CHAPTER - 4

\textit{n-TH ROOTS}
This chapter is devoted to characterizations of graphs and digraphs having $n$-th roots and different types of $n$-th root of graphs and digraphs.

1. GRAPHS

In 1974, F. Escalante, L. Montejano and T. Rojano [5] gave necessary and sufficient conditions for existence of $n$-th roots for graphs. Hamada Takashi [14] followed it with necessary and sufficient conditions for the existence of tree $n$-th roots of graphs. In this section we deal with the question of existence of regular $n$-th roots, hamiltonian $n$-th roots, bipartite $n$-th roots, eulerian $n$-th roots and cyclic $n$-th roots of graphs. These results are analogous to those of F. Escalante, L. Montejano and T. Rojano [5].

$n$-TH ROOTS OF $K_p$

In Chapter 3 we saw that every complete bipartite graph $K_{s,r}$, $s+r = p$ is a square root of $K_p$. We note that $K^n_{s,r} = K_p$ for any $n \geq 2$. Hence every complete bipartite graph $K_{s,r}$ where $s+r = p$ is an $n$-th root of $K_p$, $n \geq 2$. The question arises whether $K_p$ has other $n$-th roots? The following theorem answers this question.

THEOREM 4.1.1 Every connected spanning subgraph of diameter $\leq n$ of $K_p$ is an $n$-th root of $K_p$. 

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Proof: Let \( u_1, u_2, \ldots, u_p \) be the vertices of a complete graph \( K_p \) and \( H \) be a connected spanning subgraph of \( K_p \) such that diameter \( d(H) \) of \( H \leq n \). We show that any two vertices in \( H^n \) are adjacent.

Let \( u, v \in V(H^n) \) and \( u = u_0 + u_1 + u_2 + \ldots + u_k = v \) be a shortest path of length \( k \) in \( H \) joining \( u \) and \( v \). Since \( d(H) \leq n \), \( k \leq n \). Hence \( d_H(u, v) = k \leq n \), i.e. \( uv \in E(H^n) \). Thus \( H^n = K_p \).

Corollary 4.1.2: Every spanning tree of diameter \( \leq n \) of \( K_p \) is a minimal \( n \)-th root of \( K_p \).

Remark 4.1.3: We observe that if \( T \) is a spanning tree of \( K_p \) of diameter \( d(T) \leq n \) then there are \( 2^r \) distinct \( n \)-th roots of \( K_p \) containing \( T \), where \( r = \frac{(p-1)(p-2)}{2} \).

Cyclic \( n \)-th Roots

We label the vertices of the \( p \)-cycle \( C_p \) by \( u_1, u_2, \ldots, u_p \) in a cyclic manner and let \( d \) denote the diameter of \( C_p \). The following properties are easily verified.

(a) \( d = \frac{p}{2} \) if \( p \) is even

\[ = \frac{p-1}{2} \] if \( p \) is odd.

(b) \( C_p^n \) is a regular non-complete graph of degree \( 2n \) if and only if \( n < d \).

(c) \( C_p^n \) is a complete graph if and only if \( n \geq d \).

To study the graphs having cyclic \( n \)-th roots, we need only to consider the non-complete regular graphs of degree \( 2n \), where \( n < \frac{p}{2} \) if \( p \) is even and \( n < \frac{p-1}{2} \) if \( p \) is odd. We easily see that \( C_p^n \) is complete graph if \( p \leq 5 \), for every \( n \geq 2 \). Hence we need to consider the cyclic \( n \)-th roots of non-complete graphs of order \( p \geq 6 \). We need the following proposition.
PROPOSITION 4.1.4 Let $G$ be a regular non-complete graph of degree $2n$ with $p$ vertices, where $n < \frac{p}{2}$ if $p$ is even, $n < \frac{p-1}{2}$ if $p$ is odd and $p \geq 6$, having cyclic $n$-th root $H$. For $a \in V(H) = V(G)$. Let $V_a$ denote the neighbourhood of $a$ in $H$ and $K_a = G \setminus V_a$. Then the following hold.

(i) $K_a$ is a triangle in $G$.

(ii) $V(K_a) \cap V(K_b) = \{a, b\}$ iff $a, b \in E(H)$.

(iii) $V(K_a) \cap V(K_c) = \{c\}$ iff $ac, cb \in E(H)$.

(iv) $V(K_a) \cap V(K_b) = \emptyset$ iff every path from $a$ to $b$ in $H$ is of length $\geq 3$.

(v) For each $K_i$, there exist exactly two $K_i$'s such that $K_i$ meets each of them in exactly two vertices.

PROOF: Similar to Proposition 3.1.1. •

In the following theorem we give a set of necessary and sufficient conditions for the existence of cyclic $n$-th roots of graphs of order at least 6.

THEOREM 4.1.5 Let $G$ be a regular non-complete graph of degree $2n$ with $p$ vertices $u_1, u_2, \ldots, u_p$ ($p \geq 6$), where $n \leq \left[ \frac{p}{2} \right] - 1$. Then $G$ has a cyclic $n$-th root iff there exists a collection $F = \{K_1, K_2, \ldots, K_p\}$ of $p$ triangles of $G$ such that

(i) $u_i \in V(K_i)$ for every $i$.

(ii) $u_i \in V(K_j)$ iff $u_j \in V(K_i)$ for every $i$ and $j$.

(iii) if $u, v \in E(G)$ then there exists a $u, v$-linking of length $\leq n$ such that $u, v \in UL$.

(iv) for every $u, v$-linking $L$ with length $\leq n$, the subgraph $UL$ is complete.

(v) no two $K_i$'s intersect in more than two vertices.
(vi) for each $K_i$, there exist exactly two $K_j$ and $K_k$ such that $K_i$ meets each of them in exactly two vertices.

PROOF: Suppose $G$ has a cyclic $n$-th root $H$ (say). For each $u_i \in V(H)$, let $V_i$ denote the neighbourhood of $u_i$ in $H$ and $K_i = G(V_i)$. Now each $K_i$ is obviously complete subgraph of $G$. Since $H$ is a cycle, $|V(K_i)| = |V_i| = 3$. Thus we have a collection $F = \{K_1, K_2, \ldots, K_p\}$ of $p$ triangles of $G$. We show that the conditions (i)-(vi) are satisfied. (i) follows from the definition of $K_i$. (ii) follows from the fact that $u_j$ is in the neighbourhood of $u_i$ iff $u_i$ is in the neighbourhood of $u_j$. (v) and (vi) follow directly from the proposition 4.1.4.

