CHAPTER 3

IDEALS IN DUO F-SEMIGROUPS
Chapter 3

IDEALS IN DUO $\Gamma$-SEMIGROUPS

KRULL [29] proved that the nil-radical of an ideal $A$ in a commutative ring is equal to the intersection of all minimal prime ideals containing $A$. SATYANARAYANA [43] obtained KRULL’s theorem [29] for commutative semigroups. ANJANEYULU [4] introduced the notions of ideals in duo semigroups and exhibit some examples and some classes of duo semigroups. He obtained KRULL’s theorem [29] for pseudo symmetric semigroups which includes duo semigroups. MADHUSUDHANA RAO, ANJANEYULU and GANGADHARA RAO [32], [33], [34] and [35] introduced the notions of duo $\Gamma$-semigroups and obtained KRULL’s theorem for pseudo and semipseudo symmetric $\Gamma$-semigroups. In this thesis we introduce and made a study on ideals in duo $\Gamma$-semigroups and obtained an analogue of KRULL’s theorem [29] in duo $\Gamma$-semigroups.

This chapter is divided into 5 sections. In section 1, the terms; left duo $\Gamma$-semigroup, right duo $\Gamma$-semigroup, duo $\Gamma$-semigroup are introduced. It is proved that a $\Gamma$-semigroup $S$ is a duo $\Gamma$-semigroup if and only if $x\Gamma s^l = s^l \Gamma x$ for all $x \in S$. Further it is proved that (1) every commutative $\Gamma$-semigroup is a duo $\Gamma$-semigroup (2) every normal $\Gamma$-semigroup is a duo $\Gamma$-semigroup (3) every quasi commutative $\Gamma$-semigroup is a duo $\Gamma$-semigroup (4) every generalized $\Gamma$-semigroup is a left duo $\Gamma$-semigroup.

In section 2, it is proved that (1) if $A$ is a $\Gamma$-ideal in a left duo $\Gamma$-semigroup $S$, then $A(a) = \{ x \in S : x\Gamma a \subseteq A \}$ is a $\Gamma$-ideal of $S$ for all $a \in S$, (2) if $A$ is a $\Gamma$-ideal in a right duo $\Gamma$-semigroup $S$, then $A_r(a) = \{ x \in S : a\Gamma x \subseteq A \}$ is a $\Gamma$-ideal of $S$ for all $a \in S$, (3) if $A$ is a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$, then $A(a) = \{ x \in S : x\Gamma a \subseteq A \}$ and $A_r(a) = \{ x \in S : a\Gamma x \subseteq A \}$ are $\Gamma$-ideals of $S$ for all $a \in S$. Further it is proved that (1) if $A$ is a $\Gamma$-ideal in a left duo $\Gamma$-semigroup $S$ and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$ for all $s \in S$, (2) if $A$ is a $\Gamma$-ideal in a right duo $\Gamma$-semigroup $S$ and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$ for all $s \in S$, (3) if $A$ is a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$ and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$. It is proved that if $A$ is a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$ and $a, b \in S$, then (1) $a\Gamma b \in A$ iff $< a > \Gamma < b > \subseteq A$, (2) $a_1\Gamma a_2\Gamma \ldots a_n \Gamma a_n \subseteq A$ iff $< a_1 > \Gamma < a_2 > \ldots \Gamma < a_n > \subseteq A$, (3) for any natural number $n$, $(a\Gamma)^{n-l} \Gamma a \subseteq A$ iff $(< a > \Gamma)^{n-l} < a > \subseteq A$. It is also proved that in a duo $\Gamma$-semigroup $S$, a $\Gamma$-ideal $P$ is prime $\Gamma$-ideal if and only if $P$ is a completely prime $\Gamma$-ideal. Further it is proved that a
\(
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\)

Gamma-ideal \(A\) of a duo \(\Gamma\)-semigroup \(S\) is a completely semiprime \(\Gamma\)-ideal of \(S\) if and only if \(A\) is a semiprime \(\Gamma\)-ideal.

In section 3, it is proved that, if \(A_1 = \text{the intersection of all completely prime } \Gamma\text{-ideals of } S \text{ containing } A\), \(A_2 = \{x \in S : (x\Gamma)^{n-1}x \subseteq A \text{ for some natural number } n \}\), \(A_3 = \text{the intersection of all prime } \Gamma\text{-ideals of } S \text{ containing } A\), \(A_4 = \{x \in S : (\langle x \rangle^\Gamma)^{n-1} < x > \subseteq A \text{ for some natural number } n \}\) for a \(\Gamma\)-ideal \(A\) of a \(\Gamma\)-semigroup \(S\), then \(A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1\). If \(A\) is a \(\Gamma\)-ideal of a commutative/duo \(\Gamma\)-semigroup then it is proved that \(A_1 = A_2 = A_3 = A_4\). It is proved that if \(A\) is a \(\Gamma\)-ideal in a duo \(\Gamma\)-semigroup \(S\), then (1) \(A_2\) is the minimal completely semiprime \(\Gamma\)-ideal of \(S\) containing \(A\), (2) \(A_4\) is the minimal semiprime \(\Gamma\)-ideal of \(S\) containing \(A\). It is proved that if \(a \in \sqrt{A}\), then there exist a positive integer \(n\) such that \((a\Gamma)^{n-1}a \subseteq A\). Further if \(A\) is a \(\Gamma\)-ideal of a duo \(\Gamma\)-semigroup \(S\) then it is proved that (1) \(A_1\) = the intersection of all completely prime \(\Gamma\)-ideals of \(S\) containing \(A\), (2) \(A'_1\) = the intersection of all minimal completely prime \(\Gamma\)-ideals of \(S\) containing \(A\), (3) \(A''_1\) = the minimal completely semiprime \(\Gamma\)-ideal of \(S\) containing \(A\), (4) \(A_2\) = \(\{x \in S : (x\Gamma)^{n-1}x \subseteq A \text{ for some natural number } n \}\), (5) \(A_3\) = the intersection of all prime \(\Gamma\)-ideals of \(S\) containing \(A\), (6) \(A'_3\) = the intersection of all minimal prime \(\Gamma\)-ideals of \(S\) containing \(A\), (7) \(A''_3\) = the minimal semiprime \(\Gamma\)-ideal of \(S\) containing \(A\), (8) \(A_4\) = \(\{x \in S : (\langle x \rangle^\Gamma)^{n-1} < x > \subseteq A \text{ for some natural number } n \}\) are equal.

In section 4, the terms; Archimedean \(\Gamma\)-semigroup and strongly Archimedean \(\Gamma\)-semigroup are introduced. It is proved that if \(S\) is a duo \(\Gamma\)-semigroup, then the conditions (1) \(S\) is strongly Archimedean, (2) \(S\) is Archimedean, (3) \(S\) has no proper completely prime \(\Gamma\)-ideals and (4) \(S\) has no proper prime \(\Gamma\)-ideals; are equivalent.

