CHAPTER-4

ON DICHOTOMY AND CONDITIONING FOR TWO POINT BOUNDARY VALUE PROBLEMS ASSOCIATED WITH FIRST ORDER MATRIX LYAPUNOV SYSTEMS

Section 4.1.

The study of conditioning and dichotomy of boundary value problems is an interesting area of current research. In this direction, De Hoog and Mattheij [22], and Murty and Lakshmi [64] have obtained results of this type for two point boundary value problems associated with system of first order matrix differential equations satisfying two point boundary conditions. Further, Murty and Rao [67] studied conditioning for three point boundary value problems associated with system of first order rectangular matrix differential equations. Due to the importance of matrix Lyapunov systems in the theory of differential equations, Murty and Rao [68] studied existence and uniqueness criteria associated with two point boundary value problems by applying the technique of Kronecker product of matrices and with the help of a Green's matrix.

In this chapter we consider the general first order matrix Lyapunov system of the form

$$LX = X'(t) - (A(t)X(t) + X(t)B(t)) = F(t), \quad a \le t \le b \tag{4.1.1}$$

satisfying two point boundary conditions

$$MX(a)N + RX(b)S = Q, \tag{4.1.2}$$

where $A(t), B(t), F(t) \in [L_p(a,b)]^{n\times n}$ for some p satisfying the condition $1 \leq p < \infty$, and M, N, R, S, Q are all of constant square matrices of order n.

In this direction, we investigate the close relationship between the stability bounds of the corresponding Kronecker product two point boundary value problem on the one hand, and the growth behaviour of the fundamental matrix solution on the other hand. We show that moderate stability constants imply a dichotomy with moderate k bound. We also show that condition number is the right criterion to indicate possible error amplification of the perturbed boundary conditions.

In section 4.2 we present some basic definitions and preliminary results relating to existence and uniqueness of solutions of the corresponding Kronecker product two point boundary value problem associated with (4.1.1) satisfying (4.1.2).

In section 4.3 we define and obtain bounds for dichotomy, strong dichotomy and exponential dichotomy.

In section 4.4 we discuss about conditioning of the boundary value problems and present a stability analysis of this algorithm and also show that the condition number is an important quantity in estimating the global error.

Section 4.2.

In this section we convert the given boundary value problem into a Kronecker product two point boundary value problem and obtain existence and uniqueness of solution of two point boundary value problems with the help of a Green's matrix.

Now by applying the Vec operator to the matrix Lyapunov system (4.1.1), satisfying the boundary conditions (4.1.2), and using the properties of Kronecker product of matrices, we have

$$\hat{X}'(t) = H(t)\hat{X}(t) + \hat{F}(t) \tag{4.2.1}$$

satisfying

$$(N^* \otimes M)\hat{X}(a) + (S^* \otimes R)\hat{X}(b) = \hat{Q}, \tag{4.2.2}$$

where $H(t) = (B^* \otimes I_n) + (I_n \otimes A)$, $\hat{X} = \text{Vec}\, X$, $\hat{F} = \text{Vec}\, F$, and $\hat{Q} = \text{Vec}\, Q$.

The corresponding homogeneous system of (4.2.1) is

$$L\hat{X} = \hat{X}'(t) - H(t)\hat{X}(t) = 0. \tag{4.2.3}$$

Lemma 4.2.1. Let $Y(t)$ and $Z(t)$ be the fundamental matrices for the systems

$$X'(t) = A(t)X(t), \tag{4.2.4}$$

and

$$[X^*(t)]' = B^*(t)X^*(t) \tag{4.2.5}$$

respectively. Then the matrix $Z(t) \otimes Y(t)$ is a fundamental matrix of (4.2.3), and every solution of (4.2.3) is of the form $\hat{X}(t) = (Z(t) \otimes Y(t))c$, where c is a n^2-column vector.

Theorem 4.2.1. Let $Y(t)$ and $Z(t)$ be the fundamental matrices for the systems (4.2.4) and (4.2.5), then any solution of non-homogeneous system (4.2.1) is of the form

$$\hat{X}(t) = (Z(t) \otimes Y(t))c + (Z(t) \otimes Y(t)) \int_a^t (Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)ds.$$

Proof. First, we show that any solution of (4.2.1) is of the form $\hat{X}(t) = (Z(t) \otimes Y(t))c + \widetilde{X}(t)$, where $\widetilde{X}(t)$ is a particular solution of (4.2.1) and is given by

$$\widetilde{X}(t) = \int_a^t (Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)ds.$$

Let $u(t)$ be any other solution of (4.2.1), write $w(t) = u(t) - \widetilde{X}(t)$, then w satisfies (4.2.3), hence $w = (Z(t) \otimes Y(t))c$, $u(t) = (Z(t) \otimes Y(t))c + \widetilde{X}(t)$.

Next, we consider the vector $\widetilde{X}(t) = (Z(t) \otimes Y(t))v(t)$, where $v(t)$ is an arbitrary vector to be determined, so as to satisfy equation (4.2.1). Consider

$$\widetilde{X}'(t) = (Z(t) \otimes Y(t))'v(t) + (Z(t) \otimes Y(t))v'(t)$$

$$\Rightarrow H(t)\widetilde{X}(t) + \hat{F}(t) = H(t)(Z(t) \otimes Y(t))v(t) + (Z(t) \otimes Y(t))v'(t)$$

$$\Rightarrow (Z(t) \otimes Y(t))v'(t) = \hat{F}(t)$$

$$\Rightarrow v'(t) = (Z^{-1}(t) \otimes Y^{-1}(t))\hat{F}(t)$$

$$\Rightarrow v(t) = \int_a^t (Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)ds.$$

Hence the desired expression follows immediately.

Definition 4.2.1. The problem (4.2.3) satisfying

$$(N^* \otimes M)\hat{X}(a) + (S^* \otimes R)\hat{X}(b) = 0 \tag{4.2.6}$$

is called a homogeneous Kronecker product boundary value problem. By a solution of this problem we mean a solution of (4.2.3) whose values at 'a' and 'b' are such that the relation (4.2.6) is satisfied.

Definition 4.2.2. The dimension of the solution space of the homogeneous boundary value problem (4.2.3) satisfying (4.2.6) is called the index of compatibility of the problem. If the index of compatibility is zero then we say that the boundary value problem is incompatible.

Definition 4.2.3. If $Y(t)$, $Z(t)$ are fundamental matrices for the systems (4.2.4), (4.2.5), then the matrix

$$D = (N^* \otimes M)(Z(a) \otimes Y(a)) + (S^* \otimes R)(Z(b) \otimes Y(b)) \tag{4.2.7}$$

is called the characteristic matrix for the boundary value problem (4.2.3) and (4.2.6).

Theorem 4.2.2. Let $Y(t)$, $Z(t)$ be the fundamental matrices for the systems (4.2.4), (4.2.5) and suppose that the homogeneous boundary value problem (4.2.3) satisfying (4.2.6) is incompatible. Then there exists a unique solution to two point boundary value problem (4.2.1) satisfying (4.2.2) is of the form

$$\hat{X}(t) = (Z(t) \otimes Y(t))D^{-1}\hat{Q} + \int_a^b G(t,s)\hat{F}(s)ds, \tag{4.2.8}$$

where G is the Green's matrix for the homogeneous boundary value problem, given by

$$G(t,s) = \begin{cases} (Z(t) \otimes Y(t))D^{-1}(N^* \otimes M)(Z(a) \otimes Y(a))(Z^{-1}(s) \otimes Y^{-1}(s)), \\ \qquad a \le s < t \le b, \\ -(Z(t) \otimes Y(t))D^{-1}(S^* \otimes R)(Z(b) \otimes Y(b))(Z^{-1}(s) \otimes Y^{-1}(s)), \\ \qquad a \le t < s \le b. \end{cases} \tag{4.2.9}$$

Proof. From Theorem 4.2.1 any solution of (4.2.1) is of the form

$$\hat{X}(t) = (Z(t) \otimes Y(t))c + (Z(t) \otimes Y(t)) \int_a^t (Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)ds.$$

Substituting the general form of $\hat{X}(t)$ in the boundary condition (4.2.2) and solving for constant vector c, we have

$$c = D^{-1}\hat{Q} - D^{-1}(S^* \otimes R)(Z(b) \otimes Y(b)) \int_a^b (Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)ds.$$

Hence the solution of (4.2.1) satisfying (4.2.2) is given by

$$\hat{X}(t) = (Z(t) \otimes Y(t))D^{-1}\hat{Q} + \int_a^b G(t,s)\hat{F}(s)ds,$$

where $G(t,s)$ is the Green's matrix defined by (4.2.9).