To prove (iii) Let $uv \in E(G) = E(H^n)$. If $u = u_{i_0} \rightarrow u_{i_1} \rightarrow \ldots \rightarrow u_{i_k} = v$ is a path of length $k \leq n$ in $H$, then by definition of $K_i$'s. $L = \langle K_{i_1}, K_{i_2}, \ldots, K_{i_k} \rangle$ forms a $u,v$-linking of length $k \leq n$ and $uv \in E(L) = G(V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_k})$ because $L = G(V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_k})$. This proves (iii).

To prove (iv) : Let $L = \langle K_{i_1}, K_{i_2}, \ldots, K_{i_k} \rangle$ be any $u_{i_0}, u_{i_k}$-linking of length $k \leq n$ and $u,v \in V(L) = V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_k}$. Clearly $d(u,v) \leq K \leq n$ in $H$. Therefore $u,v \in E(H^n) = E(G)$. Hence $uv \in G(V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_k}) = UL$. Thus $UL$ is complete.

For the converse we define the graph $H$ as $V(H) = V(G) = \langle u_1, u_2, \ldots, u_p \rangle$ and $E(H) = \langle u_i, u_j \in E(G) \cap V(K_i) \cap V(K_j) = \langle u_i, u_j \rangle \rangle$. We claim that $H$ is a $n$-th root of $G$. We only have to show that $E(H^n) = E(G)$. Let $uv \in E(H^n)$. If $u = u_{i_0} \rightarrow u_{i_1} \rightarrow \ldots \rightarrow u_{i_k} = v$ is a path of length $k \leq n$ in $H$, then the collection $L = \langle K_{i_1}, K_{i_2}, \ldots, K_{i_k} \rangle$ forms a $u,v$-linking of length $k \leq n$. By condition (iv), $UL$ is a complete subgraph of $G$ and $u,v \in V(L)$. Therefore $uv \in E(G)$.
Thus $ECH^H \subseteq E(\Theta)$. For the reverse inclusion let $uv \in E(\Theta)$. From (iii) there exists a $u,v$-linking $L = \langle K_{i_1}, K_{i_2}, \ldots, K_{i_k} \rangle$ of length $k \leq n$ such that $uv \in E(L^U)$. Therefore $u = u_{i_0} \in V(K_{i_1})$, $u_{i_1} \in V(K_{i_2}), \ldots, u_{i_{k-1}} \in V(K_{i_k})$ and $u = u_{i_k}$. By conditions (ii) and (v), $V(K_{i_1}) \cap V(K_{i_2}) = \langle u_{i_0}, u_{i_1} \rangle$, $V(K_{i_1}) \cap V(K_{i_2}) = \langle u_{i_1}, u_{i_2} \rangle, \ldots,$ $V(K_{i_{k-1}}) \cap V(K_{i_k}) = \langle u_{i_{k-1}}, u_{i_k} \rangle$. Thus by definition of $ECH^H$, $u_{i_0} u_{i_1} \in ECH^H$, $u_{i_1} u_{i_2} \in ECH^H$, $\ldots$, $u_{i_{k-1}} u_{i_k} \in ECH^H$. Therefore \(d(u, v) \leq k \leq n\). Hence $uv \in ECH^H$. So that $E(\Theta) \subseteq ECH^H$. By condition (vi) and definition of $ECH^H$, degree of each vertex of $H$ is two. Hence $H$ is a cycle.

**ILLUSTRATION 4.1.6** We illustrate the Theorem 4.1.3 by the following example.

![Diagram](image)

Figure 4.1.1
A graph $G$ and its triangles $K_1 - K_8$ are shown in Figure 4.1.1. The conditions of Theorem 4.1.5 are satisfied by $K_1 - K_8$. $H$ is a cyclic cube root of $G$.

**REGULAR n-TH ROOTS**

The $n$-th power of a regular graph may not be a regular graph. For example in Figure 4.1.2 the graph $H$ is regular but $H^3$ is not regular.

Thus the irregular graphs may have regular $n$-th roots. The following theorem gives a set of necessary and sufficient conditions for the existence of a regular $n$-th root of a graph.

**THEOREM 4.1.7** A graph $G$ with $p$ vertices $u_1, u_2, \ldots, u_p$ has a $r$-regular $n$-th root, $n \geq 2$ iff $G$ contains a collection of $p$ complete subgraphs $G_1, G_2, \ldots, G_p$ such that

(i) $u_i \in V(G_i)$ for every $i$,

(ii) $u_i \in V(G_j)$ iff $u_j \in V(G_i)$ for every $i$ and $j$,

(iii) if $uv \in E(G)$ then there exists a $u,v$-linking $L$ of length $\leq n$ such that $uv \in UL$. 

Figure 4.1.2

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(iv) for every \( u,v \)-linking \( L \) with length \( \leq n \), the subgraph \( U_L \) is complete.

(v) \( |V(G_i)| = r+1 \) for every \( i \).

**PROOF:** Let \( H \) be a \( r \)-regular \( n \)-th root of \( G \). For each \( u_i \), let \( V_i \) denote the neighbourhood of \( u_i \) in \( H \) and \( G_i = G < V_i > \). Now each \( G_i \) is complete because for any \( u_k, u_m \in V(G) \), \( d(u_k, u_m) \leq 2 \leq n \) in \( H \), i.e. \( u_k u_m \in E(H^n) \). Therefore \( u_k u_m \in E(G) \), since \( G_i \) is an induced subgraph of \( G \) and \( u_k, u_m \in V(G) \). Thus we get a collection of \( p \) complete subgraphs \( G_1, G_2, \ldots, G_p \) of \( G \). It remains to show that the conditions (i)-(v) are satisfied. (i) is immediate from the definition of \( V_i \)'s. (ii) follows from the fact that \( u_j \) is in the neighbourhood of \( u_i \) iff \( u_i \) is in the neighbourhood of \( u_j \).

