CHAPTER-3

Ψ-BOUNDEDNESS, Ψ-STABILITY AND Ψ-ASYMPTOTIC STABILITY OF MATRIX LYAPUNOV SYSTEMS

Section 3.1.

Matrix Lyapunov type systems arise in a number of areas of applied mathematics such as dynamical programming, optimal filters, quantum mechanics, and systems engineering. Now we focus our attention to the first order non-homogeneous matrix Lyapunov systems of the form

$$X'(t) = A(t)X(t) + X(t)B(t) + F(t), \tag{3.1.1}$$

where $A(t), B(t), F(t)$ are all square matrices of order n, whose elements a_{ij}, b_{ij}, f_{ij} are real valued continuous functions of t on the interval $R_+ = [0, \infty)$.

Many authors [[1], [3], [16], [42], [62]] have studied Ψ-bounded solutions for system of ordinary differential equations. Recently, Diamandescu [[24], [25]] studied Ψ-bounded solutions for non-homogeneous system $x' = A(t)x + f(t)$, for every Lebesgue Ψ-integrable function f on R_+. The concept of Ψ-stability was introduced by Akinyele [1], Constantin [16] extended these concepts for solutions of ordinary differential equations. Further, Morchalo [62] introduced the notations of Ψ-(uniform) stability and Ψ-asymptotic stability of trivial solution of the nonl-inear system $x' = f(t, x)$ and also obtained new sufficient conditions for the linear system $x' = A(t)x$.

In this chapter we present a necessary and sufficient condition for the existence of at least one Ψ-bounded solution for the corresponding Kronecker product system associated with (3.1.1), where $\hat{F}$ (Vec F) is a Lebesgue Ψ-integrable function defined on R_+. Further, we also obtain sufficient conditions for Ψ-(uniform) stability and Ψ-asymptotic stability of the corresponding homogeneous and non-homogeneous Kronecker product system associated with (3.1.1).

In Section 3.2 we present some basic definitions relating to Ψ- boundedness, Lebesgue Ψ-integrability, Ψ-(uniform) stability, Ψ-asymptotic stability and obtain general solution of the corresponding homogeneous system. Further, we also obtain existence and uniqueness of solutions of initial value problems associated with the corresponding non-homogeneous system.

In Section 3.3 we prove the results relating to Ψ- boundedness of solutions of the corresponding Kronecker product system associated with (3.1.1) and illustrate the results with suitable examples. The results of this section include the results of [24] as a particular case, when $B(t) = 0$, X and F are column n-vectors.

In Section 3.4 we obtain sufficient conditions for Ψ-(uniform) stability and Ψ-asymptotic stability of trivial solution of the corresponding Kronecker product system and illustrate the results with suitable examples.

Section 3.2.

In this section we present some basic definitions and results which are useful for later discussion.

Let $\mathbf{R}^n$ be the Euclidean n-dimensional space. Elements in this space are column vectors, denoted by $u = (u_1, u_2, \ldots u_n)^*$(* denotes transpose) and their norm defined by

$$\|u\| = \max\{|u_1|, |u_2|, \ldots |u_n|\}.$$

For a $n \times n$ real matrix, we define the norm

$$|A| = \sup_{\|x\|\leq 1} \|Ax\|.$$

Let $\Psi_i : R_+ \rightarrow (0, \infty)$, $i = 1, 2, \ldots n$ be continuous functions, and let

$$\Psi = \text{diag } [\Psi_1, \Psi_2, \ldots \Psi_n].$$

Then the matrix $\Psi(t)$ is invertible for each $t \geq 0$.

Definition 3.2.1. A function $\gamma : R_+ \rightarrow \mathbf{R}^n$ is said to be Ψ-bounded on R_+, if $\Psi(t)\gamma(t)$ is bounded on R_+.

Definition 3.2.2. A function $\gamma : R_+ \rightarrow \mathbf{R}^n$ is said to be Lebesgue Ψ-integrable on R_+, if $\gamma(t)$ is measurable and $\Psi(t)\gamma(t)$ is Lebesgue integrable on R_+.

By a solution of (3.1.1), we mean an absolutely continuous function satisfying the system for almost all $t \geq 0$.

Now by applying the Vec operator to the non-homogeneous matrix Lyapunov system (3.1.1) and using the properties of Kronecker product, we have

$$\hat{X}'(t) = G(t)\hat{X}(t) + \hat{F}(t), \tag{3.2.1}$$

where $G(t) = (B^* \otimes I_n) + (I_n \otimes A)$ is a $n^2 \times n^2$ matrix.

The corresponding homogeneous system of (3.2.1) is

$$\hat{X}'(t) = G(t)\hat{X}(t). \tag{3.2.2}$$

Definition 3.2.3. The trivial solution of (3.2.1) is said to be Ψ-stable on R_+ if for every $\varepsilon > 0$ and every t_0 in R_+, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $\hat{X}(t)$ of (3.2.1) which satisfies the inequality $\|\Psi(t_0)\hat{X}(t_0)\| < \delta$, also satisfies the inequality $\|\Psi(t)\hat{X}(t)\| < \varepsilon$ for all $t \geq t_0$.

Definition 3.2.4. The trivial solution of (3.2.1) is said to be Ψ-uniformly stable on R_+, if $\delta(\varepsilon, t_0)$ in Definition 3.2.3 can be chosen independent of t_0.

Definition 3.2.5. The trivial solution of (3.2.1) is said to be Ψ-asymptotically stable on R_+, if it is Ψ-stable on R_+ and in addition, for any $t_0 \in R_+$, there exists a $\delta_0 = \delta_0(t_0) > 0$ such that any solution $\hat{X}(t)$ of (3.2.1) which satisfies the inequality $\|\Psi(t_0)\hat{X}(t_0)\| < \delta_0$, also satisfies the condition $\lim_{t\to\infty} \Psi(t)\hat{X}(t) = 0$.

Lemma 3.2.1. Let $Y(t)$ and $Z(t)$ be the fundamental matrices for the systems

$$X'(t) = A(t)X(t). \tag{3.2.3}$$

and

$$[X^*(t)]' = B^*(t)X^*(t) \tag{3.2.4}$$

respectively. Then the matrix $Z(t) \otimes Y(t)$ is a fundamental matrix of (3.2.2) and every solution of (3.2.2) is of the form $\hat{X}(t) = (Z(t) \otimes Y(t))c$, where c is a n^2-column vector.

Proof. Consider

$$\begin{aligned}(Z(t) \otimes Y(t))' &= (Z'(t) \otimes Y(t)) + (Z(t) \otimes Y'(t)) \\ &= (B^*(t)Z(t) \otimes Y(t)) + (Z(t) \otimes A(t)Y(t)) \\ &= (B^*(t) \otimes I_n)(Z(t) \otimes Y(t)) + (I_n \otimes A(t))(Z(t) \otimes Y(t)) \\ &= [B^*(t) \otimes I_n + I_n \otimes A(t)](Z(t) \otimes Y(t)) \\ &= G(t)(Z(t) \otimes Y(t)).\end{aligned}$$

Hence $Z(t) \otimes Y(t)$ is a fundamental matrix of (3.2.2). Clearly $\hat{X}(t) = (Z(t) \otimes Y(t))c$ is a solution of (3.2.2) and every solution is of this form.