In section 5, the terms; left simple \(\Gamma\)-semigroup, right simple \(\Gamma\)-semigroup, simple \(\Gamma\)-semigroup are introduced. It is proved that (1) a \(\Gamma\)-semigroup \(S\) is a left simple \(\Gamma\)-semigroup if and only if \(S\Gamma a = S\) for all \(a \in S\), (2) a \(\Gamma\)-semigroup \(S\) is a right simple \(\Gamma\)-semigroup if and only if \(a\Gamma S = S\) for all \(a \in S\), (3) a \(\Gamma\)-semigroup \(S\) is a simple \(\Gamma\)-semigroup if and only if \(S\Gamma a\Gamma S = S\) for all \(a \in S\). It is also proved that if \(S\) is a left simple \(\Gamma\)-semigroup or a right simple \(\Gamma\)-semigroup then \(S\) is a simple \(\Gamma\)-semigroup. Further it is proved that if \(S\) is a duo \(\Gamma\)-semigroup and \(a \in S\) then (1) \(a\) is regular, (2) \(a\) is
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left regular, (3) \( a \) is right regular, (4) \( a \) is intra regular and (5) \( a \) is semisimple are equivalent.

The contents of chapter 3 are published in "International eJournal of Mathematics and Engineering" under the title 'Prime Γ-ideals in duo Γ-semigroups' [17].

3.1. DUO Γ-SEMIGROUPS

Duo Γ-semigroups played an important role in the theory of Γ-semigroups. In this section the terms; left duo Γ-semigroup, right duo Γ-semigroup, duo Γ-semigroup are introduced. It is proved that a Γ-semigroup \( S \) is a duo Γ-semigroup if and only if \( x \Gamma S^1 = S^1 \Gamma x \) for all \( x \in S \). Further it is proved that (1) every commutative Γ-semigroup is a duo Γ-semigroup (2) every normal Γ-semigroup is a duo Γ-semigroup (3) every quasi commutative Γ-semigroup is a duo Γ-semigroup (4) every generalized Γ-semigroup is a left duo Γ-semigroup.

We now introduce a left duo Γ-semigroup, right duo Γ-semigroup and duo Γ-semigroup.

**DEFINITION 3.1.1 :** A Γ-semigroup \( S \) is said to be a **left duo Γ-semigroup** provided every left Γ-ideal of \( S \) is a two sided Γ-ideal of \( S \).

**DEFINITION 3.1.2 :** A Γ-semigroup \( S \) is said to be a **right duo Γ-semigroup** provided every right Γ-ideal of \( S \) is a two sided Γ-ideal of \( S \).

**DEFINITION 3.1.3 :** A Γ-semigroup \( S \) is said to be a **duo Γ-semigroup** provided it is both a left duo Γ-semigroup and a right duo Γ-semigroup.

**THEOREM 3.1.4 :** A Γ-semigroup \( S \) is a duo Γ-semigroup if and only if \( x \Gamma S^1 = S^1 \Gamma x \) for all \( x \in S \).

**Proof :** Suppose that \( S \) is a duo Γ-Semigroup and \( x \in S \).

Let \( t \in x \Gamma S^1 \). Then \( t = x \gamma s \) for some \( s \in S^1, \gamma \in \Gamma \).

Since \( S^1 \Gamma x \) is a left Γ-ideal of \( S \), \( S^1 \Gamma x \) is a Γ-ideal of \( S \).

So \( x \in S^1 \Gamma x, \gamma \in \Gamma, s \in S, S^1 \Gamma x \) is a Γ-ideal \( \Rightarrow x \gamma s \in S^1 \Gamma x \Rightarrow t \in S^1 \Gamma x \).

Therefore \( x \Gamma S^1 \subseteq S^1 \Gamma x \). Similarly we can prove that \( S^1 \Gamma x \subseteq x \Gamma S^1 \). Therefore \( S^1 \Gamma x = x \Gamma S^1 \).

Conversely suppose that \( S^1 \Gamma x = x \Gamma S^1 \) for all \( x \in S \). Let \( A \) be a left Γ-ideal of \( S \).

Let \( x \in A, s \in S \) and \( a \in \Gamma \). Then \( xas \in x \Gamma S^1 = S^1 \Gamma x \Rightarrow xas = t \beta x \) for some \( t \in S^1, \beta \in \Gamma \).
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\( x \in A, t \in S, \beta \in \Gamma, A \) is a left \( \Gamma \)-ideal of \( S \Rightarrow t\beta x \in A \Rightarrow x\alpha s \in A \).

Therefore \( A \) is a right \( \Gamma \)-ideal of \( S \) and hence \( A \) is a \( \Gamma \)-ideal of \( S \).

Therefore \( S \) is left duo \( \Gamma \)-semigroup.

Similarly we can prove that \( S \) is a right duo \( \Gamma \)-semigroup. Hence \( S \) is duo \( \Gamma \)-semigroup.

**THEOREM 3.1.5 :** Every commutative \( \Gamma \)-semigroup is a duo \( \Gamma \)-semigroup.

**Proof :** Suppose that \( S \) is a commutative \( \Gamma \)-semigroup. Therefore \( x\Gamma S^1 = S^1\Gamma x \) for all \( x \in S \). By theorem 3.1.4, \( S \) is a duo \( \Gamma \)-semigroup.

**THEOREM 3.1.6 :** Every normal \( \Gamma \)-semigroup is a duo \( \Gamma \)-semigroup.

**Proof :** Suppose that \( S \) is normal \( \Gamma \)-semigroup.

Then \( a\Gamma S = S\Gamma a \) for all \( a \in S \Rightarrow a\Gamma S^1 = S^1\Gamma a \) for all \( a \in S \).

By theorem 3.1.4, \( S \) is a duo \( \Gamma \)-semigroup.

**THEOREM 3.1.7 :** Every quasi commutative \( \Gamma \)-semigroup is a duo \( \Gamma \)-semigroup.

**Proof :** Suppose that \( S \) is a quasi commutative \( \Gamma \)-semigroup. Then for \( a, b \in S \), there exists \( n \in \mathbb{N} \) such that \( ayb = (by)^n a \) for all \( y \in \Gamma \). Let \( A \) be a left \( \Gamma \)-ideal of \( S \).

Therefore \( S\Gamma A \subseteq A \). Let \( a \in A \) and \( s \in S \). Since \( S \) is a quasi commutative \( \Gamma \)-semigroup, there exists a natural number \( n \) such that \( a\Gamma s = (s\Gamma)^n a \subseteq S\Gamma A \subseteq A \). Therefore \( a\Gamma s \subseteq A \) for all \( a \in A \) and \( s \in S \) and hence \( A\Gamma S \subseteq A \). Thus \( A \) is right \( \Gamma \)-ideal of \( S \).

Therefore \( S \) is a left duo \( \Gamma \)-semigroup. Similarly we can prove that \( S \) is a right duo \( \Gamma \)-semigroup. Therefore every quasi commutative \( \Gamma \)-semigroup is a duo \( \Gamma \)-semigroup.

**THEOREM 3.1.8 :** Every generalized commutative \( \Gamma \)-semigroup is a left duo \( \Gamma \)-semigroup.

**Proof :** Let \( S \) be a generalized commutative \( \Gamma \)-semigroup. Therefore \( 1 \) is an \( r \)-element.

Let \( A \) be a left \( \Gamma \)-ideal of \( S \). Let \( x \in A \) and \( s \in S \).