To prove (iii) : let \( u, v \in E(G) = E(H^n) \). If \( u = u_{i_0} + u_{i_1} + u_{i_2} + \ldots + u_{i_k} = v \) in \( H \) is a path of length \( \leq n \), then the collection \( L = \langle G_{i_0}, G_{i_1}, \ldots, G_{i_k} \rangle \) is clearly a \( u,v \)-linking and \( uv \in E(UL) \), since \( G < V_{i_0} U V_{i_1} U \ldots U V_{i_k} > = UL \). Hence (iii) is proved.

To prove (iv) : Let \( L = \langle G_{i_0}, G_{i_2}, \ldots, G_{i_k} \rangle \) be any \( u_{i_0}, u_{i_k} \)-linking with \( k \leq n \) and \( u, v \in V(UL) \). Clearly \( d(u, v) \leq k \leq n \) in \( H \), by definition of \( V(G) = V_i \). Therefore \( uv \in E(H^n) = E(G) \). Hence \( uv \in E(UL) \). Thus UL is complete. Condition (v) is satisfied because \( H \) is a \( r \)-regular graph and \( |V(G_i)| = |V_i| = r+1 \) for every \( i \).

Conversely, suppose there exists a collection \( \langle G_{i_1}, G_{i_2}, \ldots, G_p \rangle \) of \( p \) complete subgraphs of \( G \) satisfying (i) -(v). We define a graph \( H \) as \( V(H) = V(G) = \langle u_{i_1}, u_{i_2}, \ldots, u_p \rangle \) and \( E(H) = \langle u_j u_j \in E(G) / G_i \) contains \( u_j \rangle \). We show that \( E(H^n) = E(G) \). Let \( uv \in E(H^n) \). Then there exists a path \( u = u_{i_0} + u_{i_1} + \ldots + u_{i_k} = v \), \( k \leq n \) in \( H \).
Hence the collection \( L = \langle G_{i_1}, G_{i_2}, \ldots, G_{i_k} \rangle \) forms a \( u,v \)-linking of length \( k \leq n \). By (iv), \( UL \) is a complete subgraph of \( G \) and \( u,v \in V(UL) \).

Therefore \( uv \in E(G) \). Thus \( E(CH^H) \subseteq E(G) \). For the reverse inclusion, let \( uv \in E(G) \). From (iii) there exists a \( u,v \)-linking

\[ L = \langle G_{i_1}, G_{i_2}, \ldots, G_{i_k} \rangle \]

of length \( k \leq n \) such that \( uv \in E(UL) \).

Therefore \( u = u_{i_0} \in V(G_{i_1}), u_{i_1} \in V(G_{i_2}), \ldots, u_{i_{k-1}} \in V(G_{i_k}) \) and \( u_{i_k} = v \). By definition of \( E(UL) \), \( uu_{i_1}, u_{i_1}u_{i_2}, \ldots, u_{i_{k-1}}v \in E(UL) \).

Hence \( d(u,v) \leq k \leq n \). So that \( u,v \in E(CH^H) \). Thus \( E(G) \subseteq E(CH^H) \). That \( H \) is \( r \)-regular follows from (v). Therefore \( H \) is a \( r \)-regular \( n \)-th root of \( G \). 

**ILLUSTRATION 4.1.8** We illustrate the Theorem 4.1.7 by the following example.

![Illustration of Theorem 4.1.7](image-url)

Figure 4.1.3
A graph $G$ and its complete subgraphs $G_{i12}$ are shown in Figure 4.1.3. The conditions of Theorem 4.1.7 are satisfied by $G_{i12}$. $H$ is a 3-regular cube root of $G$.

**HAMILTONIAN n-TH ROOTS**

In this section we consider the question of existence of hamiltonian $n$-th root for a graph. Clearly the $n$-th power of a hamiltonian graph is hamiltonian. Thus only hamiltonian graphs may have hamiltonian $n$-th roots. The converse of this is not true. In fact J.J. Karaganis [10] has shown that the cube of every tree is hamiltonian. Thus the $n$-th power of a tree is hamiltonian for every $n \geq 3$. Also note that not every hamiltonian graph need have a hamiltonian $n$-th root. For example a cycle $C_p$ on $p$ vertices has no $n$-th root, $n \geq 2$. Characterizations of graphs with hamiltonian $n$-th roots can be given using the notion of closed linking.

**THEOREM 4.1.9** A hamiltonian graph $G$ with $p$ vertices $u_1, u_2, \ldots, u_p$ has a hamiltonian $n$-th root iff there exists a collection $F = \langle G_1, G_2, \ldots, G_p \rangle$ of $p$ complete subgraphs of $G$ such that

(i) $u_i \in V(G_i)$ for every $i$.

(ii) $u_i \in V(G_j)$ iff $u_j \in V(G_i)$ for every $i \neq j$.

(iii) for every pair $u, v$ of adjacent vertices in $G$, there exists a $u, v$-linking $L$ (w.r.t $F$) of length $\leq n$ such that $uv \in E(UL)$.

(iv) for every $u, v$-linking $L$ with length $\leq n$, the subgraph $UL$ is complete.

(v) $F = \langle G_1, G_2, \ldots, G_p \rangle$ forms a closed $p$-linking.
PROOF: Suppose $G$ has a hamiltonian square root $H$ and $u_1 + u_2 + \ldots + u_p$ is a hamiltonian cycle in $H$. For each $i$, let $V_i$ denote the neighbourhood of $u_i$ in $H$ and $G_i = G \langle V_i \rangle$. Now each $G_i$ is complete because for any $u_k, u_m \in V_i$, $d(u_k, u_m) \leq 2 \leq n$, in $H$, i.e. $u_k u_m \in E(H^n) = E(G)$. Therefore $u_k u_m \in E(G)$, since $G$ is induced subgraph of $G$ and $u_k, u_m \in V(G)$. Thus we get a collection of $p$ complete subgraphs $G_1, G_2, \ldots, G_p$ of $G$. It remains to show that the conditions $C_{i} - C_{v_+}$ are satisfied. (i) is immediate from the definition of $V$'s. (ii) follows from the fact that $u_j$ is in the neighbourhood of $u_i$ iff $u_i$ is in the neighbourhood of $u_j$. Proof of (iii) and (iv) are on the same lines as that of Theorem 4.1.7.

To prove (v): As $u_1 + u_2 + u_3 + \ldots + u_p$ is a hamiltonian cycle in $H$, so that $F = (G_1, G_2, \ldots, G_p)$ forms a closed $p$-linking. Thus codition (v) is satisfied.