We shall now see how the expression (4.2.8) can be used to examine the conditioning of (4.2.1), (4.2.2). We make use of the following notations. Let

$$\|v\|_p = \left[\int_a^b |v(s)|^p ds\right]^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

and

$$\|v\|_\infty = \sup_{t\in[a,b]} |v(t)|$$

be its limiting value as $p \to \infty$. Then we have from (4.2.8)

$$\|\hat{X}\| = \|\hat{X}\|_\infty \le \eta|\hat{Q}| + \gamma_q\|\hat{F}\|_p, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{4.2.10}$$

where

$$\eta = \|(Z(t) \otimes Y(t))D^{-1}\,|, \tag{4.2.11}$$

and

$$\gamma_q = \sup_{t\in[a,b]} \left[\int_a^b |G(t,s)|^q ds\right]^{\frac{1}{q}}. \tag{4.2.12}$$

The most appropriate norm in (4.2.10) actually depends on the problem under consideration. We shall discuss the case when $p = 1$, and all the arguments used here can be extended easily to an arbitrary p, $1 < p < \infty$.

When $p = 1$, (4.2.10)-(4.2.12) reduce to

$$\|\hat{X}\| \le \eta|\hat{Q}| + \gamma\|\hat{F}\|, \tag{4.2.13}$$

$$\eta = \|(Z(t) \otimes Y(t))D^{-1}\|, \tag{4.2.14}$$

and

$$\gamma = \sup_{t,s} |G(t,s)|. \tag{4.2.15}$$

If in addition, we assume that the boundary conditions (4.2.2) are scaled in such a way that

$$(N^*N \otimes MM^*) + (S^*S \otimes RR^*) = I_{n^2},$$

then

$$|(Z(t) \otimes Y(t))D^{-1}|^2 = |G(t,a)G^*(t,a) + G(t,b)G^*(t,b)|,$$

and hence

$$\eta^2 \leq \gamma^2 + \gamma^2$$

$$\eta \leq \sqrt{2}\gamma.$$

Hence the stability constant γ gives a measure for the sensitivity of (4.2.1) satisfying (4.2.2) to the changes in the data. Further, we note from (4.2.14), (4.2.15) that both the fundamental matrix, and the boundary conditions (4.2.2) will actually determine the magnitude of the stability constants η and γ. Thus it is possible to construct systems for which no boundary conditions exist such that η and γ are of moderate size; it is also possible to find boundary conditions for (4.2.1) so that η and γ are large. Hence, if system (4.2.1) can support a well conditioned problem then the conditioning is intimately related to the choice of the boundary conditions.

To simplify the algebra, we investigate the fundamental matrix $(Z(t) \otimes Y(t))$, whose characteristic matrix is the identity. Thus $(Z(t) \otimes Y(t))$

is the fundamental matrix for $L\hat{X} = 0$ for which

$$D = (N^* \otimes M)(Z(a) \otimes Y(a)) + (S^* \otimes R)(Z(b) \otimes Y(b)) = I_{n^2}. \qquad (4.2.16)$$

Then the Green's matrix is given by

$$G(t,s) = \begin{cases} (Z(t) \otimes Y(t))(N^* \otimes M)(Z(a) \otimes Y(a))(Z^{-1}(s) \otimes Y^{-1}(s)), \\ \qquad a \leq s < t \leq b, \\ -(Z(t) \otimes Y(t))(S^* \otimes R)(Z(b) \otimes Y(b))(Z^{-1}(s) \otimes Y^{-1}(s)), \\ \qquad a \leq t < s \leq b. \end{cases} \qquad (4.2.17)$$

Result 4.2.1. The fundamental matrix $(Z(t) \otimes Y(t))$ of $L\hat{X} = 0$, satisfies the following relations ;

(i) $Z(t) \otimes Y(t) = G(t,s)(Z(s) \otimes Y(s)) - G(t,u)(Z(u) \otimes Y(u)),$

$$a \leq s < t \leq u \leq b$$

(ii) $Z^{-1}(t) \otimes Y^{-1}(t) = (Z^{-1}(u) \otimes Y^{-1}(u))G(u,t) - (Z^{-1}(s) \otimes Y^{-1}(s))G(s,t),$

$$a \leq s < t \leq u \leq b.$$

Proof. (i) From (4.2.16), we have

$$Z(t) \otimes Y(t) = (Z(t) \otimes Y(t))[(N^* \otimes M)(Z(a) \otimes Y(a)) + (S^* \otimes R)(Z(b) \otimes Y(b))]$$

$$= (Z(t) \otimes Y(t))(N^* \otimes M)(Z(a) \otimes Y(a))(Z^{-1}(s) \otimes Y^{-1}(s))(Z(s) \otimes Y(s))$$

$$+(Z(t) \otimes Y(t))(S^* \otimes R)(Z(b) \otimes Y(b))(Z^{-1}(u) \otimes Y^{-1}(u))(Z(u) \otimes Y(u))$$

$$= G(t,s)(Z(s) \otimes Y(s)) - G(t,u)(Z(u) \otimes Y(u)).$$

(ii) Again from (4.2.16), we have

$$Z^{-1}(t) \otimes Y^{-1}(t) = [(N^* \otimes M)(Z(a) \otimes Y(a)) + (S^* \otimes R)(Z(b) \otimes Y(b))](Z^{-1}(t) \otimes Y^{-1}(t))$$

$$= (Z^{-1}(u) \otimes Y^{-1}(u))(Z(u) \otimes Y(u))(N^* \otimes M)(Z(a) \otimes Y(a))(Z^{-1}(t) \otimes Y^{-1}(t))$$

$$+(Z^{-1}(s) \otimes Y^{-1}(s))(Z(s) \otimes Y(s))(S^* \otimes R)(Z(b) \otimes Y(b))(Z^{-1}(t) \otimes Y^{-1}(t))$$

$$= (Z^{-1}(u) \otimes Y^{-1}(u))G(u,t) - (Z^{-1}(s) \otimes Y^{-1}(s))G(s,t).$$

The result now follows from the fact that any fundamental matrix $(Z(t) \otimes Y(t))$ of $L\hat{X} = 0$ can be represented as $(Z(t) \otimes Y(t)) = (Z_1(s) \otimes Y_1(s))(C_1 \otimes C_2)$ for some constant non-singular matrices C_1 and C_2.

Section 4.3.

In this section first, we give basic definitions about dichotomy, strong dichotomy, and exponential dichotomy. Next, we show that the difference between dichotomy and strong dichotomy. Further, we obtain bounds for dichotomy, strong dichotomy, and exponential dichotomy.

Definition 4.3.1. We say that the solution space Ω of $L\hat{X} = 0$ is dichotomic, if there exists a splitting $\Omega = \Omega_1 \oplus \Omega_2$, and a constant k such that

$$\phi \in \Omega_1 \quad \Rightarrow \quad \frac{|\phi(t)|}{|\phi(s)|} \le k, \quad \text{for} \quad t \ge s,$$

$$\phi \in \Omega_2 \quad \Rightarrow \quad \frac{|\phi(t)|}{|\phi(s)|} \le k, \quad \text{for} \quad t \le s.$$

Note 4.3.1. If P_1, P_2 are projections for the corresponding fundamental matrices $Z(t)$, $Y(t)$ of (4.2.4) and (4.2.5) respectively, then $(P_1 \otimes P_2)$ is the projection matrix corresponding to $(Z(t) \otimes Y(t))$.

Equivalently, if for every fundamental matrix $(Z(t) \otimes Y(t))$, there exists a projection $P_1 \otimes P_2 \in \mathbf{R}^{n^2 \times n^2}$ such that the solution space has the form $\Omega = \Omega_1 \oplus \Omega_2$, with

$$\Omega_1 = \{(Z(t) \otimes Y(t))(P_1 \otimes P_2)c / \; c \in \mathbf{R}^{n^2}\}, \tag{4.3.1}$$

$$\Omega_2 = \{(Z(t) \otimes Y(t))\,(I_{n^2} - (P_1 \otimes P_2))\,c /\ c \in \mathbf{R}^{n^2}\}, \tag{4.3.2}$$

then we say that the two point boundary value problem is dichotomic.