Theorem 3.2.1. Let $Y(t)$ and $Z(t)$ be the fundamental matrices for the systems (3.2.3) and (3.2.4), then the unique solution of (3.2.1), subject to the initial condition $\hat{X}(t_0) = \hat{X}_0$, is

$$\hat{X}(t) = (Z(t) \otimes Y(t))(Z^{-1}(t_0) \otimes Y^{-1}(t_0))X_0$$
$$+ \int_{t_0}^{t} (Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)ds.$$

Proof. First, we show that any solution of (3.2.1) is of the form $\hat{X}(t) = (Z(t) \otimes Y(t))c + \widetilde{X}(t)$, where $\widetilde{X}(t)$ is a particular solution of (3.2.1) and is given by

$$\widetilde{X}(t) = \int_{t_0}^{t} (Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)ds.$$

Here we observe that, $\hat{X}(t_0) = (Z(t_0) \otimes Y(t_0))c = \hat{X}_0$, $c = (Z^{-1}(t_0) \otimes Y^{-1}(t_0))\hat{X}_0$. Let $u(t)$ be any other solution of (3.2.1), write $w(t) = u(t) - \widetilde{X}(t)$, then w satisfies (3.2.2), hence $w = (Z(t) \otimes Y(t))c$, $u(t) = (Z(t) \otimes Y(t))c + \widetilde{X}(t)$.

Next, we consider the vector $\widetilde{X}(t) = (Z(t) \otimes Y(t))v(t)$, where $v(t)$ is an arbitrary vector to be determined, so as to satisfy equation (3.2.1). Consider

$$\widetilde{X}'(t) = (Z(t) \otimes Y(t))'v(t) + (Z(t) \otimes Y(t))v'(t)$$

$$\Rightarrow G(t)\widetilde{X}(t) + \hat{F}(t) = G(t)(Z(t) \otimes Y(t))v(t) + (Z(t) \otimes Y(t))v'(t)$$

$$\Rightarrow (Z(t) \otimes Y(t))v'(t) = \hat{F}(t)$$

$$\Rightarrow v'(t) = (Z^{-1}(t) \otimes Y^{-1}(t))\hat{F}(t)$$

$$\Rightarrow v(t) = \int_{t_0}^{t} (Z^{-1}(s) \otimes Y^{-1}(s))\hat{F}(s)ds.$$

Hence the desired expression follows immediately.

Section 3.3.

This section mainly deals with obtaining a necessary and sufficient condition for the existence of at least one Ψ-bounded solution of the system (3.2.1). The results in this section are illustrated with suitable examples.

Let $Y(t)$ and $Z(t)$ denote the fundamental matrices of (3.2.3) and (3.2.4) satisfying $Y(0) = I_n$, $Z(0) = I_n$ respectively, then $Z(0) \otimes Y(0) = I_n \otimes I_n = I_{n^2}$.

Let P_1 denote the subspace of $\mathbf{R}^{n^2}$ consisting of all vectors which are values of Ψ-bounded solutions of (3.2.2) at $t = 0$. Let P_2 be an arbitrary closed subspace of $\mathbf{R}^{n^2}$, supplementary to P_1. Let Q_1, Q_2 denote the corresponding projections of $\mathbf{R}^{n^2}$ onto P_1 and P_2 respectively.

Theorem 3.3.1. If G is a continuous $n^2 \times n^2$ real matrix, then (3.2.1) has at least one Ψ-bounded solution on R_+ for every Lebesgue Ψ-integrable function $\hat{F}$ on R_+ if and only if there is a positive constant M such that

$$\left.\begin{aligned} &|\Psi(t)(Z(t) \otimes Y(t))Q_1(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)| \leq M, \text{ for } 0 \leq s \leq t, \\ &|\Psi(t)(Z(t) \otimes Y(t))Q_2(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)| \leq M, \text{ for } 0 \leq t \leq s. \end{aligned}\right\} \quad (3.3.1)$$

Proof. First, we suppose that the non-homogeneous system (3.2.1) has at least one Ψ-bounded solution on R_+ for every Lebesgue Ψ-integrable function $\hat{F}$ on R_+. We define the sets;

$H_\Psi = \{\hat{X} : R_+ \to \mathbf{R}^{n^2} : \hat{X}$ is Ψ-bounded and continuous on $R_+\}$,

$J_\Psi = \{\hat{X} : R_+ \to \mathbf{R}^{n^2} : \hat{X}$ is Lebesgue Ψ-integrable on $R_+\}$,

$K_\Psi = \{\hat{X} : R_+ \to \mathbf{R}^{n^2} : \hat{X}$ is absolutely continuous on all intervals $J \subset R_+$, Ψ-bounded on R_+, $\hat{X}(0) \in P_2$, $\hat{X}'(t) - G(t)\hat{X}(t) \in J_\Psi\}$.

It is well-known that H_Ψ and J_Ψ are real Banach spaces with the norms

$$\|\hat{X}\|_{H_\Psi} = \sup_{t\geq 0} \|\Psi(t)\hat{X}(t)\|,$$

$$\|\hat{X}\|_{J_\Psi} = \int_0^\infty \|\Psi(t)\hat{X}(t)\|dt,$$

respectively. Clearly the set K_Ψ is a real linear space with the following norm, defined by

$$\|\hat{X}\|_{K_\Psi} = \sup_{t\geq 0} \|\Psi(t)\hat{X}(t)\| + \|\hat{X}' - G(t)\hat{X}\|_{J_\Psi}.$$

Claim. $(K_\Psi, \|\cdot\|_{K_\Psi})$ is a real Banach space.

Let $\{\hat{X}_n\}$ be any sequence in K_Ψ. Then $\{\hat{X}_n\} \in H_\Psi$ and $\{\hat{X}_n'(t) - G(t)\hat{X}_n(t)\} \in J_\Psi$. Since H_Ψ is a real Banach space, there exists a continuous and bounded function $\hat{X} : R_+ \to \mathbf{R}^{n^2}$ such that

$$\lim_{n\to\infty} \Psi(t)\hat{X}_n(t) = \hat{X}(t), \quad \text{uniformly on } R_+.$$

Let $\widetilde{X}(t) = \Psi^{-1}(t)\hat{X}(t) \in H_\Psi$. Then

$$\begin{aligned} \|\hat{X}_n(t) - \widetilde{X}(t)\| &= \|\Psi^{-1}(t)\Psi(t)\hat{X}_n(t) - \Psi^{-1}(t)\hat{X}(t)\| \\ &\leq |\Psi^{-1}(t)|\|\Psi(t)\hat{X}_n(t) - \hat{X}(t)\|, \end{aligned}$$

it follows that $\lim_{n\to\infty} \hat{X}_n(t) = \widetilde{X}(t)$, uniformly on every compact subset of R_+. Thus $\widetilde{X}(0) \in P_2$.