Conversely, suppose there exists a collection of $p$ complete subgraphs $G_1, G_2, \ldots, G_p$ of $G$ satisfying $C_{i} - C_{v_+}$. We define a graph $H$ as $V(H) = V(G) = \{u_1, u_2, \ldots, u_p\}$ and $E(H) = \{u_j u_i \in E(G)/u_j \in V(G)\}$. As $V(H) = V(H^n) = V(G)$, to show that $H^n = G$ we have only to show that $E(H^n) = E(G)$ and this is on the same line as the proof of Theorem 4.1.7. Now we claim that $H$ is hamiltonian. By codition (v), $F = (G_1, G_2, \ldots, G_p)$ forms a closed $p$-linking. Therefore $u_1 \in V(G_1), u_2 \in V(G_2), \ldots, u_p \in V(G_p)$ and $u_i \in V(G)$. By definition of $E(H)$, $u_{i_1}, u_{i_2}, u_{i_3}, \ldots, u_{i_p}$, $u_{i_{p-1}}, u_{i_p}$ $\in E(H)$. Hence $u_1 + u_2 + u_3 + \ldots + u_p + u_1$ is a hamiltonian cycle of length $p$ in $H$, i.e. $H$ is a hamiltonian $n$-th root of $G$. 


ILLUSTRATION 4.1.10 A graph $G$ and its complete subgraphs $G - G$ are shown in Figure 4.1.4. The conditions of Theorem 4.1.9 are satisfied by $G - G$. $H$ is a hamiltonian cube root of $G$.

![Figure 4.1.4](image)

EULERIAN $n$-TH ROOTS

The $n$-th power of an eulerian graph may not be a eulerian graph. Hence non-eulerian graphs may have eulerian $n$-th roots. The following theorem gives necessary and sufficient conditions for the existence of an eulerian $n$-th root for a graph.

THEOREM 4.1.11 A graph $G$ with $p$ vertices $u_1, u_2, \ldots, u_p$ has an eulerian $n$-th root ($n \geq 2$) iff $G$ contains a collection $F=(G_1, G_2, \ldots, G_p)$ of $p$ complete subgraphs such that

1. $u_i \in V(G_i)$ for every $i$,
2. $u_i \in V(G_j)$ iff $u_j \in V(G_i)$ for every $i \neq j$,
3. $uv \in E(G) \Rightarrow$ there exists a $u,v$-linking $L$ (w.r.t. $F$) of length $\leq n$ such that $uv \in E(L)$.
(iv) for all $u,v$-linking $L$ with length $\leq n$, the subgraph $UL$ is complete.

(v) each $G_i$ contains an odd number of vertices.

PROOF: Let $H$ be an eulerian $n$-th root of $G$. For each $i$, let $V_i$ denote the neighbourhood of $u_i$ in $H$ and $G_i = G \langle V_i \rangle$. Now each $G_i$ is complete because for any $u_k, u_m \in V(G) = V_i$, $d(u_k, u_m) \leq 2 \leq n$ in $H$, i.e. $u_k u_m \in E(H^n) = E(G)$. Therefore $u_k u_m \in E(G_i)$. Since $G_i = G \langle V_i \rangle$ and $u_k, u_m \in V(G_i)$. Thus we get a collection of $p$ complete subgraph $G_i, G_2, \ldots, G_p$ of $G$. It remains to show that the conditions (iv) - (v) are satisfied. Proofs of (i) - (iv) are on the same lines as in the proof of Theorem 4.1.7.

To prove (v): By Theorem 1.1.5 degree of each $u_i$ in $H$ is even. Hence each $V_i$ contains an odd number of vertices. Therefore $G_i = G \langle V_i \rangle$ contains an odd number of vertices for every $i$. Thus condition (v) is satisfied.

Conversely, suppose there exists a collection of $p$ complete subgraphs $F = \{G_i, G_2, \ldots, G_p\}$ of $G$ satisfying (i) - (v). We define a graph $H$ as $V(H) = V(G) = \{u_1, u_2, \ldots, u_p\}$ and $E(H) = \{u_i u_j \in E(G) / u_j \in V(G_i)\}$. The proof of $H^n = G$ is on the same lines as in the proof of Theorem 4.1.7. Finally, we claim that $H$ is eulerian. By condition (v), each $G_i$ contains an odd number of vertices and by (i), $u_i \in V(G_i)$ for every $i$. Therefore by definition of $H$, degree of $u_i$ in $H = |\{u_j / u_i u_j \in E(H)\}| = |V(G_i)| - 1$.

Hence degree of each vertex of $H$ is even in $H$. Thus $H$ is an eulerian graph. Therefore $H$ is an eulerian $n$-th root of $G$. ■
A graph $G$ and its complete subgraphs $G - G$ are shown in Figure 4.1.5. The conditions of Theorem 4.1.11 are satisfied by $G - G$. $H$ is an eulerian cube root of $G$.

**BIPARTITE $n$-TH ROOTS**

An $n$-th power of a bipartite graph is never a bipartite graph because an $n$-th power of a bipartite graph contains triangles. Also we observe that bipartite graphs have no $n$-th roots $n \geq 2$.

In this section our main aim is to characterize the graphs having bipartite $n$-th roots. First we consider the case of complete bipartite graphs. By Proposition 3.1.14 every complete bipartite graph $K_{s,r}$ is a $n$-th root of $K_p$ where $p = s + r$, $s < p$, $r < p$. Therefore there are \( \left\lfloor \frac{p}{2} \right\rfloor \) complete bipartite $n$-th roots of $K_p$, $n \geq 2$. Clearly we see that if $H$ is a bipartite graph of diameter $n$ then
H^n is complete graph. Hence complete graphs may have the bipartite n-th root. The following theorem gives necessary and sufficient conditions for the existence of a bipartite n-th roots of graphs.

**THEOREM 4.1.13** A graph G with p vertices \( u_1, u_2, \ldots, u_p \) has a bipartite n-th root, \( n \geq 2 \) iff there exists a collection \( F = \{ G_1, G_2, \ldots, G_p \} \) of p complete subgraphs of G and two disjoint subsets X and Y of \( V(G) \) such that

1. \( V(G) = X \cup Y \),
2. \( u_i \in V(G_i) \) for every \( i \),
3. \( uv \in E(G) \) iff there exists a \( u,v \)-linking \( L \) (w.r.t. \( F \)) of length \( \leq n \) such that \( uv \in UL \),
4. for every \( u,v \)-linking \( L \) with length \( \leq n \), the subgraph \( UL \) is complete,
5. \( u_j \in V(G_j) \cap X \) iff \( u_j \in V(G_j) \cap Y \) for every \( i \neq j \).

