the grade of the membership can be determined by $\mu_{x_{f_i}}(\widetilde{x}_i(T))$, the grade of the membership of $\widetilde{x}_i(T)$ in x_{f_i}. Hence based on the above discussion, fuzzy rule base for the control $\hat{U}$ is given by (5.4.8) and (5.4.9).

Remark 5.4.1. The non-singularity of the symmetric controllability matrix $W(0,T)$ in Theorem 5.4.1 is only a sufficient condition but not necessary.

The significance of the Theorem 5.4.1 is that given a deterministic system and a fuzzy state with rule base, we can determine the rule base for the control which transfers the initial state to the given final state in a finite time.

Example 5.4.1. Consider the fuzzy dynamical matrix Lyapunov system (5.1.1) satisfying (5.1.2) with

$$A(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix},$$

$$C(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, T = \frac{\pi}{2}, \text{ and } X_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let $\hat{X}_f = (x_{f_1}, x_{f_2}, x_{f_3}, x_{f_4}) \in E^4$, where

$$\left[\hat{X}_f\right]^\alpha = \begin{bmatrix} [x_{f_1}]^\alpha \\ [x_{f_2}]^\alpha \\ [x_{f_3}]^\alpha \\ [x_{f_4}]^\alpha \end{bmatrix} = \begin{bmatrix} [\alpha-1, 1-\alpha] \\ [\alpha-1, 1-\alpha] \\ [0.1(\alpha-1), 0.1(1-\alpha)] \\ [0.1(\alpha-1), 0.1(1-\alpha)] \end{bmatrix}.$$

We select the points $\widetilde{x}_1 = 0.5$, $\widetilde{x}_2 = 0.25$, $\widetilde{x}_3 = 0.05$, and $\widetilde{x}_4 = 0.025$ which are in x_{f_1}, x_{f_2}, x_{f_3}, and x_{f_4} with grades of the membership are 0.5, 0.75, 0.5, and 0.75 respectively. The fundamental matrices of (5.2.5), (5.2.6) are

$$\phi(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad \psi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}.$$

Now the fundamental matrix of (5.2.4) is

$$\psi(t) \otimes \phi(t) = \begin{bmatrix} e^t \cos t & -e^t \sin t & 0 & 0 \\ e^t \sin t & e^t \cos t & 0 & 0 \\ 0 & 0 & e^t \cos t & -e^t \sin t \\ 0 & 0 & e^t \sin t & e^t \cos t \end{bmatrix}.$$

Consider

$$(\psi(\tau) \otimes \phi(\tau))(I_n \otimes F(t))(I_n \otimes F(t))^*(\psi(\tau) \otimes \phi(\tau))^*$$

$$= e^{\tau} \begin{bmatrix} \cos(\tau) & -\sin(\tau) & 0 & 0 \\ \sin(\tau) & \cos(\tau) & 0 & 0 \\ 0 & 0 & \cos(\tau) & -\sin(\tau) \\ 0 & 0 & \sin(\tau) & \cos(\tau) \end{bmatrix} e^t \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^t \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} e^{\tau} \begin{bmatrix} \cos(\tau) & \sin(\tau) & 0 & 0 \\ -\sin(\tau) & \cos(\tau) & 0 & 0 \\ 0 & 0 & \cos(\tau) & \sin(\tau) \\ 0 & 0 & -\sin(\tau) & \cos(\tau) \end{bmatrix}$$

$$= \begin{bmatrix} e^{\pi} & 0 & 0 & 0 \\ 0 & e^{\pi} & 0 & 0 \\ 0 & 0 & e^{\pi} & 0 \\ 0 & 0 & 0 & e^{\pi} \end{bmatrix},$$

where $\tau = \frac{\pi}{2} - t$. Therefore

$$W(0, \frac{\pi}{2}) = \int_0^{\frac{\pi}{2}} \begin{bmatrix} e^{\pi} & 0 & 0 & 0 \\ 0 & e^{\pi} & 0 & 0 \\ 0 & 0 & e^{\pi} & 0 \\ 0 & 0 & 0 & e^{\pi} \end{bmatrix} dt = \frac{\pi}{2} e^{\pi} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Clearly, it is non-singular.