Now \( x\Gamma s = I\Gamma x\Gamma s = b\Gamma s\Gamma x = (b\Gamma s)\Gamma x \subseteq A \). Therefore \( A \) is a \( \Gamma \)-ideal of \( S \).

Therefore \( S \) is a left duo \( \Gamma \)-semigroup.

**3.2. \( \Gamma \)-IDEALS IN DUO \( \Gamma \)-SEMIGROUPS**

In this section, it is proved that (1) if \( A \) is a \( \Gamma \)-ideal in a left duo \( \Gamma \)-semigroup \( S \), then \( A(a) = \{ x \in S : x\Gamma a \subseteq A \} \) is a \( \Gamma \)-ideal of \( S \) for all \( a \in S \), (2) if \( A \) is a \( \Gamma \)-ideal in a
right duo $\Gamma$-semigroup $S$, then $A_\Gamma(a) = \{ x \in S : a\Gamma x \subseteq A \}$ is a $\Gamma$-ideal of $S$ for all $a \in S$.

(3) if $A$ is a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$, then $A_\Gamma(a) = \{ x \in S : a\Gamma x \subseteq A \}$ and $A_\Gamma(a) = \{ x \in S : a\Gamma x \subseteq A \}$ are $\Gamma$-ideals of $S$ for all $a \in S$. Further it is proved that (1) if $A$ is a $\Gamma$-ideal in a left duo $\Gamma$-semigroup $S$ and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$ for all $s \in S$, (2) if $A$ is a $\Gamma$-ideal in a right duo $\Gamma$-semigroup $S$ and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$ for all $s \in S$, (3) if $A$ is a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$ and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$. It is proved that if $A$ is a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$ and $a, b \in S$, then (1) $a\Gamma b \in A$ iff $< a > \Gamma < b > \subseteq A$, (2) $a_1\Gamma a_2\Gamma \ldots \Gamma a_n a_n \subseteq A$ iff $< a_1 > \Gamma < a_2 > \Gamma \ldots \Gamma < a_n > \subseteq A$, (3) for any natural number $n$, $(a\Gamma)^n \subseteq A$ iff $(< a > \Gamma)^n \subseteq A$. It is also proved that in a duo $\Gamma$-semigroup $S$, a $\Gamma$-ideal $P$ is prime $\Gamma$-ideal if and only if $P$ is a completely prime $\Gamma$-ideal. Further it is proved that a $\Gamma$-ideal $A$ of a duo $\Gamma$-semigroup $S$ is a completely semiprime $\Gamma$-ideal of $S$ if and only if $A$ is a semiprime $\Gamma$-ideal.

We now characterize left duo $\Gamma$-semigroups.

**Theorem 3.2.1**: If $A$ is a $\Gamma$-ideal in a left duo $\Gamma$-semigroup $S$, then $A_\Gamma(a) = \{ x \in S : x\Gamma a \subseteq A \}$ is a $\Gamma$-ideal of $S$ for all $a \in S$.

**Proof**: Let $x \in A_\Gamma(a)$ and $s \in S$. $x \in A_\Gamma(a) \Rightarrow x\Gamma a \subseteq A$.

$x\Gamma a \subseteq A$, $s \in S$, $A$ is a $\Gamma$-ideal $\Rightarrow s\Gamma x\Gamma a \subseteq A \Rightarrow s\Gamma x \subseteq A(\Gamma a)$.

Therefore $A_\Gamma(a)$ is a left $\Gamma$-ideal of $S$. Since $S$ is a left duo $\Gamma$-semigroup, $A_\Gamma(a)$ is a $\Gamma$-ideal of $S$.

**Theorem 3.2.2**: If $A$ is a $\Gamma$-ideal in a left duo $\Gamma$-semigroup $S$ and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$ for all $s \in S$.

**Proof**: Suppose that $x\Gamma y \subseteq A$. Let $s \in S$.

$x\Gamma y \subseteq A \Rightarrow x \in A_\Gamma(y)$.

$x \in A_\Gamma(y)$, $s \in S$, $A_\Gamma(y)$ is a $\Gamma$-ideal of $S \Rightarrow x\Gamma s \subseteq A_\Gamma(y) \Rightarrow x\Gamma s\Gamma y \subseteq A$.

We now characterize right duo $\Gamma$-semigroups.

**Theorem 3.2.3**: If $A$ is a $\Gamma$-ideal in a right duo $\Gamma$-semigroup $S$, then $A_\Gamma(a) = \{ x \in S : a\Gamma x \subseteq A \}$ is a $\Gamma$-ideal of $S$ for all $a \in S$.

**Proof**: Let $x \in A_\Gamma(a)$ and $s \in S$. $x \in A_\Gamma(a) \Rightarrow a\Gamma x \subseteq A$.

$a\Gamma x \subseteq A$, $s \in S$, $A$ is a $\Gamma$-ideal $\Rightarrow a\Gamma x\Gamma s \subseteq A \Rightarrow x\Gamma s \subseteq A_\Gamma(a)$.
Therefore $A_r(a)$ is a right $\Gamma$-ideal of $S$.
Since $S$ is a right duo $\Gamma$-semigroup, $A_r(a)$ is a $\Gamma$-ideal of $S$.

**Theorem 3.2.4**: If $A$ is a $\Gamma$-ideal in a right duo $\Gamma$-semigroup $S$ and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$.

**Proof**: Suppose that $x\Gamma y \subseteq A$. Let $s \in S$. $x\Gamma y \subseteq A \Rightarrow y \in A_r(x)$.

Since $A_r(x)$ is a $\Gamma$-ideal of $S \Rightarrow s\Gamma y \subseteq A_r(x) \Rightarrow x\Gamma s\Gamma y \subseteq A$.

We now characterize duo $\Gamma$-semigroups.

**Corollary 3.2.5**: If $A$ is a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$ and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$.

**Theorem 3.2.6**: If $A$ is a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$, then $A_r(a) = \{ x \in S : xTa \subseteq A \}$ and $A_s(a) = \{ x \in S : aTx \subseteq A \}$ are $\Gamma$-ideals of $S$ for all $a \in S$.

**Proof**: Since $S$ is a duo $\Gamma$-semigroup, $S$ is left duo $\Gamma$-semigroup and hence by theorem 3.2.1, $A_r(a) = \{ x \in S : xTa \subseteq A \}$ is a $\Gamma$-ideal of $S$. Again $S$ is right duo $\Gamma$-semigroup and hence by theorem 3.2.3, $A_s(a) = \{ x \in S : aTx \subseteq A \}$ is a $\Gamma$-ideal of $S$.

**Theorem 3.2.7**: Let $A$ be a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$ and $a, b \in S$. Then $a\Gamma b \subseteq A$ if and only if $<a> \Gamma <b> \subseteq A$.

**Proof**: Suppose that $<a> \Gamma <b> \subseteq A$. Then $a\Gamma b \subseteq <a> \Gamma <b> \subseteq A$.

Conversely suppose that $a\Gamma b \subseteq A$. Since $S$ is a duo $\Gamma$-semigroup. By corollary 3.2.5, $a\Gamma b \subseteq A \Rightarrow a\Gamma s\Gamma b \subseteq A$ for all $s \in S \Rightarrow a\Gamma s\Gamma b \subseteq A$. Since $A$ is a $\Gamma$-ideal, $a\Gamma S\Gamma b \subseteq A \Rightarrow S\Gamma a \Gamma S\Gamma b \subseteq A \Rightarrow <a> \Gamma <b> \subseteq A$.