Since $\{\hat{X}_n'(t) - G(t)\hat{X}_n(t)\} \in J_\Psi$, then $\{\overline{F}_n(t)\}$, where $\overline{F}_n(t) = \Psi(t)(\hat{X}_n'(t) - G(t)\hat{X}_n(t))$, is a fundamental sequence in $\mathbf{L}$, the Banach space of all vector functions which are Lebesgue integrable on R_+ with the norm defined in J_Ψ. Thus there exists a function $\hat{F} \in \mathbf{L}$ such that $\lim_{n\to\infty} \overline{F}_n(t) = \hat{F}(t)$, uniformly on R_+.

$$i.e. \quad \lim_{n\to\infty} \int_0^\infty \|\overline{F}_n(t) - \hat{F}(t)\|dt = 0.$$

Let $\widetilde{F}(t) = \Psi^{-1}(t)\hat{F}(t)$, then $\widetilde{F}(t) \in J_\Psi$. For $t \geq 0$, we have

$$\begin{aligned}
\widetilde{X}(t) - \widetilde{X}(0) &= \lim_{n\to\infty}[\hat{X}_n(t) - \hat{X}_n(0)] \\
&= \lim_{n\to\infty}\int_0^t \hat{X}_n'(s)ds \\
&= \lim_{n\to\infty}\int_0^t [(\hat{X}_n'(s) - G(s)\hat{X}_n(s)) + G(s)\hat{X}_n(s)]ds \\
&= \lim_{n\to\infty}\int_0^t \{\Psi^{-1}(s)[\Psi(s)(\hat{X}_n'(s) - G(s)\hat{X}_n(s))] - \widetilde{F}(s) \\
&\qquad\qquad + \widetilde{F}(s) + G(s)\hat{X}_n(s)\}ds \\
&= \lim_{n\to\infty}\int_0^t \{\Psi^{-1}(s)[\overline{F}_n(s) - \hat{F}(s)] + \widetilde{F}(s) + G(s)\hat{X}_n(s)\}ds \\
&= \int_0^t [\widetilde{F}(s) + G(s)\widetilde{X}(s)]ds.
\end{aligned}$$

It follows that $\widetilde{X}'(t) - G(t)\widetilde{X}(t) = \widetilde{F}(t) \in J_\Psi$ and $\widetilde{X}(t)$ is absolutely continuous on all intervals $J \subset R_+$. Thus $\widetilde{X}(t) \in K_\Psi$. Since $\lim_{n\to\infty}\Psi(t)\hat{X}_n(t) = \Psi(t)\widetilde{X}(t)$, uniformly on R_+ and

$$\lim_{n\to\infty}\int_0^\infty \|\Psi(t)[(\hat{X}_n'(t) - G(t)\hat{X}_n(t)) - (\widetilde{X}'(t) - G(t)\widetilde{X}(t))]\|dt = 0,$$

it follows that $\lim_{n\to\infty}\|\hat{X}_n - \widetilde{X}\|_{K_\Psi} = 0$. Thus $(K_\Psi, \|\cdot\|_{K_\Psi})$ is a real Banach space.

Now we define the operator $T : K_\Psi \to J_\Psi$, by

$$T\hat{X} = \hat{X}' - G(t)\hat{X}. \tag{3.3.2}$$

Clearly, T is linear and bounded with $\|T\| \leq 1$. Let $T\hat{X} = 0$, then $\hat{X}' = G(t)\hat{X}$, $\hat{X} \in K_\Psi$. Therefore $\hat{X}$ is a Ψ-bounded solution of (3.2.2). Then $\hat{X}(0) \in P_1 \cap P_2 = \{0\}$. Thus $\hat{X} = 0$, so the operator T is one-to-one.

Let $\hat{F} \in J_\Psi$ and let $\hat{X}(t)$ be the Ψ-bounded solution of the system (3.2.1). Let $w(t)$ be the solution of

$$w' = G(t)w + \hat{F}(t), \quad w(0) = Q_2\hat{X}(0). \tag{3.3.3}$$

Then $\hat{X}(t) - w(t)$ is a solution of (3.2.2) with $Q_2\hat{X}(0) - w(0) = 0$, i.e. $\hat{X}(0) - w(0) \in P_1$. It follows that $\hat{X}(t) - w(t)$ is Ψ-bounded on R_+. Thus $w(t)$ is Ψ-bounded on R_+. It follows that $w(t) \in K_\Psi$, $Tw = \hat{F}$, and consequently the operator T is onto. Since T is a bounded one-to-one linear operator from a Banach space K_Ψ onto a Banach space J_Ψ, then the inverse operator T^{-1} is also bounded. Therefore there exists a positive constant $M = \|T^{-1}\| - 1$ such that, for $\hat{F} \in J_\Psi$ and for the solution $\hat{X} \in K_\Psi$ of (3.2.1), we have

$$\|\hat{X}\|_{K_\Psi} = \sup_{t\geq 0} \|\Psi(t)\hat{X}(t)\| + \|\hat{X}' - G(t)\hat{X}\|_{J_\Psi}$$

$$\Rightarrow \quad \|T^{-1}\hat{F}\| = \sup_{t\geq 0} \|\Psi(t)\hat{X}(t)\| + \|\hat{F}\|_{J_\Psi}$$

$$\Rightarrow \quad \sup_{t\geq 0} \|\Psi(t)\hat{X}(t)\| \leq [\|T^{-1}\| - 1] \int_0^\infty \|\Psi(t)\hat{F}(t)\| dt$$

$$\leq M \int_0^\infty \|\Psi(t)\hat{F}(t)\| dt. \tag{3.3.4}$$

For $s \geq 0$, $\delta > 0$, $\eta \in \mathbf{R}^{n^2}$, we consider the function $\hat{F} : R_+ \longrightarrow \mathbf{R}^{n^2}$,

$$\hat{F}(t) = \begin{cases} \Psi^{-1}(t)\eta, & \text{for } s \leq t \leq s + \delta \\ 0, & \text{otherwise.} \end{cases}$$

Then $\hat{F} \in J_\Psi$ and $\|\hat{F}\|_{J_\Psi} = \delta\|\eta\|$. The corresponding solution $\hat{X} \in K_\Psi$ of (3.2.1) is

$$\hat{X}(t) = \int_s^{s+\delta} \mathbf{K}(t, \xi)\hat{F}(\xi) d\xi,$$

where

$$\mathbf{K}(t, \xi) = \begin{cases} (Z(t) \otimes Y(t))Q_1(Z^{-1}(\xi) \otimes Y^{-1}(\xi)), & \text{for } 0 \leq \xi \leq t \\ -(Z(t) \otimes Y(t))Q_2(Z^{-1}(\xi) \otimes Y^{-1}(\xi)), & \text{for } 0 \leq t \leq \xi. \end{cases}$$

Clearly $\mathbf{K}(t, \xi)$ is continuous everywhere except at $t = \xi$, and has a jump discontinuity at $t = \xi$.

Consider

$$\|\int_s^{s+\delta}\Psi(t)\mathbf{K}(t,\xi)\Psi^{-1}(\xi)\eta d\xi\| = \|\Psi(t)\hat{X}(t)\|$$

$$\le \sup_{t\ge 0}\|\Psi(t)\hat{X}(t)\|$$

$$\le M\int_0^\infty \|\Psi(t)\hat{F}(t)\|dt$$

$$= M\delta\|\eta\|.$$

This implies

$$\|\Psi(t)\mathbf{K}(t,s)\Psi^{-1}(s)\eta\| \le M\|\eta\|,$$

$$|\Psi(t)\mathbf{K}(t,s)\Psi^{-1}(s)| \le M,$$

and hence (3.3.1) follows.

Conversely suppose (3.3.1) holds. Consider the function

$$\hat{X}(t) = \int_0^t (Z(t)\otimes Y(t))Q_1(Z^{-1}(s)\otimes Y^{-1}(s))\hat{F}(s)ds$$

$$-\int_t^\infty (Z(t)\otimes Y(t))Q_2(Z^{-1}(s)\otimes Y^{-1}(s))\hat{F}(s)ds,\ \ t\ge 0,$$

where $\hat{F}$ is a Lebesgue Ψ-integrable function on R_+. It is easy to see that $\hat{X}(t)$ is a Ψ-bounded solution of (3.2.1) on R_+ .