Definition 4.3.2. We say that the solution space of $L\hat{X} = 0$ is strong dichotomic, if there exists a constant k, and a projection $P_1 \otimes P_2 \in \mathbf{R}^{n^2 \times n^2}$ such that for a fixed fundamental matrix $Z(t) \otimes Y(t)$,

$$|(Z(t) \otimes Y(t))(P_1 \otimes P_2)(Z^{-1}(s) \otimes Y^{-1}(s))| \leq k,\ \ t \geq s,$$

$$|(Z(t) \otimes Y(t))\,(I_{n^2} - (P_1 \otimes P_2))\,(Z^{-1}(s) \otimes Y^{-1}(s))| \leq k,\ \ t \leq s.$$

Definition 4.3.3. The solution space of $L\hat{X} = 0$ is said to be exponentially dichotomic, if there exists a constant $k > 0$, positive constants λ, μ, and a projection $P_1 \otimes P_2 \in \mathbf{R}^{n^2 \times n^2}$ such that

$$|(Z(t) \otimes Y(t))(P_1 \otimes P_2)(Z^{-1}(s) \otimes Y^{-1}(s))| \leq ke^{\lambda(s-t)},\ \ t \geq s,$$

$$|(Z(t) \otimes Y(t))\,(I_{n^2} - (P_1 \otimes P_2))\,(Z^{-1}(s) \otimes Y^{-1}(s))| \leq ke^{\mu(t-s)},\ \ t \leq s.$$

In the analysis of numerical schemes for boundary value problems and in the construction of algorithms for their implementation, the concepts of dichotomy and strong dichotomy are used [56]. So it is useful to investigate how these two concepts differ. First, we note the following.

Lemma 4.3.1. Let Ω_1 and Ω_2 be defined as in (4.3.1) and (4.3.2). Then

$$\phi \in \Omega_1 \Rightarrow \frac{|\phi(t)|}{|\phi(s)|} \leq |(Z(t) \otimes Y(t))(P_1 \otimes P_2)(Z^{-1}(s) \otimes Y^{-1}(s))|, \qquad t \geq s,$$

$$\phi \in \Omega_2 \Rightarrow \frac{|\phi(t)|}{|\phi(s)|} \leq |(Z(t) \otimes Y(t))\,(I_{n^2} - (P_1 \otimes P_2))\,(Z^{-1}(s) \otimes Y^{-1}(s))|,\ \ t \leq s.$$

Proof. Let $\phi \in \Omega_1$, then there exists a constant $c_1 \in \mathbf{R}^{n^2}$ such that

$$\phi(t) = (Z(t) \otimes Y(t))(P_1 \otimes P_2)c_1.$$

Thus for all $t \geq s$, we have

$$\begin{aligned}
\frac{|\phi(t)|}{|\phi(s)|} &= \frac{|(Z(t) \otimes Y(t))(P_1 \otimes P_2)c_1|}{|(Z(s) \otimes Y(s))(P_1 \otimes P_2)c_1|} \\
&= \frac{|(Z(t) \otimes Y(t))(P_1 \otimes P_2)(Z^{-1}(s) \otimes Y^{-1}(s))(Z(s) \otimes Y(s))(P_1 \otimes P_2)c_1|}{|(Z(s) \otimes Y(s))(P_1 \otimes P_2)c_1|} \\
&\leq |(Z(t) \otimes Y(t))(P_1 \otimes P_2)(Z^{-1}(s) \otimes Y^{-1}(s))|.
\end{aligned}$$

Let $\phi \in \Omega_2$, then

$$\phi(t) = (Z(t) \otimes Y(t))(I_{n^2} - (P_1 \otimes P_2))c_2, \text{ for some } c_2 \in \mathbf{R}^{n^2}.$$

For $t \leq s$, we have

$$\begin{aligned}
\frac{|\phi(t)|}{|\phi(s)|} &= \frac{|(Z(t) \otimes Y(t))(I_{n^2} - (P_1 \otimes P_2))c_2|}{|(Z(s) \otimes Y(s))(I_{n^2} - (P_1 \otimes P_2))c_2|} \\
&= \frac{|(Z(t) \otimes Y(t))(I_{n^2} - (P_1 \otimes P_2))(Z^{-1}(s) \otimes Y^{-1}(s))(Z(s) \otimes Y(s))c_2|}{|(Z(s) \otimes Y(s))(I_{n^2} - (P_1 \otimes P_2))c_2|} \\
&\leq |(Z(t) \otimes Y(t))(I_{n^2} - (P_1 \otimes P_2))(Z^{-1}(s) \otimes Y^{-1}(s))|.
\end{aligned}$$

Hence strong dichotomy implies dichotomy.

Definition 4.3.4. The angle $0 \leq \theta(t) \leq \pi/2$ between Ω_1 and Ω_2 is defined by

$$\cos\theta(t) = \max_{\substack{|u|=|v|=1 \\ u \in \Omega_1, v \in \Omega_2}} |u^* v|.$$

The main difference between these two notions is that strong dichotomy implies a directional separation between the two subspaces Ω_1 and Ω_2. We state this in the following theorem.

Theorem 4.3.1. Let $|(Z(t) \otimes Y(t))(P_1 \otimes P_2)(Z^{-1}(s) \otimes Y^{-1}(s))| \leq k$, for some k. Then

$$\cot\theta(t) \leq k.$$

Proof. Let $u \in \Omega_1$ and $v \in \Omega_2$ with $|u| = |v| = 1$ be such that $\cos\theta(t) = |u^* v|$. If u is orthogonal to v, the result is obvious. So assume that this is not the case. Now define $\overline{u} = u$, $\overline{v} = -(u^* v)^{-1} v$. Clearly, $\overline{u}$ is orthogonal to $\overline{u} + \overline{v}$, and hence

$$\cot\theta(t) = \frac{|\overline{u}|}{|\overline{u} + \overline{v}|}. \tag{4.3.3}$$

Since $\overline{u} \in \Omega_1$ and $\overline{v} \in \Omega_2$, we have

$$\overline{u} = (Z(t) \otimes Y(t))(P_1 \otimes P_2)c$$

and

$$\overline{v} = (Z(t) \otimes Y(t))\,(I_{n^2} - (P_1 \otimes P_2))\,c,$$

for some $c \in \mathbf{R}^{n^2}$. Substituting these values in (4.3.3), we get

$$\begin{aligned}
\cot\theta(t) &= \frac{|(Z(t) \otimes Y(t))(P_1 \otimes P_2)c|}{|(Z(t) \otimes Y(t))c|} \\
&= \frac{|(Z(t) \otimes Y(t))(P_1 \otimes P_2)(Z^{-1}(s) \otimes Y^{-1}(s))(Z(s) \otimes Y(s))c|}{|(Z(t) \otimes Y(t))c|} \\
&\leq \max_{|\hat{Q}|} \frac{|(Z(t) \otimes Y(t))(P_1 \otimes P_2)(Z^{-1}(s) \otimes Y^{-1}(s))\hat{Q}|}{|\hat{Q}|} \\
&\leq k.
\end{aligned}$$

Note 4.3.2. From Theorem 4.3.1, we note that the angle between two subspaces Ω_1 and Ω_2 cannot become smaller than some threshold value $\cot^{-1} k$.

In general, the boundary conditions (4.2.2) must represent n^2 linearly independent constraints on $\hat{X}(a)$ and $\hat{X}(b)$. Thus it is necessary that

$$\operatorname{rank}[N^* \otimes M, S^* \otimes R] = n^2. \tag{4.3.4}$$

Suppose that rank $(S^* \otimes R) = m < n^2$, then there exists a $n^2 \times n^2$ non-singular matrix W representing an appropriate linear combination of the rows of $(S^* \otimes R)$ such that

$$W(S^* \otimes R) = \begin{pmatrix} 0 \\ T_b \end{pmatrix} \begin{matrix} \}n^2 - m \\ \}m \end{matrix}, \quad \text{where rank } T_b = m.$$

If we introduce the partitions;

$$W\hat{Q} = \begin{pmatrix} \hat{Q}_a \\ \hat{Q}_b \end{pmatrix} \begin{matrix} \}n^2 - m \\ \}m \end{matrix}, \quad W(N^* \otimes M) = \begin{pmatrix} T_a \\ T_{ba} \end{pmatrix} \begin{matrix} \}n^2 - m \\ \}m \end{matrix}:$$

where rank $T_a = n^2 - m$, then we find that

$$W\left[(N^* \otimes M)\hat{X}(a) + (S^* \otimes R)\hat{X}(b)\right] = W\hat{Q}$$

is equivalent to

$$\left.\begin{aligned} T_a\hat{X}(a) &= \hat{Q}_a, \\ T_{ba}\hat{X}(a) + T_b\hat{X}(b) &= \hat{Q}_b. \end{aligned}\right\} \tag{4.3.5}$$

Obviously, if rank $(N^* \otimes M) = q < n^2$, we obtain by an analogous procedure, but with different matrices and vectors,

$$\left.\begin{aligned} T_a\hat{X}(a) + T_{ab}\hat{X}(b) &= \hat{Q}_a, \\ T_b\hat{X}(b) &= \hat{Q}_b. \end{aligned}\right\} \tag{4.3.6}$$

Either of the forms (4.3.5), (4.3.6) consists of partially separated boundary conditions. If $T_{ab} = 0$ and $T_{ba} = 0$, then the boundary conditions are said to be separated, which are the most naturally occurring forms in applications.