**Theorem 3.2.8**: Let $A$ be a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$. Then $a_1\Gamma a_2\Gamma \ldots \Gamma a_n \subseteq A$ if and only if $<a_1> \Gamma <a_2> \ldots \Gamma <a_n> \subseteq A$.

**Proof**: Suppose that $<a_1> \Gamma <a_2> \ldots \Gamma <a_n> \subseteq A$.

Then $a_1\Gamma a_2\Gamma \ldots \Gamma a_n \subseteq <a_1> \Gamma <a_2> \ldots \Gamma <a_n> \subseteq A$.

Conversely suppose that $a_1\Gamma a_2\Gamma \ldots \Gamma a_n \subseteq A$.

Then for any $t \in <a_1> \Gamma <a_2> \ldots \Gamma <a_n>$, we have $t = s_1a_1a_2a_3a_4a_5a_6 \ldots a_ka_1a_2a_3a_4a_5a_6 \ldots a_n$, where $s_i \in S$ and $a_i, \beta_i \in \Gamma$.

Since $x, y \in S$, $x\Gamma y \subseteq A \Rightarrow x\Gamma s\Gamma y \subseteq A$, we have $t \in A$. 

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COROLLARY 3.2.9: Let \( A \) be a \( \Gamma \)-ideal in a duo \( \Gamma \)-semigroup \( S \). Then for any natural number \( n \), \( (a \, \Gamma)^{n-1}a \subseteq A \) if and only if \( (\langle a \rangle \, \Gamma)^{n-1} \, a \subseteq A \).

*Proof*: The proof follows from theorem 3.2.8, by taking \( a_1 = a_2 = a_3 = \ldots = a_n = a \).

THEOREM 3.2.10: Let \( S \) be a duo \( \Gamma \)-semigroup. A \( \Gamma \)-ideal \( P \) of \( S \) is prime \( \Gamma \)-ideal if and only if \( P \) is a completely prime \( \Gamma \)-ideal.

*Proof*: Suppose that \( P \) is a prime \( \Gamma \)-ideal of \( \Gamma \)-semigroup \( S \). Let \( x, y \in S \) and \( x \Gamma y \subseteq P \). Now \( x \Gamma y \subseteq P \), \( P \) is a \( \Gamma \)-ideal \( \Rightarrow x \Gamma y \, S^1 \subseteq P \).

Since \( S \) is duo \( \Gamma \)-semigroup, \( x \Gamma S^1 \, \Gamma y = x \Gamma y \, \Gamma S^1 \subseteq P \).

By corollary 2.2.7, either \( x \in P \) or \( y \in P \). Hence \( P \) is a completely prime \( \Gamma \)-ideal.

Conversely suppose that \( P \) is a completely prime \( \Gamma \)-ideal of \( S \).

By theorem 2.2.8, \( P \) is a prime \( \Gamma \)-ideal of \( S \).

COROLLARY 3.2.11: Let \( S \) be a commutative \( \Gamma \)-semigroup. A \( \Gamma \)-ideal \( P \) of \( S \) is prime \( \Gamma \)-ideal if and only if \( P \) is a completely prime \( \Gamma \)-ideal.

THEOREM 3.2.12: Let \( S \) be a duo \( \Gamma \)-semigroup. A \( \Gamma \)-ideal \( A \) of \( S \) is completely semiprime iff semiprime.

*Proof*: Suppose that \( A \) is a completely semiprime \( \Gamma \)-ideal of \( S \).

By theorem 2.3.7, \( A \) is a semiprime \( \Gamma \)-ideal of \( S \).

Conversely Suppose that \( A \) is a semiprime \( \Gamma \)-ideal of \( S \). Let \( x \in S \) and \( x \Gamma x \subseteq A \).

Now \( x \Gamma x \subseteq A \Rightarrow x \Gamma x \, S^1 \subseteq A \) for all \( s \in S \) \( \Rightarrow x \Gamma S^1 \, x \subseteq A \) for all \( s \in S \) \( \Rightarrow x \Gamma S^1 \, x \subseteq A \).

\( \Rightarrow x \in A \), since \( A \) is semiprime. Therefore \( A \) is a completely semiprime \( \Gamma \)-ideal of \( S \).

COROLLARY 3.2.13: Let \( S \) be a commutative \( \Gamma \)-semigroup. A \( \Gamma \)-ideal \( A \) of \( S \) is completely semiprime iff semiprime.

3.3. \( \Gamma \)-RADICALS IN DUO \( \Gamma \)-SEMIGROUPS

In this section, it is proved that, if \( A_1 = \) the intersection of all completely prime \( \Gamma \)-ideals of \( S \) containing \( A \), \( A_2 = \{ x \in S : (x \Gamma)^{n-1} x \subseteq A \) for some natural number \( n \} \), \( A_3 = \) the intersection of all prime \( \Gamma \)-ideals of \( S \) containing \( A \), \( A_4 = \{ x \in S : (x \, \Gamma)^{n-1} < x > \subseteq A \) for some natural number \( n \} \) for a \( \Gamma \)-ideal \( A \) of a \( \Gamma \)-semigroup \( S \), then
A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1. If A is a \Gamma-ideal of a commutative/duo \Gamma-semigroup then it is proved that A_1 = A_2 = A_3 = A_4. It is proved that if A is a \Gamma-ideal in a duo \Gamma-semigroup S, then (1) A_2 is the minimal completely semiprime \Gamma-ideal of S containing A, (2) A_4 is the minimal semiprime \Gamma-ideal of S containing A. It is proved that if \alpha \in \sqrt{A}$, then there exist a positive integer n such that $(\alpha \Gamma)^{n-1}a \subseteq A$. Further if A is a \Gamma-ideal of a duo \Gamma-semigroup S then it is proved that (1) $A_1 = \text{the intersection of all completely prime \Gamma-ideals of S containing A}$, (2) $A_4' = \text{the intersection of all minimal completely prime \Gamma-ideals of S containing A}$, (3) $A_4'' = \text{the minimal completely semiprime \Gamma-ideal of S containing A}$, (4) $A_2 = \{x \in S : (x \Gamma)^{n-1}x \subseteq A \text{ for some natural number } n \}$, (5) $A_3 = \text{the intersection of all prime \Gamma-ideals of S containing A}$, (6) $A_4' = \text{the intersection of all minimal prime \Gamma-ideals of S containing A}$, (7) $A_4'' = \text{the minimal semiprime \Gamma-ideal of S containing A}$, (8) $A_4 = \{x \in S : (<x > \Gamma)^{n-1} <x > \subseteq A \text{ for some natural number } n \}$ are equal.