Example 3.3.1. Consider the non-homogeneous matrix Lyapunov system (3.1.1) with

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ B(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \text{and}\ F(t) = \begin{bmatrix} \frac{e^t}{1+t^2} & \frac{-e^t}{(1+t)^2} \\ \frac{e^t}{(1+t)^2} & \frac{-e^t}{1+t^2} \end{bmatrix}.$$

Then the fundamental matrices of (3.2.3), (3.2.4) are

$$Y(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},\ Z(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}.$$

Now the fundamental matrix of (3.2.2) is

$$Z(t)\otimes Y(t)=\begin{bmatrix} e^t\cos t & e^t\sin t & 0 & 0 \\ -e^t\sin t & e^t\cos t & 0 & 0 \\ 0 & 0 & e^t\cos t & e^t\sin t \\ 0 & 0 & -e^t\sin t & e^t\cos t \end{bmatrix}.$$

Consider the matrix $\Psi(t)=e^{-t}I_4$ such that

$$Q_1=\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix},\quad Q_2=\begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}.$$

Therefore

$$\Psi(t)(Z(t)\otimes Y(t))Q_1(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)=\begin{bmatrix} \cos(t-s) & \sin(t-s) & 0 & 0 \\ -\sin(t-s) & \cos(t-s) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\Psi(t)(Z(t)\otimes Y(t))Q_2(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(t-s) & \sin(t-s) \\ 0 & 0 & -\sin(t-s) & \cos(t-s) \end{bmatrix}.$$

Clearly (3.3.1) is satisfied with $M=2$, and

$$\int_0^\infty \|\Psi(t)\hat{F}(t)\|dt=\frac{\pi}{2}.$$

i.e. $\hat{F}(t)$ is Lebesgue Ψ-integrable on R_+. Hence from Theorem 3.3.1, the system (3.2.1) has at least one Ψ-bounded solution on R_+.

Theorem 3.3.2. Suppose that;

(I) the fundamental matrices $Y(t)$, $Z(t)$ of (3.2.3), (3.2.4), satisfies the conditions;

(i) $\lim_{t\to\infty} \Psi(t)(Z(t)\otimes Y(t))Q_1 = 0$,

(ii) $|\Psi(t)(Z(t)\otimes Y(t))Q_1(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)| \leq M$, for $0 \leq s \leq t$,

$|\Psi(t)(Z(t)\otimes Y(t))Q_2(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)| \leq M$, for $0 \leq t \leq s$,

where M is a positive constant, Q_1 and Q_2 are as in Theorem 3.3.1.

(II) the function $\hat{F} : R_+ \to \mathbf{R}^{n^2}$ is Lebesgue Ψ-integrable on R_+.

Then every Ψ-bounded solution $\hat{X}(t)$ of (3.2.1) is such that

$$\lim_{t\to\infty} \|\Psi(t)\hat{X}(t)\| = 0.$$

Proof. Let $\hat{X}(t)$ be a Ψ-bounded solution of (3.2.1). Then there exists a positive constant M such that $\|\Psi(t)\hat{X}(t)\| \leq M$, for all $t \geq 0$.

We consider the function

$$\begin{aligned} v(t) = \hat{X}(t) &- (Z(t)\otimes Y(t))Q_1\hat{X}(0) \\ &- \int_0^t (Z(t)\otimes Y(t))Q_1(Z^{-1}(s)\otimes Y^{-1}(s))\hat{F}(s)ds \\ &+ \int_t^\infty (Z(t)\otimes Y(t))Q_2(Z^{-1}(s)\otimes Y^{-1}(s))\hat{F}(s)ds, \end{aligned}$$

for all t ≥ 0.

From the hypothesis, it follows that the function $v(t)$ is a Ψ-bounded solution of (3.2.2). Then $v(0) \in P_1$. On the other hand, $Q_1 v(0) = 0$. Therefore $v(0) = Q_2 v(0) \in P_2$. Thus $v(0) = 0$ and then $v(t) = 0$, for $t \geq 0$.

For $t \geq 0$, we have

$$\hat{X}(t) = (Z(t)\otimes Y(t))Q_1\hat{X}(0) + \int_0^t (Z(t)\otimes Y(t))Q_1(Z^{-1}(s)\otimes Y^{-1}(s))\hat{F}(s)ds$$

$$-\int_t^{\infty}(Z(t)\otimes Y(t))Q_2(Z^{-1}(s)\otimes Y^{-1}(s))\hat{F}(s)ds.$$

Since $\hat{F}(t)$ is Ψ-integrable on R_+, for a given $\varepsilon > 0$, there exists $t_1 \geq 0$ such that

$$\int_t^{\infty}\|\Psi(s)\hat{F}(s)\|ds < \frac{\varepsilon}{2M}, \quad \text{for } t \geq t_1.$$

Again from the hypothesis, there exists $t_2 > t_1$ such that, for $t \geq t_2$,

$$|\Psi(t)(Z(t)\otimes Y(t))Q_1| \leq \frac{\varepsilon}{2}\Big[\|\hat{X}(0)\| + \int_0^{t_1}\|(Z^{-1}(s)\otimes Y^{-1}(s))\hat{F}(s)\|ds\Big]^{-1}.$$

Then for $t \geq t_2$, we have

$$\begin{aligned}
\|\Psi(t)\hat{X}(t)\| \leq{}& |\Psi(t)(Z(t)\otimes Y(t))Q_1|\|\hat{X}(0)\| \\
&+\int_0^{t_1}|\Psi(t)(Z(t)\otimes Y(t))Q_1|\|(Z^{-1}(s)\otimes Y^{-1}(s))\hat{F}(s)\|ds \\
&+\int_{t_1}^{t}|\Psi(t)(Z(t)\otimes Y(t))Q_1(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|\|\Psi(s)\hat{F}(s)\|ds \\
&+\int_t^{\infty}|\Psi(t)(Z(t)\otimes Y(t))Q_2(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|\|\Psi(s)\hat{F}(s)\|ds \\
\leq{}& |\Psi(t)(Z(t)\otimes Y(t))Q_1|\Big[\|\hat{X}(0)\|+\int_0^{t_1}\|(Z^{-1}(s)\otimes Y^{-1}(s))\hat{F}(s)\|ds\Big] \\
&+M\int_{t_1}^{\infty}\|\Psi(s)\hat{F}(s)\|ds \\
<{}& \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

Therefore $\lim_{t\to\infty}\|\Psi(t)\hat{X}(t)\| = 0.$

Note 3.3.1. Theorem 3.3.2 is no longer true if we require that the function $\hat{F}$ be Ψ-bounded on R_+, instead of condition (II) of Theorem 3.3.2. Even if the function $\hat{F}$ is such that $\lim_{t\to\infty}\|\Psi(t)\hat{F}(t)\| = 0$, Theorem 3.3.2 does not apply. This is shown by the following example.

Example 3.3.2. Consider the Lyapunov system (3.1.1) with $A(t) = I_2$, $B(t) = -I_2$. Then $Z(t)\otimes Y(t) = I_4$ is a fundamental matrix for (3.2.2).