Theorem 4.3.2. If the boundary conditions are separable in the sense

$$\text{rank}(N^* \otimes M) = n^2 - m, \ \text{rank}(S^* \otimes R) = m,$$

then there exists a projection P such that

$$|(Z(t) \otimes Y(t))P(Z^{-1}(s) \otimes Y^{-1}(s))| \leq \gamma, \ \ t \geq s,$$

$$|(Z(t) \otimes Y(t))\,(I_{n^2} - P)\,(Z^{-1}(s) \otimes Y^{-1}(s))| \leq \gamma, \ \ t \leq s,$$

where γ is the stability constant given by (4.2.15).

Proof. First, we show that $P = (N^* \otimes M)(Z(a) \otimes Y(a))$ is a projection. Let E be an orthogonal matrix such that the last $n^2 - m$ rows of $(E \otimes I_n)(S^* \otimes R)$ are zero. Then

$$(E \otimes I_n)\,[(N^* \otimes M)(Z(a) \otimes Y(a)) + (S^* \otimes R)(Z(b) \otimes Y(b))]\,(E \otimes I_n)^* = I_{n^2}.$$

On equating the last $n^2 - m$ rows of the above equation, we find that

$$\widetilde{P} = (E \otimes I_n)(N^* \otimes M)(Z(a) \otimes Y(a))(E \otimes I_n)^*$$

has the following structure;

$$\widetilde{P} = \begin{pmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ 0 & I_{n^2-m} \end{pmatrix}.$$

Since rank $\widetilde{P} = n^2 - m$, we must have $\widetilde{P}_{11} = 0$, and hence $\widetilde{P}^2 = \widetilde{P}$. Thus

$$P^2 = (E \otimes I_n)^* \widetilde{P}^2 (E \otimes I_n) = (E \otimes I_n)^* \widetilde{P} (E \otimes I_n) = P.$$

Thus $P = (N^* \otimes M)(Z(a) \otimes Y(a))$ is a projection. The proof now follows from (4.2.16) on noting that

$$G(t,s) = \begin{cases} (Z(t) \otimes Y(t))P(Z^{-1}(s) \otimes Y^{-1}(s)), & s < t, \\ -(Z(t) \otimes Y(t))\,(I_{n^2} - P)\,(Z^{-1}(s) \otimes Y^{-1}(s)), & t < s. \end{cases}$$

From the above theorem, we note that if the boundary conditions are separable, then a strong dichotomy exists when $k = \gamma$. It follows from Lemma 4.3.1 that the same result holds with our weaker version of the dichotomy.

In order to construct separable boundary conditions, we monitor the growth of solutions over the entire interval. Let the singular value decomposition of $(Z(b)\otimes Y(b))(Z^{-1}(a)\otimes Y^{-1}(a))$ be given by

$$(Z(b)\otimes Y(b))(Z^{-1}(a)\otimes Y^{-1}(a)) = U\mathbf{D}V^*,$$

where U and V are orthogonal matrices, and $\mathbf{D}$ is a positive diagonal matrix with ordered elements. We use the following notations;

$$\mathbf{D} = \text{diag}\left(d_1^{-1}, d_2^{-1}, \ldots, d_m^{-1}, d_{m+1}, \ldots, d_{n^2}\right)$$

with $0 < d_i \leq 1$, $i = 1, 2, \ldots, n^2$,

$$\mathbf{D}_1 = \text{diag}\left(d_1, d_2, \ldots d_m, 1, 1, \ldots, 1\right),$$

$$\mathbf{D}_2 = \text{diag}\left(1, 1, \ldots, 1, d_{m+1}, \ldots, d_{n^2}\right),$$

and

$$\overline{P} = \begin{pmatrix} 0 & 0 \\ 0 & I_{n^2-m} \end{pmatrix}. \tag{4.3.7}$$

Now we define the separated boundary conditions specified by

$$\widetilde{N}^*\otimes\widetilde{M} = \overline{P}V^* \text{ and } \widetilde{S}^*\otimes\widetilde{R} = \left(I_{n^2} - \overline{P}\right)U^*. \tag{4.3.8}$$

It is easy to verify with the structure of $\overline{P}$ that

$$(\widetilde{N}^*\otimes\widetilde{M})(\widetilde{Z}(a)\otimes\widetilde{Y}(a)) + (\widetilde{S}^*\otimes\widetilde{R})(\widetilde{Z}(b)\otimes\widetilde{Y}(b)) = I_{n^2},$$

where

$$\begin{aligned}\widetilde{Z}(t)\otimes\widetilde{Y}(t) &= (Z(t)\otimes Y(t))(Z^{-1}(a)\otimes Y^{-1}(a))V\mathbf{D}_1 \\ &= (Z(t)\otimes Y(t))(Z^{-1}(b)\otimes Y^{-1}(b))U\mathbf{D}_2.\end{aligned} \tag{4.3.9}$$

The corresponding Green's matrix is

$$\widetilde{G}(t,s) = \begin{cases} (\widetilde{Z}(t)\otimes\widetilde{Y}(t))(\widetilde{N}^*\otimes\widetilde{M})(\widetilde{Z}(a)\otimes\widetilde{Y}(a))(\widetilde{Z}^{-1}(s)\otimes\widetilde{Y}^{-1}(s)), \\ \qquad\qquad t>s, \\ -(\widetilde{Z}(t)\otimes\widetilde{Y}(t))(\widetilde{S}^*\otimes\widetilde{R})(\widetilde{Z}(b)\otimes\widetilde{Y}(b))(\widetilde{Z}^{-1}(s)\otimes\widetilde{Y}^{-1}(s)), \\ \qquad\qquad t<s. \end{cases} \tag{4.3.10}$$

Now we establish the properties of the fundamental matrix $(\widetilde{Z}(t)\otimes\widetilde{Y}(t))$ in terms of the Green's matrix (4.2.17).

Result 4.3.1. For the fundamental matrix $(\widetilde{Z}(t)\otimes\widetilde{Y}(t))$ given in (4.3.9), the following relations hold good;

(i) $\widetilde{Z}(t)\otimes\widetilde{Y}(t) = G(t,s)(\widetilde{Z}(s)\otimes\widetilde{Y}(s)) - G(t,u)(\widetilde{Z}(u)\otimes\widetilde{Y}(u)),$

$$a\le s<t\le u\le b,$$

(ii) $\widetilde{Z}^{-1}(t)\otimes\widetilde{Y}^{-1}(t) = (\widetilde{Z}^{-1}(u)\otimes\widetilde{Y}^{-1}(u))G(u,t) - (\widetilde{Z}^{-1}(s)\otimes\widetilde{Y}^{-1}(s))G(s,t),$

$$a\le s<t\le u\le b,$$

(iii) $(\widetilde{Z}(t)\otimes\widetilde{Y}(t))(\widetilde{Z}^{-1}(u)\otimes\widetilde{Y}^{-1}(u))G(u,s) = G(t,s).$

Proof. (i) From Result 4.2.1 (i), we have

$$Z(t)\otimes Y(t) = G(t,s)(Z(s)\otimes Y(s)) - G(t,u)(Z(u)\otimes Y(u)).$$

Since $(Z^{-1}(a)\otimes Y^{-1}(a))V\mathbf{D}_1$ is nonsingular, from (4.3.9)

$$Z(t)\otimes Y(t) = (\widetilde{Z}(t)\otimes\widetilde{Y}(t))\left((Z^{-1}(a)\otimes Y^{-1}(a))V\mathbf{D}_1\right)^{-1},$$

we have

$$(\widetilde{Z}(t)\otimes\widetilde{Y}(t))\left((Z^{-1}(a)\otimes Y^{-1}(a))V\mathbf{D}_1\right)^{-1}$$

$$= G(t,s)(\widetilde{Z}(s)\otimes\widetilde{Y}(s))\left((Z^{-1}(a)\otimes Y^{-1}(a))V\mathbf{D}_1\right)^{-1}$$

$$-G(t,u)(\widetilde{Z}(u)\otimes\widetilde{Y}(u))\left((Z^{-1}(a)\otimes Y^{-1}(a))V\mathbf{D}_1\right)^{-1}.$$

Thus

$$\widetilde{Z}(t)\otimes\widetilde{Y}(t) = G(t,s)(\widetilde{Z}(s)\otimes\widetilde{Y}(s)) - G(t,u)(\widetilde{Z}(u)\otimes\widetilde{Y}(u)).$$

(ii) From Result 4.2.1 (ii)

$$Z^{-1}(t)\otimes Y^{-1}(t) = (Z^{-1}(u)\otimes Y^{-1}(u))G(u,t) - (Z^{-1}(s)\otimes Y^{-1}(s))G(s,t).$$