**NOTATION 3.3.1** : If A is a \Gamma-ideal of a \Gamma-semigroup S, then we associate the following four types of sets.

- $A_1 = \text{The intersection of all completely prime \Gamma-ideals of S containing A}$.
- $A_2 = \{x \in S : (x \Gamma)^{n-1}x \subseteq A \text{ for some natural number } n \}$
- $A_3 = \text{The intersection of all prime ideals of S containing A}$.
- $A_4 = \{x \in S : (<x > \Gamma)^{n-1} <x > \subseteq A \text{ for some natural number } n \}$

**NOTE 3.3.2** : If A is a \Gamma-ideal of a \Gamma-semigroup S then $\text{rad } A = A_3$ and $\text{c.rad } A = A_4$.

**THEOREM 3.3.3** : If A is a \Gamma-ideal of a \Gamma-semigroup S, then $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

**Proof** : (i) $A \subseteq A_4$ : Let $x \in A$. Then $(<x > \Gamma)^0 <x > \subseteq A$ and hence $x \in A_4$. $\therefore A \subseteq A_4$.

(ii) $A_4 \subseteq A_3$ : Let $x \in A_4$. Then $(<x > \Gamma)^{n-1} <x > \subseteq A$ for some $n \in N$.

Let P be any prime \Gamma-ideal of S containing A. Then $(<x > \Gamma)^{n-1} <x > \subseteq A \Rightarrow ( <x > \Gamma)^{n-1} <x > \subseteq P$.

Since P is prime, $<x > \subseteq P$ and hence $x \in P$.

Since this is true for all prime \Gamma-ideals P containing A, $x \in A_3$. Therefore $A_4 \subseteq A_3$.

(iii) $A_3 \subseteq A_2$ : Let $x \in A_3$. Suppose if possible $x \notin A_2$. Then $(x \Gamma)^{n-1}x \notin A$ for all $n \in N$.

Consider $T = U(x \Gamma)^{n-1}x$, where $x \in S$ and $n$ is a natural number.

Let $a, b \in T$. Then $a \in (x \Gamma)^{r-1}x$, $b \in (x \Gamma)^{s-1}x$ for some $r, s \in N$. 


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Therefore \( aTb = (x\Gamma)^{n-1}x(x\Gamma)^{n-1}x = (x\Gamma)^{n-1}x \subseteq T \).

Therefore \( T \) is a \( \Gamma \)-subsemigroup of \( S \) and \( T \) is a \( c \)-system of \( S \) and \( x \in T \).

By theorem 2.2.4, \( P = S \cap T \) is a completely prime \( \Gamma \)-ideal of \( S \) and \( x \not\in P \).

By theorem 2.2.8, \( P \) is prime \( \Gamma \)-ideal of \( S \) and \( x \not\in P \).

Therefore \( x \not\in A_3 \). It is a contradiction. \( \therefore x \in A_2 \) and hence \( A_3 \subseteq A_2 \).

(iv) \( A_2 \subseteq A_1 \): Let \( x \in A_2 \). Now \( x \in A_2 \Rightarrow (x\Gamma)^{n-1}x \subseteq A \) for some natural number \( n \).

Let \( P \) be any completely prime \( \Gamma \)-ideal of \( S \) containing \( A \).

Then \( (x\Gamma)^{n-1}x \subseteq A \subseteq P \Rightarrow (x\Gamma)^{n-1}x \subseteq P \Rightarrow x \in P \). Therefore \( x \in A_1 \). Therefore \( A_2 \subseteq A_1 \).

Hence \( A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1 \).

**THEOREM 3.3.4**: If \( A \) is a \( \Gamma \)-ideal of a commutative \( \Gamma \)-semigroup \( S \), then \( A_1 = A_2 = A_3 = A_4 \).

**Proof**: By theorem 3.3.3, \( A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1 \). By corollary 3.2.11, in a commutative \( \Gamma \)-semigroup \( S \), a \( \Gamma \)-ideal \( P \) is a prime \( \Gamma \)-ideal iff \( P \) is a completely prime \( \Gamma \)-ideal. So \( A_1 = A_3 \). By theorem 3.2.13, in a commutative \( \Gamma \)-semigroup \( S \), a \( \Gamma \)-ideal \( P \) is a semiprime \( \Gamma \)-ideal iff \( P \) is a completely semiprime \( \Gamma \)-ideal. So \( A_4 = A_2 \).

Therefore \( A_1 = A_2 = A_3 = A_4 \).

**NOTE 3.3.5**: If \( A \) is a \( \Gamma \)-ideal in a arbitrary \( \Gamma \)-semigroup, then \( A_1, A_2, A_3, A_4 \) need not be equal.

**EXAMPLE 3.3.6**: Let \( S \) be the free \( \Gamma \)-semigroup generated by two alphabets \( a, b \). It is clear that \( A = S\Gamma a\Gamma a\Gamma S \) is a \( \Gamma \)-ideal in \( S \). Since \( (a\Gamma)^3a \subseteq S\Gamma a\Gamma a\Gamma S = A \), we have \( a \in A_2 \).

Evidently \( (a\Gamma b\Gamma)^{n-1}a\Gamma b \not\subseteq S\Gamma a\Gamma a\Gamma S \) for all natural number \( n \) and thus \( a\Gamma b \not\subseteq A_2 \). Thus \( A_2 \) is not a \( \Gamma \)-ideal in \( S \). Therefore \( A_1 \neq A_2 \) and \( A_2 \neq A_3 \).

**EXAMPLE 3.3.7**: Let \( S \) be the free \( \Gamma \)-semigroup over the countable infinite alphabet \( \{ x_1, x_2, \ldots \} \) and \( \Gamma \) as \{ \( a_1, a_2, \ldots \} \). Consider the \( \Gamma \)-ideal \( A = \bigcup_{l(s)} (\langle s \rangle \Gamma^{|s|-1} < s >) \), where \( l(s) \) is the length of the word \( s \). For any \( s \in S \), \( x_1\Gamma s\Gamma x_1 > l(s)+1 < x_1\Gamma s\Gamma x_1 > \subseteq A \) and hence \( x_1\Gamma s\Gamma x_1 \subseteq A_4 \) for all \( s \in S \). If \( A_3 = A_4 \), then \( A_4 \) is a semiprime \( \Gamma \)-ideal and hence \( x_1 \in A_4 \). Therefore \( < x_1 > \Gamma^{n-1} < x_1 > \subseteq A \) for some natural number \( n \). Consider the word \( t = x_1a_1x_2a_2x_3a_3x_4a_4x_5 \ldots \ldots a_nx_1a_{n+1}x_{n+1} \).

Now \( t \in < x_1 > \Gamma^{n-1} < x_1 > \subseteq A \). So \( t \in < s \Gamma >^{l(s)-1} < s > \) for some \( s \in S \) with \( l(s) > 1 \).

Thus in \( t, s \) occurs at least two times, which is a contradiction. So \( A_3 \neq A_4 \).
THEOREM 3.3.8: If $A$ is a $\Gamma$-ideal of a duo $\Gamma$-semigroup $S$, then $A_1 = A_2 = A_3 = A_4$.

Proof: By theorem 3.3.3, $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$. By theorem 3.1.10, in a duo $\Gamma$-semigroup $S$, a $\Gamma$-ideal $P$ is a prime $\Gamma$-ideal iff $P$ is a completely prime $\Gamma$-ideal.