Consider

$$\Psi(t) = \begin{bmatrix} (1+t)^{-1} & 0 & 0 & 0 \\ 0 & (1+t)^{-1} & 0 & 0 \\ 0 & 0 & (1+t) & 0 \\ 0 & 0 & 0 & (1+t) \end{bmatrix}.$$

We have $\Psi(t)(Z(t) \otimes Y(t)) = \Psi(t)$, such that Q_1, Q_2 are as in Example 3.3.1. It follows that the first hypothesis of Theorem 3.3.2 is satisfied with $M = 1$.

First, we take

$$F(t) = \begin{bmatrix} \sqrt{1+t} & (1+t)^{-2} \\ \sqrt{1+t} & (1+t)^{-2} \end{bmatrix},$$

then

$$\lim_{t\to\infty} \|\Psi(t)\hat{F}(t)\| = 0.$$

On the other hand, the solutions of the system (3.2.1) are

$$\hat{X}(t) = \left(\frac{2}{3}\sqrt{(t+1)^3} + c_1, \frac{2}{3}\sqrt{(t+1)^3} + c_2, -(t+1)^{-1} + c_3, -(t+1)^{-1} + c_4\right)^*.$$

It follows that the solutions of the system (3.2.1) are Ψ-unbounded on R_+.

Now, we take

$$F(t) = \begin{bmatrix} \frac{1}{t+1} & \frac{1}{(t+1)^3} \\ \frac{1}{t+1} & \frac{1}{(t+1)^3} \end{bmatrix},$$

then we have $\int\limits_0^\infty \|\Psi(t)\hat{F}(t)\|dt = 1$. On the other hand, the solutions of the system (3.2.1) are

$$\hat{X}(t) = \left(\log(t+1) + c_1,\ \log(t+1) + c_2,\ -\frac{1}{2(t+1)^2} + c_3,\ -\frac{1}{2(t+1)^2} + c_4\right)^*.$$

It is easily seen that these solutions are Ψ-bounded on R_+ if and only if $c_3 = c_4 = 0$. In this case

$$\lim_{t\to\infty} \|\Psi(t)\hat{X}(t)\| = 0.$$

Section 3.4.

In this section we obtain sufficient conditions for Ψ-stability, Ψ-uniform stability and Ψ-asymptotic stability of trivial solution of system (3.2.2) and also obtain sufficient conditions for Ψ-stability and Ψ-asymptotic stability of non-homogeneous system (3.2.1). Further, the results of this section are highlighted with suitable examples.

Theorem 3.4.1. Let $Y(t)$ and $Z(t)$ be the fundamental matrices of (3.2.3) and (3.2.4). Then

(i) the trivial solution of (3.2.2) is Ψ-stable on R_+ if and only if there exists a positive constant N such that $|\Psi(t)(Z(t) \otimes Y(t))| \leq N$, for all $t \geq 0$.

(ii) the trivial solution of (3.2.2) is Ψ-uniformly stable on R_+ if and only if there exists a positive constant N such that

$$|\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)| \leq N, \text{ for all } 0 \leq s \leq t < \infty.$$

Proof. The solution of (3.2.2) with initial point at $t_0 \geq 0$ is

$$\hat{X}(t) = (Z(t) \otimes Y(t))(Z^{-1}(t_0) \otimes Y^{-1}(t_0))\hat{X}(t_0), \quad \text{for } t \geq 0.$$

First, we suppose that the trivial solution of (3.2.2) is Ψ-stable on R_+. Then for $\varepsilon = 1$ and $t_0 = 0$, there exists $\delta > 0$ such that any solution $\hat{X}(t)$ of (3.2.2) which satisfies the inequality $\|\Psi(0)\hat{X}(0)\| < \delta$, also satisfies the inequality $\|\Psi(t)\hat{X}(t)\| < 1$.

$$\text{i.e.} \quad \|\Psi(t)(Z(t) \otimes Y(t))[\Psi(0)(Z(0) \otimes Y(0))]^{-1}\Psi(0)\hat{X}(0)\| < 1, \quad \text{for } t \geq 0.$$

Let $w \in \mathbf{R}^{n^2}$ be such that $\|w\| \leq 1$. If we take $\hat{X}(0) = \frac{\delta}{2}\Psi^{-1}(0)w$, then we have $\|\Psi(0)\hat{X}(0)\| < \delta$. Hence

$$\|\Psi(t)(Z(t) \otimes Y(t))[\Psi(0)(Z(0) \otimes Y(0))]^{-1}\frac{\delta}{2}w\| < 1, \quad \text{for } t \geq 0.$$

Therefore $|\Psi(t)(Z(t)\otimes Y(t))[\Psi(0)(Z(0)\otimes Y(0))]^{-1}| < 2/\delta$, for $t \geq 0$, and hence $|\Psi(t)(Z(t)\otimes Y(t))| \leq N$, for $t \geq 0$, where $N = \frac{2}{\delta|[\Psi(0)(Z(0)\otimes Y(0))]^{-1}|}$ is a positive constant.

Conversely suppose that $|\Psi(t)(Z(t)\otimes Y(t))| \leq N$, for $t \geq 0$. For any $\varepsilon > 0$ and $t_0 \in R_+$, choose $\delta(\varepsilon, t_0) = \frac{\varepsilon}{N|[\Psi(t_0)(Z(t_0)\otimes Y(t_0))]^{-1}|} > 0$ such that $\|\Psi(t_0)\hat{X}(t_0)\| < \delta$, for $t \geq t_0$. Then we have

$$\begin{aligned}
\|\Psi(t)\hat{X}(t)\| &= \|\Psi(t)(Z(t)\otimes Y(t))[\Psi(t_0)(Z(t_0)\otimes Y(t_0))]^{-1}\Psi(t_0)\hat{X}(t_0)\| \\
&\leq |\Psi(t)(Z(t)\otimes Y(t))[\Psi(t_0)(Z(t_0)\otimes Y(t_0))]^{-1}|\ \|\Psi(t_0)\hat{X}(t_0)\| \\
&< |\Psi(t)(Z(t)\otimes Y(t))|\ |[\Psi(t_0)(Z(t_0)\otimes Y(t_0))]^{-1}|\ \delta \\
&\leq \ N.\frac{\varepsilon}{N} = \varepsilon.
\end{aligned}$$

Thus the trivial solution of (3.2.2) is Ψ-stable on R_+. Similarly we prove our second assertion.

Remark 3.4.1. It is easy to observe that if $|\Psi(t)|$ and $|\Psi^{-1}(t)|$ are bounded on R_+, then the Ψ-stability is equivalent to classical stability. But Ψ-stability need not imply classical stability.

The above Remark 3.4.1 and Theorem 3.4.1 are illustrated by the following examples.