Again $(Z^{-1}(a)\otimes Y^{-1}(a))V\mathbf{D}_1$ is nonsingular and from (4.3.9)

$$Z^{-1}(t)\otimes Y^{-1}(t) = \left((Z^{-1}(a)\otimes Y^{-1}(a))V\mathbf{D}_1\right)(\widetilde{Z}^{-1}(t)\otimes\widetilde{Y}^{-1}(t)).$$

Then

$$\left((Z^{-1}(a)\otimes Y^{-1}(a))V\mathbf{D}_1\right)(\widetilde{Z}^{-1}(t)\otimes\widetilde{Y}^{-1}(t))$$

$$= \left((Z^{-1}(a)\otimes Y^{-1}(a))V\mathbf{D}_1\right)(\widetilde{Z}^{-1}(u)\otimes\widetilde{Y}^{-1}(u))G(u,t)$$

$$- \left((Z^{-1}(a)\otimes Y^{-1}(a))V\mathbf{D}_1\right)(\widetilde{Z}^{-1}(s)\otimes\widetilde{Y}^{-1}(s))G(s,t).$$

Thus

$$\widetilde{Z}^{-1}(t)\otimes\widetilde{Y}^{-1}(t) = (\widetilde{Z}^{-1}(u)\otimes\widetilde{Y}^{-1}(u))G(u,t) - (\widetilde{Z}^{-1}(s)\otimes\widetilde{Y}^{-1}(s))G(s,t).$$

(iii) Since $\widetilde{Z}(t)\otimes\widetilde{Y}(t) = (Z(t)\otimes Y(t))(Z^{-1}(a)\otimes Y^{-1}(a))V\mathbf{D}_1$, we have

$$(\widetilde{Z}(t)\otimes\widetilde{Y}(t))(\widetilde{Z}^{-1}(u)\otimes\widetilde{Y}^{-1}(u))G(u,s)$$

$$= (Z(t)\otimes Y(t))(Z^{-1}(a)\otimes Y^{-1}(a))V\mathbf{D}_1\mathbf{D}_1^{-1}V^{-1}(Z(a)\otimes Y(a))(Z^{-1}(u)\otimes Y^{-1}(u))$$

$$\left[(Z(u)\otimes Y(u))(N^*\otimes M)(Z(a)\otimes Y(a))(Z^{-1}(s)\otimes Y^{-1}(s))\right]$$

$$= (Z(t)\otimes Y(t))(N^*\otimes M)(Z(a)\otimes Y(a))(Z^{-1}(s)\otimes Y^{-1}(s))$$

$$= G(t,s),\ \ a\le s<t\le b.$$

Since $\widetilde{Z}(t)\otimes\widetilde{Y}(t) = (Z(t)\otimes Y(t))(Z^{-1}(b)\otimes Y^{-1}(b))U\mathbf{D}_2$, we have

$$(\widetilde{Z}(t)\otimes\widetilde{Y}(t))(\widetilde{Z}^{-1}(u)\otimes\widetilde{Y}^{-1}(u))G(u,s)$$

$$= (Z(t)\otimes Y(t))(Z^{-1}(b)\otimes Y^{-1}(b))U\mathbf{D}_2\mathbf{D}_2^{-1}U^{-1}(Z(b)\otimes Y(b))(Z^{-1}(u)\otimes Y^{-1}(u))$$

$$\left[-(Z(u)\otimes Y(u))(S^*\otimes R)(Z(b)\otimes Y(b))(Z^{-1}(s)\otimes Y^{-1}(s))\right]$$

$$= -(Z(t)\otimes Y(t))(S^*\otimes R)(Z(b)\otimes Y(b))(Z^{-1}(s)\otimes Y^{-1}(s))$$

$$= G(t,s),\ \ a\le t<s\le b.$$

The following theorem establishes the relationship between the Green's matrices $\widetilde{G}$, G given in (4.3.10) and (4.2.17) respectively.

Theorem 4.3.3.

$$\widetilde{G}(t,s) = G(t,s) - (\widetilde{Z}(t)\otimes\widetilde{Y}(t))\left[(\widetilde{N}^*\otimes\widetilde{M})G(a,s) + (\widetilde{S}^*\otimes\widetilde{R})G(b,s)\right].$$

Proof. For $t>s$, using Result 4.3.1(ii), (iii), we have

$$\widetilde{G}(t,s) = (\widetilde{Z}(t)\otimes\widetilde{Y}(t))(\widetilde{N}^*\otimes\widetilde{M})(\widetilde{Z}(a)\otimes\widetilde{Y}(a))(\widetilde{Z}^{-1}(s)\otimes\widetilde{Y}^{-1}(s))$$

$$= (\widetilde{Z}(t)\otimes\widetilde{Y}(t))(\widetilde{N}^*\otimes\widetilde{M})(\widetilde{Z}(a)\otimes\widetilde{Y}(a))[(\widetilde{Z}^{-1}(b)\otimes\widetilde{Y}^{-1}(b))G(b,s)$$
$$-(\widetilde{Z}^{-1}(a)\otimes\widetilde{Y}^{-1}(a))G(a,s)]$$

$$= (\widetilde{Z}(t)\otimes\widetilde{Y}(t))\left[I_{n^2} - (\widetilde{S}^*\otimes\widetilde{R})(\widetilde{Z}(b)\otimes\widetilde{Y}(b))\right](\widetilde{Z}^{-1}(b)\otimes\widetilde{Y}^{-1}(b))G(b,s)$$
$$-(\widetilde{Z}(t)\otimes\widetilde{Y}(t))(\widetilde{N}^*\otimes\widetilde{M})G(a,s)$$

$$= (\widetilde{Z}(t)\otimes\widetilde{Y}(t))(\widetilde{Z}^{-1}(b)\otimes\widetilde{Y}^{-1}(b))G(b,s) - (\widetilde{Z}(t)\otimes\widetilde{Y}(t))(\widetilde{S}^*\otimes\widetilde{R})G(b,s)$$
$$-(\widetilde{Z}(t)\otimes\widetilde{Y}(t))(\widetilde{N}^*\otimes\widetilde{M})G(a,s)$$

$$= G(t,s) - (\widetilde{Z}(t) \otimes \widetilde{Y}(t)) \left[(\widetilde{N^*} \otimes \widetilde{M}) G(a,s) + (\widetilde{S}^* \otimes \widetilde{R}) G(b,s) \right].$$

Similarly for $t < s$, we have

$$\widetilde{G}(t,s) = -(\widetilde{Z}(t) \otimes \widetilde{Y}(t))(\widetilde{S}^* \otimes \widetilde{R})(\widetilde{Z}(b) \otimes \widetilde{Y}(b))(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))$$

$$= -(\widetilde{Z}(t) \otimes \widetilde{Y}(t))(\widetilde{S}^* \otimes \widetilde{R})(\widetilde{Z}(b) \otimes \widetilde{Y}(b))[(\widetilde{Z}^{-1}(b) \otimes \widetilde{Y}^{-1}(b))G(b,s)$$
$$-(\widetilde{Z}^{-1}(a) \otimes \widetilde{Y}^{-1}(a))G(a,s)]$$

$$= (\widetilde{Z}(t) \otimes \widetilde{Y}(t)) \left[I_{n^2} - (\widetilde{N^*} \otimes \widetilde{M})(\widetilde{Z}(a) \otimes \widetilde{Y}(a)) \right] (\widetilde{Z}^{-1}(a) \otimes \widetilde{Y}^{-1}(a))G(a,s)$$
$$-(\widetilde{Z}(t) \otimes \widetilde{Y}(t))(\widetilde{S}^* \otimes \widetilde{R})G(b,s)$$

$$= (\widetilde{Z}(t) \otimes \widetilde{Y}(t))(\widetilde{Z}^{-1}(a) \otimes \widetilde{Y}^{-1}(a))G(a,s) - (\widetilde{Z}(t) \otimes \widetilde{Y}(t))(\widetilde{N^*} \otimes \widetilde{M})G(a,s)$$
$$-(\widetilde{Z}(t) \otimes \widetilde{Y}(t))(\widetilde{S}^* \otimes \widetilde{R})G(b,s)$$

$$= G(t,s) - (\widetilde{Z}(t) \otimes \widetilde{Y}(t)) \left[(\widetilde{N^*} \otimes \widetilde{M}) G(a,s) + (\widetilde{S}^* \otimes \widetilde{R}) G(b,s) \right].$$

Hence the theorem.