**PROOF:** Let \( H \) be a bipartite n-th root of \( G \). Then there exist two disjoint subsets X and Y of \( V(H) \) such that \( V(H) = X \cup Y \) and every edge in \( H \) joins a vertex in \( X \) to a vertex in \( Y \). As \( H^n = G \) therefore \( V(H) = V(H^n) = V(G) = X \cup Y = \{ u_1, u_2, \ldots, u_p \} \) and \( X \cap Y = \emptyset \). For each \( u_i \), let \( V_i \) be the neighbourhood of \( u_i \) in \( H \) and \( G_i = G \langle V_i \rangle \). As in the proof of Theorem 4.1.7 each \( G_i \) is a complete subgraph of \( G \). Thus we get a collection of p complete subgraphs \( G_1, G_2, \ldots, G_p \) of \( G \) and two disjoint subsets \( X \) and \( Y \) of \( V(G) \). We show that the conditions (i) - (v) are satisfied (i) and (ii) are obvious. The proofs of (ii) and (iv) are same as that of Theorem 4.1.7.

To prove (v): We see that \( u_j \in V(G_j) \cap X \) iff \( u_j \in V(G_j) \) and \( u_j \in X \) iff \( u_j \in E(H) \) and \( u_j \in X \) iff \( u_j \in E(H) \) and \( u_j \in Y \), since
H is bipartite with partition X and Y iff \( u_i \in V(G) \) and \( u_i \in Y \) iff \( u_i \in V(G) \cap Y \).

For the converse we define a graph \( H \) as \( V(H) = V(G) = X \cup Y = \{ u_1, u_2, \ldots, u_p \} \), \( X \cap Y = \emptyset \) and \( E(H) = \{ u_j \in E(G) / u_j \in V(G) \cap X \} \). We show that \( E(H^n) = E(G) \). This will prove that \( H^n = G \). Let \( uv \in E(H^n) \). If \( u = u_{i_0} \rightarrow u_{i_1} \rightarrow u_{i_2} \rightarrow \ldots \rightarrow u_{i_k} = v \) is a path of length \( k \leq n \) in \( H \) then the collection \( L = \{ G_{i_1}, G_{i_2}, \ldots, G_{i_k} \} \) form a \( u,v \)-linking of length \( k \leq n \). By (iv), \( UL \) is a complete subgraph of \( G \) and \( u,v \in V(UL) \). Therefore \( uv \in E(G) \). Thus \( E(H^n) \subseteq E(G) \). For the reverse inclusion, let \( uv \in E(G) \). From (iii) there exists a \( u,v \)-linking \( L = \{ G_{i_1}, G_{i_2}, \ldots, G_{i_k} \} \) of length \( k \leq n \) such that \( uv \in E(UL) \). Therefore \( u_{i_r} \in V(G) \cap X \cup Y, X \cap Y = \emptyset, v = u_{i_k} \) and \( r = 0,1, \ldots, k-1 \). Hence \( u_{i_r} \in V(G) \cap X \) or \( u_{i_r} \in V(G) \cap Y, v = u_{i_k} \) and \( r = 0,1, \ldots, k-1 \). By condition (v), \( u_{i_r} \in V(G) \cap X \) or \( u_{i_r} \in V(G) \cap Y, v = u_{i_k} \) and \( r = 0,1, \ldots, k-1 \). Hence by definition of \( H \), \( u_{i_r} \in E(H), u_{i_r+1} \in E(H), v = u_{i_k} \) and \( r = 0,1, \ldots, k-1 \). In any case \( u_{i_r} \in E(H), v = u_{i_k} \) and \( r = 0,1, \ldots, k-1 \). Hence \( u_{i_r} \in E(H), v = u_{i_k} \) and \( r = 0,1, \ldots, k-1 \). Hence \( d(u,v) \leq k \leq n \) in \( H \), i.e \( uv \in E(H^n) \). Thus \( E(G) \subseteq E(H^n) \).

Further \( H \) is bipartite. For, if \( u_j \in E(H) \), then \( u_j \in E(G) \) and \( u_j \in V(G) \cap X \), i.e. \( u_j \in X \) and \( u_j \in V(G) \cap X \). By (v), \( u_j \in X \) and \( u_j \in V(G) \cap Y \). Therefore, \( u_j \in X \) and \( u_j \in Y \). Thus every edge in \( H \) joins to a vertex of \( X \) to a vertex of \( Y \). Also \( V(H) = X \cup Y, X \cap Y = \emptyset \). Hence \( H \) is a bipartite \( n \)-th root of \( G \).
ILLUSTRATION 4.1.14 We illustrate the Theorem 4.1.11 by the following example.

A graph $G$ and its complete subgraphs $G_i$ are shown in Figure 4.1.6. The conditions of Theorem 4.1.13 are satisfied for $X = \{u_1, u_2, u_3, u_4\}$, $Y = \{u_5, u_6, u_7, u_8\}$ and $G - G_i$. $H$ is a bipartite 4-th root of $G$.

2. DIGRAPH

In 1974, F. Escalante, L. Montejano and T. Rojano [3] gave necessary and sufficient conditions for the existence of $n$-th roots of digraphs. Motivated by this paper and Hamada Takashi's [14] work we study the existence of tree $n$-th roots, regular $n$-th roots, hamiltonian $n$-th roots, eulerian $n$-th roots, bipartite $n$-th roots and directed cyclic $n$-th roots of digraphs. The results resamble in many respect to those of F. Escalante, L. Montejano and T. Rojano's [3].

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n-TH ROOTS OF TRANSITIVE DIGRAPHS

Recall that a simple connected digraph D is transitive if no vertex of D is contained in a circuit (i.e. a directed cycle) of D. First we give a characterization of (simple connected) transitive digraphs.

**THEOREM 4.2.1** A (connected) digraph D on p vertices $u_1, u_2, \ldots, u_p$ is transitive iff $D^n = D$ for all $n \geq 2$.