Example 3.4.1. Consider the homogeneous matrix Lyapunov system corresponding to (3.1.1) with

$$A(t) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\quad B(t) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Then the fundamental matrices of (3.2.3), (3.2.4) are

$$Y(t) = \begin{bmatrix} e^t \sin t & e^t \cos t \\ -e^t \cos t & e^t \sin t \end{bmatrix},\quad Z(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}.$$

Now the fundamental matrix of (3.2.2) is

$$Z(t)\otimes Y(t)=\begin{bmatrix} e^{2t}\sin t & e^{2t}\cos t & 0 & 0 \\ -e^{2t}\cos t & e^{2t}\sin t & 0 & 0 \\ 0 & 0 & e^{-t}\sin t & e^{-t}\cos t \\ 0 & 0 & -e^{-t}\cos t & e^{-t}\sin t \end{bmatrix}.$$

Clearly $Z(t)\otimes Y(t)$ is unbounded on R_+, it follows that the system (3.2.2) is not stable on R_+. Consider

$$\Psi(t)=\begin{bmatrix} e^{-2t} & 0 & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & e^{t} & 0 \\ 0 & 0 & 0 & e^{t} \end{bmatrix},$$

for all $t\geq 0$, we have

$$\Psi(t)(Z(t)\otimes Y(t))=\begin{bmatrix} \sin t & \cos t & 0 & 0 \\ -\cos t & \sin t & 0 & 0 \\ 0 & 0 & \sin t & \cos t \\ 0 & 0 & -\cos t & \sin t \end{bmatrix}.$$

It is easily seen from Theorem 3.4.1 with $N=2$, the system (3.2.2) is Ψ-stable on R_+. •

Example 3.4.2. In Example 3.4.1, if we take $B(t)=I_2$. Then $Z(t)=e^{t}I_2$, and

$$Z(t)\otimes Y(t)=\begin{bmatrix} e^{2t}\sin t & e^{2t}\cos t & 0 & 0 \\ -e^{2t}\cos t & e^{2t}\sin t & 0 & 0 \\ 0 & 0 & e^{2t}\sin t & e^{2t}\cos t \\ 0 & 0 & -e^{2t}\cos t & e^{2t}\sin t \end{bmatrix}$$

is the fundamental matrix of (3.2.2). Here $Z(t) \otimes Y(t)$ is unbounded and hence the system (3.2.2) is unstable on R_+. Let $\Psi(t) = e^{-2t} I_4$. Then for all $0 \le s \le t < \infty$, we have

$$\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)$$

$$= \begin{bmatrix} \cos(t-s) & -\sin(t-s) & 0 & 0 \\ \sin(t-s) & \cos(t-s) & 0 & 0 \\ 0 & 0 & \cos(t-s) & -\sin(t-s) \\ 0 & 0 & \sin(t-s) & \cos(t-s) \end{bmatrix}.$$

Thus from Theorem 3.4.1 with $N = 2$, the system (3.2.2) is Ψ-uniformly stable on R_+.

Theorem 3.4.2. Let $Y(t)$, $Z(t)$ be the fundamental matrices for (3.2.3), (3.2.4) respectively. If there exists a continuous function $\phi : R_+ \to (0, \infty)$ and constants $p \ge 1$, $L > 0$ which satisfies any one of the following conditions;

(i) $\int_0^t \phi(s)|\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|^p \, ds \le L$, for all $t \ge 0$,

(ii) $\int_0^t \phi(s)|(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)\Psi(t)(Z(t) \otimes Y(t))|^p \, ds \le L$, for all $t \ge 0$,

then the system (3.2.2) is Ψ-stable on R_+.

Proof. We prove this theorem using condition (i), the proof for the other case follows along similar lines. First, we take $p = 1$ and $b(t) = |\Psi(t)(Z(t) \otimes Y(t))|^{-1}$, for $t \ge 0$. From the identity

$$\Big(\int_0^t \phi(s)b(s)\, ds\Big)\Psi(t)(Z(t) \otimes Y(t))$$

$$= \int_0^t \phi(s)\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)$$

$$\Psi(s)(Z(s) \otimes Y(s))b(s)\, ds, \tag{3.4.1}$$

it follows that

$$\Big(\int_0^t \phi(s)b(s)\,ds\Big)|\Psi(t)(Z(t)\otimes Y(t))|$$
$$\leq \int_0^t \phi(s)|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|$$
$$|\Psi(s)(Z(s)\otimes Y(s))|b(s)\,ds.$$

Thus the scalar function $a(t) = \int_0^t \phi(s)b(s)ds$ satisfies the inequality

$$a(t)b^{-1}(t) \leq L, \quad \text{for} \quad t \geq 0.$$

We have $a'(t) = \phi(t)b(t) \geq L^{-1}\phi(t)a(t)$, for $t \geq 0$. It follows that

$$a(t) \geq a(t_1)e^{L^{-1}\int_{t_1}^t \phi(s)\,ds}, \quad \text{for } t \geq t_1 > 0$$

and hence

$$|\Psi(t)(Z(t)\otimes Y(t))| = b^{-1}(t) \leq La^{-1}(t_1)e^{-L^{-1}\int_{t_1}^t \phi(s)\,ds}, \quad \text{for } t \geq t_1 > 0.$$

Since $|\Psi(t)(Z(t)\otimes Y(t))|$ is a continuous function on the compact interval $[0, t_1]$, there exists a positive constant N such that $|\Psi(t)(Z(t)\otimes Y(t))| \leq N$, for $t \geq 0$. From Theorem 3.4.1, the trivial solution of (3.2.2) is Ψ-stable on R_+. Hence the theorem is proved for $p = 1$.

Now suppose that $p > 1$ and $c(t) = |\Psi(t)(Z(t)\otimes Y(t))|^{-p}$, for $t \geq 0$. From the identity (3.4.1), we have

$$\Big(\int_0^t \phi(s)c(s)\,ds\Big)|\Psi(t)(Z(t)\otimes Y(t))|$$
$$\leq \int_0^t \phi(s)|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|$$
$$|\Psi(s)(Z(s)\otimes Y(s))|c(s)\,ds.$$

Since $\phi(s)|\Psi(s)(Z(s)\otimes Y(s))|c(s) = [\phi(s)]^{\frac{1}{p}}[\phi(s)c(s)]^{\frac{1}{q}}$, (where $\frac{1}{p}+\frac{1}{q}=1$) we have

$$\Big(\int_0^t \phi(s)c(s)\,ds\Big)|\Psi(t)(Z(t)\otimes Y(t))|$$
$$\le \int_0^t [\phi(s)]^{\frac{1}{p}}|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|[\phi(s)c(s)]^{\frac{1}{q}}\,ds\,.$$

By using the Hölder inequality, we obtain

$$\Big(\int_0^t \phi(s)c(s)\,ds\Big)|\Psi(t)(Z(t)\otimes Y(t))|$$
$$\le \Big(\int_0^t \phi(s)|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|^p\,ds\Big)^{\frac{1}{p}}$$
$$\Big(\int_0^t \phi(s)c(s)\,ds\Big)^{\frac{1}{q}}$$
$$= L^{\frac{1}{p}}\Big(\int_0^t \phi(s)c(s)\,ds\Big)^{1-\frac{1}{p}},\quad t\ge 0.$$

Therefore

$$|\Psi(t)(Z(t)\otimes Y(t))| \le L^{\frac{1}{p}}\Big(\int_0^t \phi(s)c(s)\,ds\Big)^{-\frac{1}{p}},\quad \text{for all}\quad t\ge 0\,.$$

Taking $d(t)=\int_0^t \phi(s)c(s)\,ds$ for $t\ge 0$, we have

$$|\Psi(t)(Z(t)\otimes Y(t))| \le L^{\frac{1}{p}}[d(t)]^{-\frac{1}{p}},\quad \text{for all}\quad t\ge 0. \tag{3.4.2}$$

Hence $d'(t)=\phi(t)c(t)=\phi(t)|\Psi(t)(Z(t)\otimes Y(t))|^{-p}\ge L^{-1}\phi(t)d(t)$, implies

$$d(t)\ge d(1)e^{L^{-1}\int_1^t \phi(s)\,ds}\quad t\ge 1. \tag{3.4.3}$$

From (3.4.2) and (3.4.3), it follows that

$$|\Psi(t)(Z(t)\otimes Y(t))| \le L^{\frac{1}{p}}[d(1)]^{-\frac{1}{p}}e^{-\frac{1}{pL}\int_1^t \phi(s)\,ds},\quad t\ge 1. \tag{3.4.4}$$

Here $|\Psi(t)(Z(t)\otimes Y(t))|$ is a continuous function on the compact interval $[0,1]$, there exists a positive constant N such that $|\Psi(t)(Z(t)\otimes Y(t))|\le N$, for $t\ge 0$. Hence the theorem follows immediately from Theorem 3.4.1.