Thus from Theorem 5.4.1, the input $\hat{U}$ can be chosen by the following α-level sets.

$$\hat{U}^{\alpha}(t) = \frac{2e^{-t}}{\pi}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} e^{-\tau}\begin{bmatrix} \cos(\tau) & \sin(\tau) & 0 & 0 \\ -\sin(\tau) & \cos(\tau) & 0 & 0 \\ 0 & 0 & \cos(\tau) & \sin(\tau) \\ 0 & 0 & -\sin(\tau) & \cos(\tau) \end{bmatrix}$$

$$\left[\begin{bmatrix} [\alpha-1,1-\alpha] \\ [\alpha-1,1-\alpha] \\ [0.1(\alpha-1),0.1(1-\alpha)] \\ [0.1(\alpha-1),0.1(1-\alpha)] \end{bmatrix} - e^{t}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right.$$

$$e^{\tau}\begin{bmatrix} \cos(\tau) & \sin(\tau) & 0 & 0 \\ -\sin(\tau) & \cos(\tau) & 0 & 0 \\ 0 & 0 & \cos(\tau) & \sin(\tau) \\ 0 & 0 & -\sin(\tau) & \cos(\tau) \end{bmatrix} \frac{2e^{-\pi}}{\pi}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left. e^{\frac{\pi}{2}}\begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) & 0 & 0 \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & 0 & 0 \\ 0 & 0 & \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ 0 & 0 & \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix}\right]\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence the α-level sets of fuzzy control is given by

$$\hat{U}^{\alpha}(t) = \frac{2e^{-\frac{\pi}{2}}}{\pi}\begin{bmatrix} (\sin t+\cos t)[\alpha-1,1-\alpha] \\ (\sin t-\cos t)[\alpha-1,1-\alpha] \\ (\sin t+\cos t)[0.1(\alpha-1),0.1(1-\alpha)] \\ (\sin t-\cos t)[0.1(\alpha-1),0.1(1-\alpha)] \end{bmatrix} - \frac{2}{\pi}\begin{bmatrix} \cos t-\sin t \\ \sin t+\cos t \\ \cos t-\sin t \\ \sin t+\cos t \end{bmatrix},$$

and the corresponding control function at the point $(0.5, 0.25, 0.05, 0.025)$ is given by

$$\hat{U}(t) = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \end{bmatrix} = \frac{2e^{-\frac{\pi}{2}}}{\pi} \begin{bmatrix} 0.5(\sin t + \cos t) \\ 0.75(\sin t - \cos t) \\ 0.5(\sin t + \cos t) \\ 0.75(\sin t - \cos t) \end{bmatrix} - \frac{2}{\pi} \begin{bmatrix} \cos t - \sin t \\ \sin t + \cos t \\ \cos t - \sin t \\ \sin t + \cos t \end{bmatrix}.$$

Now, we state the following theorem relating to controllability of fuzzy dynamical Lyapunov system (5.3.14), (5.3.15), when the input $\hat{U}(t)$ is a fuzzy set in E^{n^2}, with out using fuzzy rule base.

Theorem 5.4.2. The fuzzy system (5.3.14), (5.3.15) is completely controllable if the $n^2 \times n^2$ symmetric controllability matrix

$$\widehat{W(0,T)} = \int_0^T (\psi(T-t)\otimes\phi(T-t))(I_n\otimes F(t))(I_n\otimes F(t))^*(\psi(T-t)\otimes\phi(T-t))^* dt.$$

is non-singular. Furthermore, the fuzzy control $\hat{U}(t)$ transfer the state of the system from $\hat{X}(0) = \hat{X}_0$ to a fuzzy state $\hat{X}(T) = \hat{X}_f$, can be chosen as

$$\hat{U}(t) = \tfrac{1}{T}(I_n \otimes F(t))^{-1}(\psi(T-t) \otimes \phi(T-t))^{-1}\hat{X}_f$$
$$-(I_n \otimes F(t))^*(\psi(T-t) \otimes \phi(T-t))^* W^{-1}(0,T)(\psi(T) \otimes \phi(T))\hat{X}_0.$$