**PROOF:** Suppose D is transitive, i.e. $D^* = D$. As $V(D^n) = V(D) = \{u_1, u_2, \ldots, u_p\}$ for all $n \geq 2$, to show that $D^n = D$ we have only to show that $A(D^n) = A(D)$. But $A(D) \subseteq A(D^n)$ always for all $n \geq 2$. Therefore we show that $A(D^n) \subseteq A(D)$. For, let $(u, v) \in A(D^n)$, $n \geq 2$. Then there exists a dipath $u = u_{i_0} \rightarrow u_{i_1} \rightarrow \ldots \rightarrow u_{i_k} = v$ of length $k \leq n$ in D. By definition of $D^*$, $(u, v) \in A(D^*) = A(D)$.

Conversely, suppose $D^n = D$, $n \geq 2$. As $V(D^*) = V(D) = \{u_1, u_2, \ldots, u_p\}$ and $D^*$ is the smallest transitive digraph containing D. We have only to show that $A(D^*) \subseteq A(D)$. Let $(u, v) \in A(D^*)$. Then $u = u_{i_0} \rightarrow u_{i_1} \rightarrow \ldots \rightarrow u_{i_k} = v$ is a dipath of length $k \leq n$ in D. Therefore, $u = u_{i_0} \rightarrow u_{i_1} \rightarrow \ldots \rightarrow u_{i_k} = v$ is a dipath of length $k$ in D and there is no addition of new arcs in D when we take the n-th power of D for all $n \geq 2$. Therefore $(u, v) \in A(D)$. Thus $D^* = D$. Therefore D is transitive. 

From above theorem we see that if D is transitive then D itself is an n-th root of D for all $n \geq 2$. Following theorem gives a minimal n-th root of a transitive digraph when $n \geq$ length of a longest dipath in D. In fact all the distinct n-th roots of transitive digraph when $n \geq$ length of a longest dipath in D can
be found out. An arc \((u,v)\) in \(D\) is said to be \(k\)-restored if there is a \(u,v\)-dipath of length \(k\) in \(D\). Let \(l\) denote the length of a longest dipath of \(D\).

**THEOREM 4.2.2** If \(D_v\) denotes the reduction (see p. 59) of a transitive digraph \(D\) at each of its vertices and \(n \geq l\) then the following holds \((V = V(D))\).

(i) \(D_v\) is connected.

(ii) There is a \(u,v\)-dipath in \(D\) iff there is a \(u,v\)-dipath in \(D_v\).

(iii) \((u,v) \in A(D_v)\) iff \((u,v)\) is not \(k\)-restored in \(D\) for all \(k \leq n\).

(iv) \(D_v\) is a unique minimal \(n\)-th root of \(D\).

**PROOF:** Let \(D\) be a transitive digraph on \(p\) vertices \(u_1,u_2,\ldots,u_p\) and \(n \geq l\).

(i): As \(D_v\) is a digraph obtained by reducing \(D\) at each vertex of \(D\) one by one. In the process of reduction of \(D_{i-1}\) at vertex \(u_i\), we remove only those arcs \((u_j,u_k)\) for which \(u_j + u_i + u_k\) is a dipath in \(D_{i-1}\). Therefore even removing \((u_j,u_k)\) there exists a dipath \(u_j + u_i + u_k\) in \(D_{i-1}\). Hence at each step of reduction digraph remains connected. Thus \(D_v\) is connected.

(ii): Let \(u = u_{i_0} + u_{i_1} + \ldots + u_{i_k} = v\) is a diapath in \(D\). Therefore \((u_{i_0},u_{i_1}), (u_{i_1},u_{i_2}),\ldots, (u_{i_{k-1}},u_{i_k}) \in A(D)\). If \((u_{i_0},u_{i_1}), (u_{i_1},u_{i_2}),\ldots, (u_{i_{k-1}},u_{i_k}) \in A(D_v)\) then there is nothing to prove. If one of these arcs say \((u_{i_r},u_{i_{r+1}}), r \in (0,1,\ldots,k)\), is not in \(D_v\) then there is a dipath from \(u_{i_r}\) to \(u_{i_{r+1}}\) in \(D_v\). Hence there is a dipath from \(u\) to \(v\) in \(D_v\). Inductively, there is a \(u,v\)-dipath in \(D_v\). As \(A(D_v) \subseteq A(D)\) every \(u,v\)-dipath in \(D_v\) is a \(u,v\)-dipath in \(D\).
(iii): Let \((u,v) \in A(D_v)\). Suppose \((u,v)\) is restored by a dipath
\[ u = u_0 \rightarrow u_1 \rightarrow ... \rightarrow u_k = v \] in \(D\) then \((u_0,u_1)\) and \((u_k,u_0)\)
eq A(D) since \(D\) is transitive. Therefore \((u_0,u_k) \notin A(D_u)\)
and \(D_v \leq D_u\). Hence \((u,v) \notin A(D_v)\), a contradiction. Converse
follows as we remove only \(k\)-restored arcs.

(iv): As \(V(D_v) = V(D)\) to show \(D_v^n = D\) we have only to show that
\(A(D_v^n) = A(D)\). Let \((u,v) \in A(D_v^n)\) then there is a \(u,v\)-dipath
of length \(\leq n\) in \(D_v\). By (ii) there is a \(u,v\)-dipath in \(D\).
Therefore \((u,v) \in A(D_v^n) = A(D)\). For the reverse inclusion
let \((u,v) \in A(D) = A(D_v^n)\) then there exists a \(u,v\)-dipath in \(D\).
By (ii), there exists a \(u,v\)-dipath in \(D_v\) of length \(k \leq n\).
Therefore \((u,v) \in A(D_v^n)\). Thus \(D_v\) is a \(n\)-th root of \(D\). Now
suppose \(H\) is any other \(n\)-th root of \(D\). We claim that
\(A(D_v^n) \leq A(H)\). For, let \((u,v) \in A(D_v^n)\). By (iii), \((u,v)\) is not
\(k\)-restored in \(D\), \((k \leq n)\). Suppose \((u,v) \notin A(H)\) then there is
a \(u,v\)-dipath of length \(k\) in \(H\). Hence there is a \(u,v\)-dipath
of length \(k\) in \(D\). Therefore \((u,v)\) is \(k\)-restored in \(D\), a
contradiction. Thus \((u,v) \in A(D_v^n) \Rightarrow (u,v) \in A(H)\). Hence for
any other \(n\)-th root \(H\) of \(D\), \(A(D_v^n) \leq A(H)\), so that \(D_v\) is a
unique minimal \(n\)-th root of \(D\). \(\blacksquare\)

THEOREM 4.2.3 Let \(A_v\) be the set of arcs removed to obtain \(D_v\)
from a transitive digraph then there are exactly \(2^m\) distinct \(n\)-th
roots of \(D\), where \(n \geq l\), \(|A_v| = m\).