From (4.3.8) and (4.3.9), we have

$$(\widetilde{N^*} \otimes \widetilde{M})(\widetilde{Z}(a) \otimes \widetilde{Y}(a)) = \overline{P}V^*(Z(a) \otimes Y(a))(Z^{-1}(a) \otimes Y^{-1}(a))V\mathbf{D}$$
$$= \overline{P} \tag{4.3.11}$$

and

$$(\widetilde{S}^* \otimes \widetilde{R})(\widetilde{Z}(b) \otimes \widetilde{Y}(b)) = I_{n^2} - \overline{P}. \tag{4.3.12}$$

Hence the associated Green's matrix for the boundary conditions

$$(\widetilde{N^*} \otimes \widetilde{M})\hat{X}(a) + (\widetilde{S}^* \otimes \widetilde{R})\hat{X}(b) = \hat{Q}$$

is obtained by substituting (4.3.11) and (4.3.12) in (4.3.10);

$$\widetilde{G}(t,s) = \begin{cases} (\widetilde{Z}(t) \otimes \widetilde{Y}(t))\overline{P}(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s)), & t > s \\ -(\widetilde{Z}(t) \otimes \widetilde{Y}(t))\left(I_{n^2} - \overline{P}\right)(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s)), & t < s. \end{cases}$$

Now we are in a position to give the following estimates;

Theorem 4.3.4. For $\gamma = \sup_{t,s} |G(t,s)|$

(i) $|(\widetilde{Z}(b) \otimes \widetilde{Y}(b))\overline{P}(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))| = |\widetilde{G}(b,s)| \leq 2\gamma$,

(ii) $|(\widetilde{Z}(a) \otimes \widetilde{Y}(a))\left(I_{n^2} - \overline{P}\right)(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))| = |\widetilde{G}(a,s)| \leq 2\gamma$, and

(iii) $|\widetilde{Z}(t) \otimes \widetilde{Y}(t)| \leq 2\gamma\ell$, where $\ell = \max\left\{|\widetilde{Z}(a)||\widetilde{Y}(a)|, |\widetilde{Z}(b)||\widetilde{Y}(b)|\right\}$.

Proof. (i) Consider

$$\begin{aligned}
|\widetilde{G}(b,s)| &= |(\widetilde{Z}(b)\otimes\widetilde{Y}(b))\overline{P}(\widetilde{Z}^{-1}(s)\otimes\widetilde{Y}^{-1}(s))| \\
&= |(\widetilde{Z}(b)\otimes\widetilde{Y}(b))\overline{P}[(\widetilde{Z}^{-1}(b)\otimes\widetilde{Y}^{-1}(b))G(b,s) - (\widetilde{Z}^{-1}(a)\otimes\widetilde{Y}^{-1}(a))G(a,s)]| \\
&\leq |G(b,s)| + |(\widetilde{Z}(b) \otimes \widetilde{Y}(b))\overline{P}(\widetilde{Z}^{-1}(a) \otimes \widetilde{Y}^{-1}(a))G(a,s)| \\
&\leq \gamma + |U\mathbf{D}_2\overline{P}\mathbf{D}_1^{-1}V^{-1}|\gamma \\
&= \gamma + |U\mathbf{D}\overline{P}V^{-1}|\gamma \leq \gamma + \gamma = 2\gamma.
\end{aligned}$$

(ii) Consider

$$\begin{aligned}
|\widetilde{G}(a,s)| &= |-(\widetilde{Z}(a)\otimes\widetilde{Y}(a))(I_{n^2}-\overline{P})(\widetilde{Z}^{-1}(s)\otimes\widetilde{Y}^{-1}(s))| \\
&= |(\widetilde{Z}(a) \otimes \widetilde{Y}(a))(I_{n^2} - \overline{P})(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))| \\
&= |(\widetilde{Z}(a)\otimes\widetilde{Y}(a))(I_{n^2}-\overline{P})[(\widetilde{Z}^{-1}(b)\otimes\widetilde{Y}^{-1}(b))G(b,s) - (\widetilde{Z}^{-1}(a)\otimes\widetilde{Y}^{-1}(a))G(a,s)]| \\
&\leq |(I_{n^2}-\overline{P})G(a,s)| + |(\widetilde{Z}(a)\otimes\widetilde{Y}(a))(I_{n^2}-\overline{P})(\widetilde{Z}^{-1}(b)\otimes\widetilde{Y}^{-1}(b))G(b,s)| \\
&\leq |G(a,s)| + |V\mathbf{D}_1(I_{n^2} - \overline{P})\mathbf{D}_2^{-1}U^{-1}||G(b,s)|
\end{aligned}$$

$= \gamma + |V\mathbf{D}(I_{n^2} - \overline{P})U^{-1}|\gamma \leq \gamma + \gamma = 2\gamma.$

(iii) From Result 4.3.1(i), we have

$$\widetilde{Z}(t) \otimes \widetilde{Y}(t) = G(t,a)(\widetilde{Z}(a) \otimes \widetilde{Y}(a)) - G(t,b)(\widetilde{Z}(b) \otimes \widetilde{Y}(b)),$$

and hence

$$\begin{aligned}|\widetilde{Z}(t) \otimes \widetilde{Y}(t)| &\leq |G(t,a)||\widetilde{Z}(a)||\widetilde{Y}(a)| + |G(t,b)||\widetilde{Z}(b)||\widetilde{Y}(b)| \\ &\leq \gamma\left(|\widetilde{Z}(a)||\widetilde{Y}(a)| + |\widetilde{Z}(b)||\widetilde{Y}(b)|\right) \\ &\leq 2\gamma\ell.\end{aligned}$$

To establish results on strong dichotomy, we need the following result.

Result 4.3.2.

(i) $|\widetilde{G}(t,s)| \leq \gamma + 4\gamma^2\ell,$

(ii) $|\widetilde{G}(t,s)| \leq \gamma + 2\gamma\xi$, where $\xi = \eta\max\{|\widetilde{N}^*||\widetilde{M}|, |\widetilde{S}^*||\widetilde{R}|\}$.

Proof. (i) From Theorem 4.3.3, we have

$$\begin{aligned}\widetilde{G}(t,s) &= G(t,s) - (\widetilde{Z}(t) \otimes \widetilde{Y}(t))\left[(\widetilde{N}^* \otimes \widetilde{M})G(a,s) + (\widetilde{S}^* \otimes \widetilde{R})G(b,s)\right] \\ &= G(t,s) - (\widetilde{Z}(t) \otimes \widetilde{Y}(t))\left[\overline{P}V^*G(a,s) + \left(I_{n^2} - \overline{P}\right)U^*G(b,s)\right].\end{aligned}$$

Since $|\widetilde{Z}(t) \otimes \widetilde{Y}(t)| \leq 2\gamma\ell$, $|G(a,s)| \leq \gamma$, $|G(b,s)| \leq \gamma$, we have

$$\begin{aligned}|\widetilde{G}(t,s)| &\leq \gamma + 2\gamma^2\ell + 2\gamma^2\ell \\ &= \gamma + 4\gamma^2\ell.\end{aligned}$$

(ii) Since $\eta = |\widetilde{Z}(t) \otimes \widetilde{Y}(t)|$, we have

$$|\widetilde{G}(t,s)| = |G(t,s) - (\widetilde{Z}(t) \otimes \widetilde{Y}(t))\left[(\widetilde{N}^* \otimes \widetilde{M})G(a,s) + (\widetilde{S}^* \otimes \widetilde{R})G(b,s)\right]|$$

$$\le |G(t,s)| + |\widetilde{Z}(t) \otimes \widetilde{Y}(t)| \left[|(\widetilde{N}^* \otimes \widetilde{M})||G(a,s)| + |(\widetilde{S}^* \otimes \widetilde{R})||G(b,s)|\right]$$

$$\le \gamma + \eta \left[|N^*||M|\gamma + |S^*||R|\gamma\right]$$

$$\le \gamma + 2\gamma\xi.$$

From this result, we have the following estimates for the strong dichotomy.

Theorem 4.3.5.

(i) $|(\widetilde{Z}(t) \otimes \widetilde{Y}(t))\overline{P}(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))| \le \gamma + 4\gamma^2\ell, t > s$

(ii) $|(\widetilde{Z}(t) \otimes \widetilde{Y}(t)) \left(I_{n^2} - \overline{P}\right) (\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))| \le \gamma + 4\gamma^2\ell, t < s$

(iii) $|(\widetilde{Z}(t) \otimes \widetilde{Y}(t))\overline{P}(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))| \le \gamma + 2\gamma\xi, t > s$

(iv) $|(\widetilde{Z}(t) \otimes \widetilde{Y}(t)) \left(I_{n^2} - \overline{P}\right) (\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))| \le \gamma + 2\gamma\xi, t < s.$

Now we are in a position to investigate the stability bounds for exponential dichotomy. For that we replace the condition (4.2.15) by the following conditions;

$$|G(t,s)| \le \gamma e^{\lambda(s-t)}, \qquad t > s, \qquad \lambda > 0, \tag{4.3.13}$$

$$|G(t,s)| \le \gamma e^{\mu(t-s)}, \qquad t < s, \qquad \mu > 0, \tag{4.3.14}$$

and using similar techniques discussed above, we can show that (4.3.13) and (4.3.14) imply an exponentially dichotomic solution space for the two point boundary value problem.