So $A_1 = A_3$. By theorem 3.2.12, in a duo $\Gamma$-semigroup $S$, a $\Gamma$-ideal $P$ is a semiprime $\Gamma$-ideal iff $P$ is a completely semiprime $\Gamma$-ideal. So $A_4 = A_2$.

Therefore $A_1 = A_2 = A_3 = A_4$.

THEOREM 3.3.9: If $A$ is a $\Gamma$-ideal of a duo $\Gamma$-semigroup $S$, then $\text{rad} A = c.\text{rad} A$.

Proof: By theorem 3.3.8, $\text{rad} A = c.\text{rad} A$.

THEOREM 3.3.10: If $A$ is a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$, then $A_2 = \{x \in S : (x\Gamma)^{n-1} x \subseteq A \text{ for some } n \in \mathbb{N} \}$ is the minimal completely semiprime $\Gamma$-ideal of $S$ containing $A$.

Proof: Clearly $A \subseteq A_2$ and hence $A_2$ is nonempty subset of $S$. Let $x \in A_2$ and $s \in S$.

Since $x \in A_2$, $(x\Gamma)^{n-1} x \subseteq A$ for some $n \in \mathbb{N}$. Now $(x\Gamma s)^{n-1} x \Gamma s \subseteq A$ and $(s\Gamma x)^{n-1} s\Gamma x \subseteq A$ implies $x\Gamma s, s\Gamma x \in A_2$. Therefore $A_2$ is a $\Gamma$-ideal of $S$ containing $A$. Let $x \in S$ such that $x\Gamma x \subseteq A_2$. Then $(x\Gamma x\Gamma)^{n-1} x \Gamma x \subseteq A$.

Thus $A_2$ is a completely semiprime $\Gamma$-ideal of $S$ containing $A$. Let $P$ be a completely semi prime $\Gamma$-ideal of $S$ containing $A$. Let $x \in A_2$. Then $(x\Gamma)^{n-1} x \subseteq A$ for some $n \in \mathbb{N}$. Since $A \subseteq P$, then $(x\Gamma)^{n-1} x \subseteq P$ for some $n \in \mathbb{N}$. Since $P$ is completely semiprime $\Gamma$-ideal of $S$, $(x\Gamma)^{n-1} x \subseteq P \Rightarrow x \in P$. Therefore $A_2 \subseteq P$ and hence $A_2$ is the minimal completely semiprime $\Gamma$-ideal of $S$ containing $A$.

THEOREM 3.3.11: If $A$ is a $\Gamma$-ideal in a duo $\Gamma$-semigroup $S$, then $A_4 = \{x \in S : (< x >\Gamma)^{n-1} < x > \subseteq A \text{ for some } n \in \mathbb{N} \}$ is the minimal semiprime $\Gamma$-ideal of $S$ containing $A$.

Proof: Clearly $A \subseteq A_4$ and hence $A_4$ is nonempty subset of $S$. Let $x \in A_4$ and $s \in S$.

Since $x \in A_4$, $(< x >\Gamma)^{n-1} < x > \subseteq A$ for some $n \in \mathbb{N}$.

Now $(< x s \Gamma >\Gamma)^{n-1} < x s \Gamma > \subseteq (< x >\Gamma)^{n-1} < x > \subseteq A$ and $(s\Gamma x >\Gamma)^{n-1} < s\Gamma x > \subseteq (< x >\Gamma)^{n-1} < x > \subseteq A$ implies $x s \Gamma, s\Gamma x \subseteq A_4$.

Therefore $A_4$ is a $\Gamma$-ideal of $S$ containing $A$. Let $x \in S$ such that $(< x >\Gamma) < x > \subseteq A_4$. 

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Then \((x \Gamma^n x) \leq A\) implies \((x \Gamma)^n \leq A\) \(\Rightarrow x \in A_4\).

Thus \(A_4\) is semiprime \(\Gamma\)-ideal of \(S\) containing \(A\).

Let \(Q\) be a semiprime \(\Gamma\)-ideal of \(S\) containing \(A\). Let \(x \in A_4\). Then \((x \Gamma^n x) \leq A\) for some \(n \in \mathbb{N}\). Since \(A \subseteq Q\), then \((x \Gamma^n x) \leq Q\) for some \(n \in \mathbb{N}\).

Since \(Q\) is a semiprime \(\Gamma\)-ideal of \(S\), \((x \Gamma^n x) \leq Q\) \(\Rightarrow x \in Q\).

Therefore \(A_4 \subseteq Q\) and hence \(A_4\) is the minimal semiprime \(\Gamma\)-ideal of \(S\) containing \(A\).

**COROLLARY 3.3.12** : If \(A\) is a \(\Gamma\)-ideal of a duo \(\Gamma\)-semigroup \(S\) then

1. \(A_1 = \) the intersection of all completely prime \(\Gamma\)-ideals of \(S\) containing \(A\),
2. \(A_2 = \) the intersection of all minimal completely prime \(\Gamma\)-ideals of \(S\) containing \(A\),
3. \(A_3 = \) the minimal completely semiprime \(\Gamma\)-ideal of \(S\) containing \(A\),
4. \(A_4 = \) all \((x \Gamma^n x) \leq A\) for some natural number \(n\),
5. \(A_5 = \) the intersection of all prime \(\Gamma\)-ideals of \(S\) containing \(A\),
6. \(A_6 = \) the intersection of all minimal prime \(\Gamma\)-ideals of \(S\) containing \(A\),
7. \(A_7 = \) the minimal semiprime \(\Gamma\)-ideal of \(S\) containing \(A\),
8. \(A_8 = \) all \((x \Gamma^n x) \leq A\) for some natural number \(n\) are equal.

**THEOREM 3.3.13** : If \(a \in \sqrt{A}\), then there exist a positive integer \(n\) such that \((a \Gamma)^n a \subseteq A\).

**Proof** : By theorem 3.3.3, \(A_3 \subseteq A_2\) and hence \(a \in \sqrt{A} = A_3 \subseteq A_2\).

Therefore \((a \Gamma)^n a \subseteq A\) for some \(n \in \mathbb{N}\).

### 3.4. ARCHIMEDEAN \(\Gamma\)-SEMIGROUPS

In this section, the terms; Archimedean \(\Gamma\)-semigroup and strongly Archimedean \(\Gamma\)-semigroup are introduced. It is proved that if \(S\) is a duo \(\Gamma\)-semigroup, then the conditions (1) \(S\) is strongly Archimedean, (2) \(S\) is Archimedean, (3) \(S\) has no proper completely prime \(\Gamma\)-ideals and (4) \(S\) has no proper prime \(\Gamma\)-ideals; are equivalent.

We now introduce the notions of archimedean \(\Gamma\)-semigroup and strongly archimedean \(\Gamma\)-semigroup.

**DEFINITION 3.4.1** : A \(\Gamma\)-semigroup \(S\) is said to be an **archimedean \(\Gamma\)-semigroup** provided for any \(a, b \in S\), there exists a natural number \(n\) such that \((a \Gamma)^n a \subseteq b\).
DEFINITION 3.4.2: A $\Gamma$-semigroup $S$ is said to be a **strongly archimedean** $\Gamma$-semigroup provided for any $a, b \in S$, there is a natural number $n$ such that $(\langle a \rangle \Gamma)^{n-1} < a > \subseteq < b >$.