PROOF: Let \(B \subseteq A_v\) and \(H_B\) be the digraph defined as
\[ V(H_B) = V(D_v) = V(D) \text{ and } A(H_B) = A(D_v^n) \cup B. \] To show that \(H_B^n = D\)
we have only to show that \(A(H_B^n) = A(D)\). But \((u,v) \in A(D) = A(D_v^n)\)
iff there is a \(u,v\)-dipath of length \(\leq n\) in \(D_v \leq H_B\) (by (ii)) and
this is true iff $(u,v) \in A(H^p_r)$. Thus for every distinct subset of $A_v$ there is a distinct $n$-th root of $D$. But there are exactly $2^m$ distinct subsets of $A_v$. Hence there are $2^m$ distinct $n$-th roots of $D$. Now if $K$ is any $n$-th root of $D$ obtained from $D$ by removing $R$ arcs and $D_v$ is the only minimal $n$-th root of $D$ therefore $R \subseteq A_v$.

Hence there is an $n$-th root say $H$ such that $H = K$. Thus there are only $2^m$ distinct $n$-th roots of $D$ where $n \geq l$. 

**TREE n-TH ROOTS OF DIGRAPHS**

Homada Takashi [14] stated and proved a set of necessary and sufficient conditions for the existence of a tree $n$-th roots of graphs. Here we extend the result of Hamada Takashi to digraphs, using the notion of carrier-complete subdigraphs. We need the following definition of a weak $u,v$-carrier linking.

Given a family $\Omega = \langle K_1, K_2, \ldots, K_p \rangle$ of carrier-complete subdigraphs of a digraph $D$ with vertices $u_1, u_2, \ldots, u_p$, weak $u,v$-carrier linking (w.r.t $\Omega$) is a subcollection $L = \langle K_{i_1}, K_{i_2}, \ldots, K_{i_n} \rangle$ of $n \geq 1$ elements of $\Omega$ such that $v = u_{i_n}$ and $u = u_{i_0} \in S_{i_1} \cup T_{i_1}$, $u_{i_1} \in S_{i_2} \cup T_{i_2}$, $\ldots$, $u_{i_{n-1}} \in S_{i_n} \cup T_{i_n}$, $n$ is called the length of the weak $u,v$-carrier-linking and the digraph spanned by $\bigcup_{r=1}^{n-1} V(K_{i_r})$ is called the union of the linking $L$ and is denoted by $UL$, where $K_{i_j} = K(S_{i_j}, u_{i_j}, T_{i_j})$.

**PROPOSITION 4.2.4** Let $D$ be a digraph with $p$ vertices $u_1, u_2, \ldots, u_p$ and $T$ be a ditree such that $T^n = D$, $n \geq 2$. For each $i$, let $K_i$ be the complete-carrier $K(S_i, u_i, T)$, where $S_i$ and $T_i$ are the in and out-neighborhoods of a vertex $u_i$ in $T$, respectively and $\Omega = \langle K_1, K_2, \ldots, K_p \rangle$. Following properties hold
(i) \( u = u_0 \to u_1 \to u_2 \to \ldots \to u_k = v \) is a dipath in \( T \) iff
\[ L = \langle K_{t_i, t_{i_2}, \ldots, t_{i_k}} \rangle \] is a \( u, v \)-carrier linking w.r.t \( \Omega \).

(ii) \( u = u_0 \to u_1 \to u_{i_2} \to \ldots \to u_k = v \) is a path in \( T \) iff
\[ L = \langle K_{t_{i_1}, t_{i_2}, \ldots, t_{i_k}} \rangle \] is a weak \( u, v \)-carrier linking w.r.t \( \Omega \).

**PROOF:**

(i) Let \( u = u_0 \to u_1 \to u_{i_2} \to \ldots \to u_k = v \) be a dipath in \( T \). Then
\[ u = u_0 \in S_{t_{i_0}}, u_i \in S_{t_{i_1}}, \ldots, u_{i_{k-1}} \in S_{t_{i_k}} \] and \( u_k = v \) and
\[ L = \langle K_{t_{i_1}, t_{i_2}, \ldots, t_{i_k}} \rangle \] forms a \( u, v \)-carrier linking w.r.t \( \Omega \).

Conversely, if \( L = \langle K_{t_{i_1}, t_{i_2}, \ldots, t_{i_k}} \rangle \) is a \( u, v \)-carrier linking w.r.t \( \Omega \), then \( u = u_0 \in S_{t_{i_0}} \), \( u_i \in S_{t_{i_1}} \), \ldots, \( u_{i_{k-1}} \in S_{t_{i_k}} \) and \( u_k = v \). By definition of the \( S_i \)'s,
\[ (u_{i_0}, u_{i_1}), (u_{i_1}, u_{i_2}), \ldots, (u_{i_{k-1}}, v) \in ACT \). Hence
\[ u = u_0 \to u_1 \to u_{i_2} \to \ldots \to u_k = v \] is a dipath in \( T \).

(ii) Let \( u = u_{i_0} \to u_1 \to u_{i_2} \to \ldots \to u_k = v \) be a path in \( T \). Then
\[ (u_{i_0}, u_{i_1}) \text{ or } (u_{i_1}, u_{i_2}), (u_{i_2}, u_{i_3}) \text{ or } (u_{i_3}, u_{i_4}), \ldots, \]
\[ (u_{i_{k-1}}, u_k) \text{ or } (u_k, u_{i_k}) \in ACT \). Therefore \( u = u_{i_0} \in S_{t_{i_1}} \text{ or } T_{i_1}, u_{i_1} \in S_{t_{i_2}} \text{ or } T_{i_1}, \ldots, u_{i_{k-1}} \in S_{t_{i_k}} \text{ or } T_{i_k} \text{ and } u_k = v \), i.e. \( u = u_{i_0} \in S_{t_{i_1}} \cup T_{i_1}, u_{i_1} \in S_{t_{i_2}} \cup T_{i_2}, \ldots, u_{i_{k-1}} \in S_{t_{i_k}} \cup T_{i_k} \),
\[ u_{i_k} = v \], so that \( L = \langle K_{t_{i_1}, t_{i_2}, \ldots, t_{i_k}} \rangle \) forms a weak \( u, v \)-carrier linking, w.r.t \( \Omega \).