Theorem 4.3.6. Let

$$\alpha(t) = \gamma\ell\left[e^{\lambda(a-t)} + e^{\mu(t-b)}\right],$$

$$\beta(t) = \gamma\left[e^{\lambda(t-b)} + e^{\mu(a-t)}\right],$$

$\overline{P}$ is defined in (4.3.7), $\xi = \eta \max\{|\widetilde{N}^*||\widetilde{M}|, |\widetilde{S}^*||\widetilde{R}|\}$, then the following relations hold good ;

(i) $|(\widetilde{Z}(t) \otimes \widetilde{Y}(t))\overline{P}(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))| \leq \gamma e^{\lambda(s-t)} + \alpha(t)\beta(s), t > s$

(ii) $|(\widetilde{Z}(t) \otimes \widetilde{Y}(t))\left(I_{n^2} - \overline{P}\right)(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))| \leq \gamma e^{\mu(t-s)} + \alpha(t)\beta(s), t < s$

(iii) $|(\widetilde{Z}(t) \otimes \widetilde{Y}(t))\overline{P}(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))| \leq \gamma e^{\lambda(s-t)} + \xi\beta(s), t > s$

(iv) $|(\widetilde{Z}(t) \otimes \widetilde{Y}(t))\left(I_{n^2} - \overline{P}\right)(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))| \leq \gamma e^{\mu(t-s)} + \xi\beta(s), t < s.$

Proof. (i) For $t > s$, consider

$$|(\widetilde{Z}(t) \otimes \widetilde{Y}(t))\overline{P}(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))| = |\widetilde{G}(t,s)|$$

$$= |G(t,s) - (\widetilde{Z}(t) \otimes \widetilde{Y}(t))\left[(\widetilde{N}^* \otimes \widetilde{M})G(a,s) + (\widetilde{S}^* \otimes \widetilde{R})G(b,s)\right]|$$

$$= |G(t,s) - (\widetilde{Z}(t) \otimes \widetilde{Y}(t))\left[\overline{P}V^*G(a,s) + \left(I_{n^2} - \overline{P}\right)U^*G(b,s)\right]|$$

$$\leq |G(t,s)| + |\left[G(t,a)(\widetilde{Z}(a) \otimes \widetilde{Y}(a)) - G(t,b)(\widetilde{Z}(b) \otimes \widetilde{Y}(b))\right]|$$

$$[\ |G(a,s)| + |G(b,s)|\ ]$$

$$\leq \gamma e^{\lambda(s-t)} + \left[|G(t,a)||\widetilde{Z}(a)||\widetilde{Y}(a))| + |G(t,b)||\widetilde{Z}(b)||\widetilde{Y}(b))|\right]$$

$$\left[\gamma e^{\mu(a-s)} + \gamma e^{\lambda(s-b)}\right]$$

$$\leq \gamma e^{\lambda(s-t)} + \ell\left[\gamma e^{\lambda(a-t)} + \gamma e^{\mu(t-b)}\right]\beta(s)$$

$$= \gamma e^{\lambda(s-t)} + \alpha(t)\beta(s).$$

(ii) For $t < s$, consider

$$|(\widetilde{Z}(t)\otimes\widetilde{Y}(t))\left(I_{n^2} - \overline{P}\right)(\widetilde{Z}^{-1}(s)\otimes\widetilde{Y}^{-1}(s))| = |\widetilde{G}(t,s)|$$

$$\leq |G(t,s)| + |\left[G(t,a)(\widetilde{Z}(a) \otimes \widetilde{Y}(a)) - G(t,b)(\widetilde{Z}(b) \otimes \widetilde{Y}(b))\right]|$$

$$[\ |G(a,s)| + |G(b,s)|\ ]$$

$$\leq \gamma e^{\mu(t-s)} + \left[|G(t,a)||\widetilde{Z}(a)||\widetilde{Y}(a))| + |G(t,b)||\widetilde{Z}(b)||\widetilde{Y}(b))|\right]\left[\gamma e^{\mu(a-s)} + \gamma e^{\lambda(s-b)}\right]$$

$$\leq \gamma e^{\mu(t-s)} + \ell\left[\gamma e^{\lambda(a-t)} + \gamma e^{\mu(t-b)}\right]\beta(s)$$

$$= \gamma e^{\mu(t-s)} + \alpha(t)\beta(s).$$

(iii) For $t > s$, consider

$$|(\widetilde{Z}(t) \otimes \widetilde{Y}(t))\overline{P}(\widetilde{Z}^{-1}(s) \otimes \widetilde{Y}^{-1}(s))| = |\widetilde{G}(t,s)|$$

$$= |G(t,s) - (\widetilde{Z}(t) \otimes \widetilde{Y}(t))\left[(\widetilde{N^*} \otimes \widetilde{M})G(a,s) + (\widetilde{S}^* \otimes \widetilde{R})G(b,s)\right]|$$

$$\leq |G(t,s)| + |\widetilde{Z}(t) \otimes \widetilde{Y}(t)|\left[|(\widetilde{N^*} \otimes \widetilde{M})||G(a,s)| + |(\widetilde{S}^* \otimes \widetilde{R})||G(b,s)|\right]$$

$$\leq \gamma e^{\lambda(s-t)} + \eta\left[|N^*||M|\gamma e^{\mu(a-s)} + |S^*||R|\gamma e^{\lambda(s-b)}\right]$$

$$\leq \gamma e^{\lambda(s-t)} + \gamma\xi\left[e^{\mu(a-s)} + e^{\lambda(s-b)}\right]$$

$$= \gamma e^{\lambda(s-t)} + \xi\beta(s).$$

(iv) For $t < s$, consider

$$|(\widetilde{Z}(t)\otimes\widetilde{Y}(t))\left(I_{n^2} - \overline{P}\right)(\widetilde{Z}^{-1}(s)\otimes\widetilde{Y}^{-1}(s))| = |\widetilde{G}(t,s)|$$

$$\leq |G(t,s)| + |\widetilde{Z}(t) \otimes \widetilde{Y}(t)|\left[|(\widetilde{N^*} \otimes \widetilde{M})||G(a,s)| + |(\widetilde{S}^* \otimes \widetilde{R})||G(b,s)|\right]$$

$$\leq \gamma e^{\mu(t-s)} + \eta\left[|N^*||M|\gamma e^{\mu(a-s)} + |S^*||R\,\gamma e^{\lambda(s-b)}\right]$$

$$\leq \gamma e^{\mu(t-s)} + \gamma\xi\left[e^{\mu(a-s)} + e^{\lambda(s-b)}\right]$$

$$= \gamma e^{\mu(t-s)} + \xi\beta(s).$$

Section 4.4.

In this section we show that the condition number is the right criterion to indicate possible error amplification of the perturbed boundary conditions.

If the solution of the boundary value problem

$$\hat{X}'(t) = H(t)\hat{X}(t) + \hat{F}(t) \tag{4.4.1}$$

satisfying

$$(I_n \otimes M)\hat{X}(a) + (I_n \otimes R)\hat{X}(b) = \hat{Q} \tag{4.4.2}$$

(for convenience taking $N = I_n$ and $S = I_n$ in (4.2.2)) is unique, then the characteristic matrix

$$D = (I_n \otimes M)(Z(a) \otimes Y(a)) + (I_n \otimes R)(Z(b) \otimes Y(b)) \tag{4.4.3}$$

must be non-singular, and in this case the boundary value problem is said to be well-posed.

Definition 4.4.1. The condition number η of the boundary value problem (4.4.1), (4.4.2) is defined as

$$\eta = \sup_{a \leq t \leq b} \|(Z(t) \otimes Y(t))D^{-1}\|.$$

It is easily seen that, the number η is independent of the choice of the fundamental matrix.