We now characterize archimedean $\Gamma$-semigroups.

THEOREM 3.4.3: If $S$ is a duo $\Gamma$-semigroup, then $S$ is strongly archimedean if and only if archimedean.

**Proof**: Suppose that $S$ is strongly Archimedean. Then for any $a, b \in S$, there is a natural number $n$ such that $(\langle a \rangle \Gamma)^{n-1} < a > \subseteq < b >$. Therefore $(a \Gamma)^{n-1} a \subseteq (\langle a \rangle \Gamma)^{n-1} < a > \subseteq < b >$ and hence $S$ is Archimedean.

Conversely suppose that $S$ is archimedean. Let $a, b \in S$. Since $S$ is archimedean, there exists a natural number $n$ such that $(\langle a \rangle \Gamma)^{n-1} < a > \subseteq < b > \subseteq S \Gamma b \Gamma S$. Since $S \Gamma b \Gamma S$ is a $\Gamma$-ideal of a duo $\Gamma$-semigroup $S$, by corollary 3.2.5, $(a \Gamma)^{n-1} a \subseteq S \Gamma b \Gamma S \Rightarrow (\langle a \rangle \Gamma)^{n-1} < a > \subseteq S \Gamma b \Gamma S$. Therefore $S$ is a strongly Archimedean duo $\Gamma$-semigroup.

THEOREM 3.4.4: If $S$ is a duo $\Gamma$-semigroup, then $S$ is archimedean if and only if $S$ has no proper prime $\Gamma$-ideals.

**Proof**: Suppose that $S$ is archimedean $\Gamma$-semigroup. Let $P$ be prime $\Gamma$-ideal of $S$. Let $a, b \in S$. Since $P$ is $\Gamma$-ideal, $S \Gamma a \Gamma S \subseteq P$. Since $S$ is archimedean, $(b \Gamma)^{n-1} \subseteq S \Gamma a \Gamma S$ for some natural number $n$. Thus $(b \Gamma)^{n-1} \subseteq S \Gamma a \Gamma S \subseteq P$. Since $S$ is a duo $\Gamma$-semigroup, by theorem 3.2.10, $P$ is completely prime. Thus $(b \Gamma)^{n-1} b \subseteq P \Rightarrow b \in P$. Hence $S = P$. Therefore $S$ has no proper prime $\Gamma$-ideals.

Conversely suppose that $S$ has no proper prime $\Gamma$-ideals. Then for any $b \in S$, the intersection of all prime $\Gamma$-ideals of $S$ containing $B = < b >$ is $S$ itself. Therefore $B_3 = S$. We have $B_4 = \{ x \in S : (x \Gamma)^{n-1} \subseteq < x > \subseteq < b > \text{ for some } n \in \mathbb{N} \} = S$. Therefore for any $a \in S$, $(\langle a \rangle \Gamma)^{n-1} < a > \subseteq < b > \text{ for some natural number } n$. So $(\langle a \rangle \Gamma)^{n-1} < a > \subseteq S \Gamma b \Gamma S$. Thus $S$ is strongly archimedean.

Hence by theorem 3.4.3, $S$ is archimedean.

COROLLARY 3.4.5: If $S$ is a duo $\Gamma$-semigroup, then the conditions (1) $S$ is strongly Archimedean, (2) $S$ is Archimedean, (3) $S$ has no proper completely prime $\Gamma$-ideals and (4) $S$ has no proper prime $\Gamma$-ideals are equivalent.
3.5. SIMPLE $\Gamma$-SEMIGROUPS

In this section, the terms; left simple $\Gamma$-semigroup, right simple $\Gamma$-semigroup, simple $\Gamma$-semigroup are introduced. It is proved that (1) a $\Gamma$-semigroup $S$ is a left simple $\Gamma$-semigroup if and only if $S\Gamma a = S$ for all $a \in S$, (2) a $\Gamma$-semigroup $S$ is a right simple $\Gamma$-semigroup if and only if $a\Gamma S = S$ for all $a \in S$, (3) a $\Gamma$-semigroup $S$ is a simple $\Gamma$-semigroup if and only if $S\Gamma a\Gamma S = S$ for all $a \in S$. It is also proved that if $S$ is a left simple $\Gamma$-semigroup or a right simple $\Gamma$-semigroup then $S$ is a simple $\Gamma$-semigroup. Further it is proved that if $S$ is a duo $\Gamma$-semigroup and $a \in S$ then (1) $a$ is regular, (2) $a$ is left regular, (3) $a$ is right regular, (4) $a$ is intra regular and (5) $a$ is semisimple are equivalent.

We now introduce a left simple $\Gamma$-semigroup.

DEFINITION 3.5.1 : A $\Gamma$-semigroup $S$ is said to be a left simple $\Gamma$-semigroup if $S$ is its only left $\Gamma$-ideal.

We now characterize left simple $\Gamma$-semigroups.

THEOREM 3.5.2 : A $\Gamma$-semigroup $S$ is a left simple $\Gamma$-semigroup if and only if $S\Gamma a = S$ for all $a \in S$.

Proof : Suppose that $S$ is a left simple $\Gamma$-semigroup and $a \in S$. Let $t \in S\Gamma a$, $s \in S$, $\gamma \in \Gamma$.

$t \in S\Gamma a \Rightarrow t = s_i\alpha a$ where $s_i \in S$ and $\alpha \in \Gamma$.

Now $syt = sy(s_i\alpha a) = (sy)s_i\alpha a \in S\Gamma a \Rightarrow S\Gamma a$ is a left $\Gamma$-ideal of $S$.

Since $S$ is a left simple $\Gamma$-semigroup, $S\Gamma a = S$.

Therefore $S\Gamma a = S$ for all $a \in S$.

Conversely suppose that $S\Gamma a = S$ for all $a \in S$. Let $L$ be a left $\Gamma$-ideal of $S$.

Let $l \in L$. Then $l \in S$. By assumption $S\Gamma l = S$.

Let $s \in S$. Then $s \in S\Gamma l \Rightarrow s = tal$ for some $t \in S$, $\alpha \in \Gamma$.

$l \in L$, $t \in S$, $\alpha \in \Gamma$ and $L$ is a left $\Gamma$-ideal $\Rightarrow tal \in L \Rightarrow s \in L$.

Therefore $S \subseteq L$. Clearly $L \subseteq S$ and hence $S = L$.

Therefore $S$ is the only left $\Gamma$-ideal of $S$. Hence $S$ is left simple $\Gamma$-semigroup.
We now introduce a right simple $\Gamma$-semigroup.

**DEFINITION 3.5.3 :** A $\Gamma$-semigroup $S$ is said to be a **right simple $\Gamma$-semigroup** if $S$ is its only right $\Gamma$-ideal.

We now characterize right simple $\Gamma$-semigroups.

**THEOREM 3.5.4 :** A $\Gamma$-semigroup $S$ is a right simple $\Gamma$-semigroup if and only if $a\Gamma S = S$ for all $a \in S$.