Example 5.4.2. Consider the fuzzy dynamical matrix Lyapunov system (5.1.1) satisfying (5.1.2) with

$$A(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix},$$

$$C(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, T = 1, \text{ and } X_0 = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 1 \end{bmatrix}.$$

Also assume that α-level sets of the final state

$$X_f^\alpha = \begin{bmatrix} [\alpha+1,2] & [\alpha-1,1-\alpha] \\ [2\alpha+1,3] & [0,-1.5(\alpha-1)] \end{bmatrix}.$$

Then the fundamental matrices of (5.2.5), (5.2.6) are

$$\phi(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad \psi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}.$$

Now the fundamental matrix of (5.2.4) is

$$\psi(t)\otimes\phi(t) = \begin{bmatrix} e^t & te^t & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^t & te^t \\ 0 & 0 & 0 & e^t \end{bmatrix} = e^t \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider

$$(\psi(1-t)\otimes\phi(1-t))(I_n\otimes F(t))(I_n\otimes F(t))^*(\psi(1-t)\otimes\phi(1-t))^*$$

$$= e^{1-t}\begin{bmatrix} 1 & 1-t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1-t \\ 0 & 0 & 0 & 1 \end{bmatrix} e^t \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} e^t \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{1-t}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1-t & 1 \end{bmatrix} = e^2 \begin{bmatrix} t^2-2t+2 & 1-t & 0 & 0 \\ 1-t & 1 & 0 & 0 \\ 0 & 0 & t^2-2t+2 & 1-t \\ 0 & 0 & 1-t & 1 \end{bmatrix}.$$

Therefore

$$W(0,1)=\int_0^1 e^2\begin{bmatrix} t^2-2t+2 & 1-t & 0 & 0\\ 1-t & 1 & 0 & 0\\ 0 & 0 & t^2-2t+2 & 1-t\\ 0 & 0 & 1-t & 1\end{bmatrix}dt=e^2\begin{bmatrix} \frac{4}{3} & \frac{1}{2} & 0 & 0\\ \frac{1}{2} & 1 & 0 & 0\\ 0 & 0 & \frac{4}{3} & \frac{1}{2}\\ 0 & 0 & \frac{1}{2} & 1\end{bmatrix}.$$

Clearly, it is non-singular.

Thus from Theorem 5.4.2, the input $\hat{U}(t)$ can be chosen by the following α-level sets.

$$\hat{U}^\alpha(t)=e^{-t}\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}e^{t-1}\begin{bmatrix} 1&t-1&0&0\\0&1&0&0\\0&0&1&t-1\\0&0&0&1\end{bmatrix}\begin{bmatrix}[\alpha+1,2]\\ [2\alpha+1,3]\\ [\alpha-1,1-\alpha]\\ [0,-1.5(\alpha-1)]\end{bmatrix}$$

$$-e^{t}\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}e^{1-t}\begin{bmatrix} 1&0&0&0\\1-t&1&0&0\\0&0&1&0\\0&0&1-t&1\end{bmatrix}\frac{12}{13}e^{-2}$$

$$\begin{bmatrix} 1&-\frac{1}{2}&0&0\\-\frac{1}{2}&\frac{4}{3}&0&0\\0&0&1&-\frac{1}{2}\\0&0&-\frac{1}{2}&\frac{4}{3}\end{bmatrix}e\begin{bmatrix} 1&1&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\ \frac{3}{2}\\ 1\\ 1\end{bmatrix}$$

$$=e^{-1}\begin{bmatrix}[(3t-2)+\alpha,2(t-1)\alpha+(t+1)]\\ [2\alpha+1,3]\\ [(\alpha-1)(2.5-1.5t),1-\alpha]\\ [0,-1.5(\alpha-1)]\end{bmatrix}-\begin{bmatrix}\frac{5}{4}\\ -\frac{1}{4}(5t-9)\\ \frac{3}{2}\\ \frac{11-9t}{6}\end{bmatrix}.$$