We consider the variation $\hat{X}(t)$ of (4.4.1) with respect to the small perturbation in the boundary conditions, the perturbation of (4.4.2) in the form

$$[I_n \otimes (M + \delta M)]\,\hat{X}(a) + [I_n \otimes (R + \delta R)]\,\hat{X}(b) = \hat{Q} + \delta\hat{Q}. \tag{4.4.4}$$

Then the perturbed characteristic matrix

$$D_1 = [I_n \otimes (M + \delta M)]\,(Z(a) \otimes Y(a)) + [I_n \otimes (R + \delta R)]\,(Z(b) \otimes Y(b))$$

$$= [(I_n \otimes M) + (I_n \otimes \delta M)]\,(Z(a) \otimes Y(a)) + [(I_n \otimes R) + (I_n \otimes \delta R)]\,(Z(b) \otimes Y(b))$$

$$= (I_n \otimes M)(Z(a) \otimes Y(a)) + (I_n \otimes R)(Z(b) \otimes Y(b))$$

$$+ (I_n \otimes \delta M)(Z(a) \otimes Y(a)) + (I_n \otimes \delta R)(Z(b) \otimes Y(b))$$

$= D + \delta D.$

Assume that D_1 is non-singular. Let $\widetilde{X}(t)$ be the unique solution of (4.4.1) satisfying (4.4.4).

Lemma 4.4.1. $\|\delta D D^{-1}\| \leq (\|\delta M\| + \|\delta R\|)\,\eta.$

Proof. Consider

$$\|\delta D D^{-1}\| = \|\left[(I_n \otimes \delta M)(Z(a) \otimes Y(a)) + (I_n \otimes \delta R)(Z(b) \otimes Y(b))\right] D^{-1}\|$$

$$\leq \|(I_n \otimes \delta M)\|\|(Z(a) \otimes Y(a))D^{-1}\| + \|(I_n \otimes \delta R)\|\|(Z(b) \otimes Y(b))D^{-1}\|$$

$$= \|I_n\|\|\delta M\|\|(Z(a) \otimes Y(a))D^{-1}\| + \|I_n\|\|\delta R\|\|(Z(b) \otimes Y(b))D^{-1}\|$$

$$\leq (\|\delta M\| + \|\delta R\|)\,\|(Z(t) \otimes Y(t))D^{-1}\|$$

$$\leq (\|\delta M\| + \|\delta R\|)\,\eta.$$

Theorem 4.4.1. Let $\epsilon > 0$ be such that $0 < \epsilon < \frac{1}{(1+k)\delta\eta}$, where

$$\delta = \max\left\{\|\delta M\|, \|\delta R\|, \|\delta \hat{Q}\|, \|\delta D\|\right\}$$

and

$$k = \int_a^b \|(Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)\|ds.$$

Then the solution $\widetilde{X}(t)$ of (4.4.1) satisfying (4.4.4) is such that

$$\delta\eta(1-k)\left(\|Z(a)\|\|Y(a)\| + \|Z(b)\|\|Y(b)\|\right) \leq \max_{t\in[a,b]} \|\widetilde{X}(t) - \hat{X}(t)\|$$

$$\leq \delta\eta(1+k)\left(\|Z(a)\|\|Y(a)\| + \|Z(b)\|\|Y(b)\|\right).$$

Proof. Any solution $\hat{X}(t)$ of (4.4.1) satisfying (4.4.2) is given by

$$\hat{X}(t) = (Z(t) \otimes Y(t))D^{-1}\hat{Q} - \int_a^b G(t,s)\hat{F}(s)ds,$$

where $G(t,s)$ is the Green's matrix given by

$$G(t,s) = \begin{cases} (Z(t)\otimes Y(t))D^{-1}(I_n\otimes M)(Z(a)\otimes Y(a))(Z^{-1}(s)\otimes Y^{-1}(s)), \\ \qquad\qquad\qquad a\le s<t\le b, \\ -(Z(t)\otimes Y(t))D^{-1}(I_n\otimes R)(Z(b)\otimes Y(b))(Z^{-1}(s)\otimes Y^{-1}(s)), \\ \qquad\qquad\qquad a\le t<s\le b, \end{cases}$$

and any solution $\widetilde{X}(t)$ of (4.4.1) satisfying (4.4.3) is given by

$$\widetilde{X}(t) = (Z(t)\otimes Y(t))D_1^{-1}\left(\hat{Q}+\delta\hat{Q}\right) + \int_a^b G_1(t,s)\hat{F}(s)ds,$$

where

$$G_1(t,s) = \begin{cases} (Z(t)\otimes Y(t))D_1^{-1}(I_n\otimes M_1)(Z(a)\otimes Y(a))(Z^{-1}(s)\otimes Y^{-1}(s)), \\ \qquad\qquad\qquad a\le s<t\le b, \\ -(Z(t)\otimes Y(t))D_1^{-1}(I_n\otimes R_1)(Z(b)\otimes Y(b))(Z^{-1}(s)\otimes Y^{-1}(s)), \\ \qquad\qquad\qquad a\le t<s\le b, \end{cases}$$

$M_1 = M+\delta M$ and $R_1 = R+\delta R$.

Now consider

$$\begin{aligned} \|\widetilde{X}(t)-\hat{X}(t)\| \le{}& \|(Z(t)\otimes Y(t))\left[D_1^{-1}(\hat{Q}+\delta\hat{Q})-D^{-1}\hat{Q}\right]\| \\ &+\int_a^t \|(Z(t)\otimes Y(t))\left[D_1^{-1}(I_n\otimes M_1)-D^{-1}(I_n\otimes M)\right] \\ &\qquad (Z(a)\otimes Y(a))(Z^{-1}(s)\otimes Y^{-1}(s))\hat{F}(s)\|ds \qquad (4.4.5) \\ &+\int_t^b \|(Z(t)\otimes Y(t))\left[D_1^{-1}(I_n\otimes R_1)-D^{-1}(I_n\otimes R)\right] \\ &\qquad (Z(b)\otimes Y(b))(Z^{-1}(s)\otimes Y^{-1}(s))\hat{F}(s)\|ds. \end{aligned}$$

In accordance with the linear terms, we have the following rough estimates;

$$D_1^{-1}(\hat{Q}+\delta\hat{Q})-D^{-1}\hat{Q} = (D+\delta D)^{-1}(\hat{Q}+\delta\hat{Q})-D^{-1}\hat{Q}$$

$$= D^{-1}\left(I_{n^2} + D^{-1}\delta D\right)^{-1}(\hat{Q} + \delta\hat{Q}) - D^{-1}\hat{Q}$$

$$\cong D^{-1}\left[I_{n^2} - D^{-1}\delta D\right](\hat{Q} + \delta\hat{Q}) - D^{-1}\hat{Q}$$

$$\cong D^{-1}\delta\hat{Q}.$$

Similarly

$$D_1^{-1}(I_n \otimes M_1) - D^{-1}(I_n \otimes M) \cong D^{-1}(I_n \otimes \delta M)$$

and

$$D_1^{-1}(I_n \otimes R_1) - D^{-1}(I_n \otimes R) \cong D^{-1}(I_n \otimes \delta R).$$

Using these estimates in (4.4.5), we get

$$\|\widetilde{X}(t) - \hat{X}(t)\| \leq \|(Z(t) \otimes Y(t))D^{-1}\delta\hat{Q}\|$$

$$+ \int_a^t \|(Z(t) \otimes Y(t))D^{-1}(I_n \otimes \delta M)(Z(a) \otimes Y(a))(Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)\|ds$$

$$+ \int_t^b \|(Z(t) \otimes Y(t))D^{-1}(I_n \otimes \delta R)(Z(b) \otimes Y(b))(Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)\|ds$$

$$\leq \|(Z(t) \otimes Y(t))D^{-1}\delta\hat{Q}\| + \|(Z(t) \otimes Y(t))D^{-1}[(I_n \otimes \delta M)(Z(a) \otimes Y(a))$$

$$+(I_n \otimes \delta R)(Z(b) \otimes Y(b))]\| \int_a^b \|(Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)\|ds$$

$$\leq \|(Z(t) \otimes Y(t))D^{-1}\delta\hat{Q}\| + \|(Z(t) \otimes Y(t))D^{-1}\|$$

$$\| [(I_n \otimes \delta M)(Z(a) \otimes Y(a)) + (I_n \otimes \delta R)(Z(b) \otimes Y(b))] \|k$$

$$\leq \delta\eta + \delta\eta k\left[\|Z(a)\|\|Y(a)\| + \|Z(b)\|\|Y(b)\|\right]$$

$$\leq (1+k)\delta\eta\left[\|Z(a)\|\|Y(a)\| + \|Z(b)\|\|Y(b)\|\right].$$

The reverse inequality follows by noting the fact that

$$\|\widetilde{X}(t) - \hat{X}(t)\| \geq \|(Z(t) \otimes Y(t))D^{-1}\delta\hat{Q}\| - \int_a^b \|G_1(t,s) - G(t,s)\|\|\hat{F}(s)\|ds.$$

One may choose η such that

$$\eta = \sup_{a \leq t \leq b} \|(Z(t) \otimes Y(t))\| \|D^{-1}\|,$$

to obtain a more reliable quantity for η. The estimate in the above theorem depends on well-known quantities and on the value of the fundamental matrix at the boundary points.