**Proof :** Suppose that $S$ is a right simple $\Gamma$-semigroup and $a \in S$. Let $t \in a\Gamma S$, $s \in S$, $x \in \Gamma$.

$t \in a\Gamma S \Rightarrow t = axs$, where $s \in S$ and $a \in \Gamma$.

Now $t\gamma = (axs)\gamma \in a\Gamma \gamma S \Rightarrow a\Gamma S$ is a right $\Gamma$-ideal of $S$.

Since $S$ is a right simple $\Gamma$-semigroup, $a\Gamma S = S$.

Therefore $a\Gamma S = S$ for all $a \in S$.

Conversely suppose that $a\Gamma S = S$ for all $a \in S$.

Let $R$ be a right $\Gamma$-ideal of a $\Gamma$-semigroup $S$.

Let $r \in R$. Then $r \in S$. By assumption $r\Gamma S = S$.

Let $s \in S$. Then $s \in r\Gamma S \Rightarrow s = rat$ for some $t \in S$, $a \in \Gamma$.

$r \in R$, $t \in S$, $a \in \Gamma$ and $R$ is a right $\Gamma$-ideal $\Rightarrow rat \in R \Rightarrow s \in R$.

Therefore $S \subseteq R$. Clearly $R \subseteq S$ and hence $S = R$.

Therefore $S$ is the only right $\Gamma$-ideal of $S$. Hence $S$ is right simple $\Gamma$-semigroup.

We now introduce a simple $\Gamma$-semigroup.

**DEFINITION 3.5.5 :** A $\Gamma$-semigroup $S$ is said to be **simple $\Gamma$-semigroup** if $S$ is its only two-sided $\Gamma$-ideal.

We now characterize simple $\Gamma$-semigroups

**THEOREM 3.5.6 :** If $S$ is a left simple $\Gamma$-semigroup or a right simple $\Gamma$-semigroup then $S$ is a simple $\Gamma$-semigroup.

**Proof :** Suppose that $S$ is a left simple $\Gamma$-semigroup. Then $S$ is the only left $\Gamma$-ideal of $S$.

If $A$ is a $\Gamma$-ideal of $S$, then $A$ is a left $\Gamma$-ideal of $S$ and hence $A = S$.

Therefore $S$ itself is the only $\Gamma$-ideal of $S$ and hence $S$ is a simple $\Gamma$-semigroup.

Suppose that $S$ is a right simple $\Gamma$-semigroup. Then $S$ is the only right $\Gamma$-ideal of $S$.

If $A$ is a $\Gamma$-ideal of $S$, then $A$ is a right $\Gamma$-ideal of $S$ and hence $A = S$.

Therefore $S$ itself is the only $\Gamma$-ideal of $S$ and hence $S$ is a simple $\Gamma$-semigroup.
THEOREM 3.5.7: A $\Gamma$-semigroup $S$ is simple $\Gamma$-semigroup if and only if $S \Gamma a S = S$ for all $a \in S$.

**Proof:** Suppose that $S$ is a simple $\Gamma$-semigroup and $a \in S$.

Let $i \in S \Gamma a S$, $s \in S$ and $y \in \Gamma$.

$t \in S \Gamma a S \Rightarrow t = s_1 \alpha a \beta s_2$ where $s_1, s_2 \in S$ and $\alpha, \beta \in \Gamma$.

Now $t y s = (s_1 \alpha a \beta s_2) y s = s_1 \alpha a \beta (s_2 y s) \in S \Gamma a S$

and $s y t = s y (s_1 \alpha a \beta s_2) = (s y s_1) \alpha a \beta s_2 \in S \Gamma a S$. Therefore $S \Gamma a S$ is a $\Gamma$-ideal of $S$.

Since $S$ is a simple $\Gamma$-semigroup, $S$ itself is the only $\Gamma$-ideal of $S$ and hence $S \Gamma a S = S$.

Conversely suppose that $S \Gamma a S = S$ for all $a \in S$. Let $I$ be a $\Gamma$-ideal of $S$.

Let $a \in I$. Then $a \in S$. So $S \Gamma a S = S$.

Let $s \in S$. Then $s \in S \Gamma a S \Rightarrow s = t_1 \alpha a \beta t_2$ for some $t_1, t_2 \in S$, $\alpha, \beta \in \Gamma$.

$a \in I$, $t_1, t_2 \in S$, $\alpha, \beta \in \Gamma$, $I$ is a $\Gamma$-ideal of $S \Rightarrow t_1 \alpha a \beta t_2 \in I \Rightarrow s \in I$.

Therefore $S \subseteq I$. Clearly $I \subseteq S$ and hence $S = I$.

Therefore $S$ is the only $\Gamma$-ideal of $S$. Hence $S$ is a simple $\Gamma$-semigroup.

THEOREM 3.5.8: If $S$ is a duo $\Gamma$-semigroup, then the following are equivalent for any element $a \in S$.

1) $a$ is regular.
2) $a$ is left regular.
3) $a$ is right regular.
4) $a$ is intra regular.
5) $a$ is semisimple.

**Proof:** Since $S$ is duo $\Gamma$-semigroup, $a S^1 \Gamma a = S^1 \Gamma a$.

We have $a S^1 \Gamma a = a S \Gamma a S = S^1 \Gamma a \Gamma a = <a \Gamma a> = <a > \Gamma <a >$.

(1) $\Rightarrow$ (2): Suppose that $a$ is regular. Then $a = a x a y a$ for some $x \in S$ and $\alpha, \beta \in \Gamma$.

Therefore $a \in a S^1 \Gamma a = a S \Gamma a S $ $\Rightarrow a = a \gamma a \delta a$ for some $\gamma \in S^1$, $\gamma, \delta \in \Gamma$.

Therefore $a$ is left regular.

(2) $\Rightarrow$ (3): Suppose that $a$ is left regular. Then $a = a x a \beta a$ for some $x \in S$ and $\alpha, \beta \in \Gamma$.

Therefore $a \in a S \Gamma a S^1 = S \Gamma a S \Gamma a \Rightarrow a = \gamma \gamma a \alpha a a$ for some $\gamma \in S^1$, $\gamma, \delta \in \Gamma$.

Therefore $a$ is right regular.

(3) $\Rightarrow$ (4): Suppose that $a$ is right regular. Then for some $x \in S$, $\alpha, \beta \in \Gamma$; $a = x a a \beta a$.

Therefore $a \in S^1 \Gamma a S = <a \Gamma a > \Rightarrow a = x a a \beta a y \gamma a$ for some $x, y \in S^1$ and $\alpha, \beta, \gamma \in \Gamma$.

Therefore $a$ is intra regular.
(4) ⇒ (5): Suppose that \( a \) is intra regular. Then \( a = xaaβαyy \) for some \( x, y \in S^1 \) and \( a, β, y \in Γ \). Therefore \( a \in < a > Γ < a > \). Therefore \( a \) is semisimple.

(5) ⇒ (1): Suppose that \( a \) is semisimple. Then \( a \in < a > Γ < a > = aΓS^1Γa \) \( ⇒ a \in aαβαa \) for some \( x \in S^1 \) and \( α, β \in Γ \). Therefore \( a \) is a regular element.