Theorem 3.4.3. Let $Y(t)$ and $Z(t)$ be the fundamental matrices of (3.2.3) and (3.2.4). Then the trivial solution of (3.2.2) is Ψ-asymptotically stable on R_+ if and only if $\lim_{t\to\infty} \Psi(t)(Z(t)\otimes Y(t)) = 0$.

Proof. The solution of (3.2.2) with the initial point at $t_0 \geq 0$ is

$$\hat{X}(t) = (Z(t)\otimes Y(t))(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\hat{X}(t_0), \quad \text{for } t \geq 0.$$

First, we suppose that the trivial solution of (3.2.2) is Ψ-asymptotically stable on R_+. Then, the trivial solution of (3.2.2) is Ψ-stable on R_+ and for any $t_0 \in R_+$, there exists a $\delta_0 = \delta(t_0) > 0$ such that any solution $\hat{X}(t)$ of (3.2.2) which satisfies the inequality $\|\Psi(t_0)\hat{X}(t_0)\| < \delta_0$, and satisfies the condition $\lim_{t\to\infty} \Psi(t)\hat{X}(t) = 0$.

Therefore, for any $\epsilon > 0$ and $t_0 \geq 0$, there exists a $\delta_0 > 0$ such that $\|\Psi(t_0)\hat{X}(t_0)\| < \delta_0$ and also satisfies

$$\|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)\Psi(t_0)\hat{X}(t_0)\| < \epsilon \quad \text{for all } t \geq t_{\epsilon,t_0}.$$

Let $v \in \mathbf{R}^{n^2}$ be such that $\|v\| \leq 1$. For $\hat{X}(t_0) = \frac{\delta_0}{2}\Psi^{-1}(t_0)v$, we have $\|\Psi(t_0)\hat{X}(t_0)\| < \delta_0$ and hence,

$$\|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(t_0)\otimes Y^{-1}(t_0)\Psi^{-1}(t_0)\frac{\delta_0}{2}v\| < \epsilon.$$

$$\Rightarrow |\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)| < \frac{2\epsilon}{\delta_0}$$

$$\Rightarrow |\Psi(t)(Z(t)\otimes Y(t))| \leq \frac{2\epsilon}{\delta_0|(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|}, \quad \text{for } t \geq t_{\epsilon,t_0}$$

Therefore, $\lim_{t\to\infty} \Psi(t)(Z(t)\otimes Y(t)) = 0$.

Conversely suppose that $\lim_{t\to\infty} \Psi(t)(Z(t)\otimes Y(t)) = 0$. Hence, there exists $N > 0$ such that $|\Psi(t)(Z(t)\otimes Y(t))| \leq N$ for $t \geq 0$. It is easy to see that the trivial solution of (3.2.2) is Ψ-stable on R_+. For any $\hat{X}(t_0) \in \mathbf{R}^{n^2}$, we have

$$\lim_{t\to\infty} \Psi(t)\hat{X}(t) = \lim_{t\to\infty} \Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\hat{X}(t_0) = 0.$$

Thus, the trivial solution of (3.2.2) is Ψ-asymptotically stable on R_+.

The above Theorem 3.4.3 is illustrated by the following example.

Example 3.4.3. In Example 3.4.1, consider

$$\Psi(t) = \begin{bmatrix} \frac{e^{-2t}}{t+1} & 0 & 0 & 0 \\ 0 & \frac{e^{-2t}}{t+1} & 0 & 0 \\ 0 & 0 & \frac{e^{t}}{\sqrt{t+1}} & 0 \\ 0 & 0 & 0 & \frac{e^{t}}{\sqrt{t+1}} \end{bmatrix},$$

for all $t \geq 0$, we have

$$\Psi(t)(Z(t) \otimes Y(t)) = \begin{bmatrix} \frac{\sin t}{t+1} & \frac{\cos t}{t+1} & 0 & 0 \\ -\frac{\cos t}{t+1} & \frac{\sin t}{t+1} & 0 & 0 \\ 0 & 0 & \frac{\sin t}{\sqrt{t+1}} & \frac{\cos t}{\sqrt{t+1}} \\ 0 & 0 & -\frac{\cos t}{\sqrt{t+1}} & \frac{\sin t}{\sqrt{t+1}} \end{bmatrix}.$$

It is easily seen from Theorem 3.4.3, the system (3.2.2) is Ψ-asymptotically stable on R_+.

Remark 3.4.2. Ψ-asymptotic stability need not imply classical asymptotic stability.

The above Remark 3.4.2 is illustrated by the following example.

Example 3.4.4. Consider the homogeneous matrix Lyapunov system corresponding to (3.1.1) with

$$A(t) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad B(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then the fundamental matrices of (3.2.3), (3.2.4) are

$$Y(t) = \begin{bmatrix} e^{t}\sin t & e^{t}\cos t \\ -e^{t}\cos t & e^{t}\sin t \end{bmatrix}, \quad Z(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Now the fundamental matrix of (3.2.2) is

$$Z(t) \otimes Y(t) = \begin{bmatrix} \sin t & \cos t & 0 & 0 \\ -\cos t & \sin t & 0 & 0 \\ 0 & 0 & \sin t & \cos t \\ 0 & 0 & -\cos t & \sin t \end{bmatrix}.$$

Clearly the system (3.2.2) is stable, but it is not asymptotically stable on R_+.

Consider

$$\Psi(t) = \begin{bmatrix} \frac{1}{t+1} & 0 & 0 & 0 \\ 0 & \frac{1}{t+1} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{t+1}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{t+1}} \end{bmatrix},$$

for all $t \geq 0$, we have

$$\Psi(t)(Z(t) \otimes Y(t)) = \begin{bmatrix} \frac{\sin t}{t+1} & \frac{\cos t}{t+1} & 0 & 0 \\ -\frac{\cos t}{t+1} & \frac{\sin t}{t+1} & 0 & 0 \\ 0 & 0 & \frac{\sin t}{\sqrt{t+1}} & \frac{\cos t}{\sqrt{t+1}} \\ 0 & 0 & -\frac{\cos t}{\sqrt{t+1}} & \frac{\sin t}{\sqrt{t+1}} \end{bmatrix}.$$

Thus, from Theorem 3.4.3 the system (3.2.2) is Ψ-asymptotically stable on R_+.

Theorem 3.4.4. Let $Y(t)$, $Z(t)$ be the fundamental matrices of (3.2.3), (3.2.4) respectively. If there exists a continuous function $\phi : R_+ \to (0, \infty)$ such that $\int_0^\infty \phi(s)ds = \infty$, and a positive constant L satisfying

$$\int_0^t \phi(s)|\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|ds \leq L, \quad \text{for all} \quad t \geq 0$$

then the homogeneous system (3.2.2) is Ψ-asymptotically stable on R_+.

Proof. From Theorem 3.4.2, the trivial solution of (3.2.2) is Ψ-stable on R_+. And also from (3.4.4),

$$\int_0^\infty \phi(s)ds = \infty, \text{ it follows that } \lim_{t\to\infty} \Psi(t)(Z(t)\otimes Y(t)) = 0.$$

Hence by using Theorem 3.4.3, system (3.2.2) is Ψ-asymptotically stable.

Theorem 3.4.5. Suppose that there exists a continuous function $\phi : R_+ \longrightarrow (0,\infty)$ and a positive constant L, such that the fundamental matrix $Z(t)\otimes Y(t)$ of the system (3.2.2) satisfies the condition

$$\int_0^t \phi(s)|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|ds \le L,\ \forall\, t\ge 0 \quad (3.4.5)$$

and the function $\hat{F} : R_+ \to \mathbf{R}^{n^2}$ is continuous, such that

$$\sup_{t\ge 0} \phi^{-1}(t)\|\Psi(t)\hat{F}(t)\| \quad (3.4.6)$$

is sufficiently a small number. Then the system (3.2.1) is Ψ-stable on R_+.

Proof. From the condition (3.4.5), Theorems 3.4.1 and 3.4.2, there exists a positive constant N such that

$$|\Psi(t)(Z(t)\otimes Y(t))| \le N, \quad \forall\ t\ge 0.$$

By using method of variation of parameters, the solution of (3.2.1) with initial condition $\hat{X}(t_0) = \hat{X}_0$ is given by

$$\hat{X}(t) = (Z(t)\otimes Y(t))(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\hat{X}_0 + \int_{t_0}^t (Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\hat{F}(s)ds,$$

for all $t\ge 0$. For a given $\varepsilon > 0$ and $t_0 \ge 0$, we choose

$\delta(t_0,\varepsilon) = \frac{\varepsilon}{2N|(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|}$ such that $\|\Psi(t_0)\hat{X}_0\| < \delta$.

Consider

$$\|\Psi(t)\hat{X}(t)\| \leq \|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)\Psi(t_0)\hat{X}_0\|$$

$$+\int_{t_0}^{t} \|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)\Psi(s)\hat{F}(s)\|ds$$

$$\leq |\Psi(t)(Z(t)\otimes Y(t))|\ |(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|\ \|\Psi(t_0)\hat{X}_0\|$$

$$+\int_{t_0}^{t} \phi(s)|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|$$

$$\left(\sup_{s\geq 0}\phi^{-1}(s)\|\Psi(s)\hat{F}(s)\|\right) ds.$$

From hypothesis $b = \sup_{t\geq 0}\phi^{-1}(t)\|\Psi(t)\hat{F}(t)\|$ is sufficiently small, so we can choose $Lb < \frac{\varepsilon}{2}$. Therefore

$$\|\Psi(t)\hat{X}(t)\| \leq N|(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|\ \|\Psi(t_0)\hat{X}_0\| + Lb$$

$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Hence the system (3.2.1) is Ψ-stable on R_+.

Remark 3.4.3. Theorem 3.4.5 holds, even if the condition (3.4.6) is replaced by the following condition

$$\lim_{t\to\infty}\phi^{-1}(t)\|\Psi(t)\hat{F}(t)\| = 0.$$

Remark 3.4.4. Theorem 3.4.5 fails, if we replace the condition (3.4.5) by the fact that the system (3.2.2) is Ψ-uniformly stable on R_+.

The following example illustrates the Remark 3.4.4.

Example 3.4.5. Consider the system (3.1.1) with $A(t) = I_2$, $B(t) = -I_2$, and

$$F(t) = \begin{bmatrix} \frac{b}{t+1} & 0 \\ 0 & b \end{bmatrix},$$

where b is a small positive number. Then $Z(t) \otimes Y(t) = I_4$ is a fundamental matrix of (3.2.2). Let

$$\Psi(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{t+1} & 0 \\ 0 & 0 & 0 & \frac{1}{t+1} \end{bmatrix},$$

we have

$$\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{s+1}{t+1} & 0 \\ 0 & 0 & 0 & \frac{s+1}{t+1} \end{bmatrix}$$

is bounded for $0 \le s \le t < \infty$, it follows that the system (3.2.2) is Ψ-uniformly stable on R_+. On the other hand the solutions of the system (3.2.1) are

$$\hat{X}(t) = (b\log(t+1) + c_1, \ c_2, \ c_3, \ bt + c_4)^*.$$

Clearly the system (3.2.1) is not Ψ-stable on R_+.

Finally, if we take $\phi(t) = 1$ on R_+, then

$$\sup_{t\ge 0} \|\Psi(t)\hat{F}(t)\| = b, \text{ and } \lim_{t\to\infty} \|\Psi(t)\hat{F}(t)\| = \lim_{t\to\infty} \frac{b}{t+1} = 0.$$

Theorem 3.4.6. Let $Y(t)$, $Z(t)$ be the fundamental matrices of (3.2.3), (3.2.4) respectively. If there exists a continuous function $\phi : R_+ \to (0, \infty)$ such that $\int_0^\infty \phi(s)ds = \infty$, and a positive constant L satisfying

$$\int_0^t \phi(s)|\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|ds \le L, \quad \text{for all } t \ge 0$$

and a function $\hat{F} : R_+ \to R^{n^2}$ is continuous, such that

$$\lim_{t\to\infty} \phi^{-1}(t)\|\Psi(t)\hat{F}(t)\| = 0.$$

Then the system (3.2.1) is Ψ-asymptotically stable on R_+.

Proof. From Theorem 3.4.4 and Remark 3.4.3, it follows that the homogeneous system (3.2.2) is Ψ-asymptotically stable on R_+ and the non-homogeneous system (3.2.1) is Ψ-stable on R_+. Therefore $\lim_{t\to\infty} \Psi(t)(Z(t) \otimes Y(t)) = 0$. Let $\hat{X}(t)$ be the solution of (3.2.1), then it is easily seen that $\lim_{t\to\infty} \Psi(t)\hat{X}(t) = 0$. Hence the system (3.2.1) is Ψ-asymptotically stable on R_+.

The following example illustrates the Remark 3.4.3 and Theorem 3.4.6.

Example 3.4.6. Consider the system (3.1.1) with $A(t) = I_2$, $B(t) = -I_2$, and

$$F(t) = \begin{bmatrix} \frac{t}{t+1} & 0 \\ 0 & 0 \end{bmatrix},$$

Then $Z(t) \otimes Y(t) = I_4$ is a fundamental matrix of (3.2.2). Let $\Psi(t) = e^{-t}I_4$, then $\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s) = e^{s-t}I_4$. If we take $\phi(t) = 1$ on R_+, then

$$\int_0^t \phi(s)|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|ds = \int_0^t e^{s-t}ds = 1-e^{-t} \leq 1$$

and

$$\lim_{t\to\infty} \phi^{-1}(t)\|\Psi(t)\hat{F}(t)\| = \lim_{t\to\infty} \frac{te^{-t}}{t+1} = 0.$$

From Theorem 3.4.6, the system (3.2.1) is Ψ-asymptotically stable on R_+.