Conversely, suppose \( L = \langle K_{t_{i_1}, t_{i_2}, \ldots, t_{i_k}} \rangle \) forms a weak \( u, v \)-carrier linking w.r.t \( \Omega \). Then \( u = u_{i_0} \in S_{t_{i_1}} \cup T_{i_1}, \).
\[ u_{i_1} \in S_{t_{i_2}} \cup T_{i_2}, \ldots, u_{i_{k-1}} \in S_{t_{i_k}} \cup T_{i_k} \text{ and } u_k = v \], i.e. \( u_{i_1} \in S_{t_{i_1}} \text{ or } T_{i_1}, u_{i_2} \in S_{t_{i_2}} \text{ or } T_{i_2}, \ldots, u_{i_{k-1}} \in S_{t_{i_k}} \text{ or } T_{i_k} \) and \( u_k = v \). Therefore \( (u_{i_0}, u_{i_1}) \) or \( (u_{i_1}, u_{i_2}), (u_{i_2}, u_{i_3}) \) or \( (u_{i_3}, u_{i_4}), \ldots, (u_{i_{k-1}}, u_k) \text{ or } (u_k, u_{i_k}) \in ACT \). Hence
\[ u_{i_0} \to u_{i_1} \to u_{i_2} \to \ldots \to u_{i_k} = v \] is a path in \( T \).
If \( T \) is a dltree then between any two distinct vertices there is a unique path joining them. Therefore by Proposition 4.2.4 there is a unique weak \( u,v \)-carrier linking of subdigraphs of \( T^n, n \geq 2 \), w.r.t \( \Omega \).

**Theorem 4.2.5** A digraph \( D \) with \( p \) vertices \( u_1, u_2, \ldots, u_p \) has a directed tree \( n \)-th root, \( n \geq 2 \) iff \( D \) contains a collection 
\[ \Omega = \langle K_1, K_2, \ldots, K_p \rangle \]
such that

(i) \( u_i \in T \) iff \( u_i \in S_i \) for every \( i \neq j \).

(ii) \( (u,v) \in A(D) \) iff there exists a \( u,v \)-carrier linking \( L \) w.r.t \( \Omega \) of length \( \leq n \).

(iii) for any \( u,v \in V(D) \) there exists exactly one weak \( u,v \)-carrier linking \( L \) between \( u \) and \( v \).

**Proof:** Suppose \( D \) has a directed tree \( n \)-th root \( T \). For each \( u_i \), let \( K_i = K(S_i, u_i, T_i) \), where \( S_i \) and \( T_i \) are the in and out-neighbourhood of \( u_i \) in \( T \), respectively. We then have a collection 
\[ \Omega = \langle K_1, K_2, \ldots, K_p \rangle \]
of carrier complete digraphs. Moreover, each \( K_i \) is subdigraph of \( D \). In fact, if \( (u_k, u_j) \in A(K_i) \) for some \( i \), then 
\[ d(u_k, u_j) \leq 2 \leq n \] in \( T \). Therefore \( (u_k, u_j) \in A(T^n) = A(D) \). Now, it remains to show that conditions (i) - (iii) are satisfied. 

Condition (i) follows from the definition of \( S_i \) and \( T_i \).

Proof of (ii) : \( (u,v) \in A(D) = A(T^n) \) iff there exists a unique 

dipath \( u = u_i_0 + u_i_1 + \ldots + u_i_k = v \) of length \( k \leq n \) in \( T \). By 

**proposition 4.2.4** this is equivalent to that there exists a 
\( u,v \)-carrier linking \( L = \langle K_i_1, K_i_2, \ldots, K_i_k \rangle \) of length \( k \leq n \).

Proof of (iii) : Let \( u,v \in V(D) = V(T) \) be any two vertices. Then 
there exists a unique path joining \( u \) and \( v \) in \( T \). Therefore by
Proposition 4.2.4, there exist exactly one weak $u,v$-carrier linking between $u$ and $v$.

Conversely, suppose there exists a collection $\Omega = \{K_1, K_2, \ldots, K_p\}$ of $p$ complete-carrier $K_i = K(S_i, u_i, T_i)$ satisfying the conditions (i)-(iii). Define the digraph $T$ as $V(T) = V(D) = \{u_1, u_2, \ldots, u_p\}$ and $A(T) = \{(u_i, u_j) \in A(D) \mid u_i \in S_j\}$. Let $(u, v) \in A(T^n)$. Then $d(u,v) \leq n$ in $T$. Therefore there exists a unique dipath $u = u_{i_0} \rightarrow u_{i_1} \rightarrow \ldots \rightarrow u_{i_k} = v$, $k \leq n$ in $T$. Hence $(u_{i_0}, u_{i_1}), (u_{i_1}, u_{i_2}), \ldots, (u_{i_{k-1}}, u_{i_k}) \in A(T)$. So that $u = u_{i_0} \in S_{i_1}, u_{i_1} \in S_{i_2}, \ldots, u_{i_{k-1}} \in S_{i_k}$ and $u_{i_k} = v$. Hence by definition of $T$, $(u, u_{i_1}), (u_{i_1}, u_{i_2}), \ldots, (u_{i_{k-1}}, u_{i_k}) \in A(T)$ and $u_{i_k} = v$. Thus $u = u_{i_0} \rightarrow u_{i_1} \rightarrow \ldots \rightarrow u_{i_k} = v$ is a dipath of length $k \leq n$ in $T$. Therefore $d(u,v) \leq n$ in $T$. So that $(u, v) \in A(T^n)$.

Hence $A(D) \subseteq A(T^n)$. Finally, $T$ is a directed tree, since if $u, v \in V(T) = V(D)$ then by condition (iii), there exists exactly one weak $u,v$-carrier linking $L$, w.r.t $\Omega$, between $u$ and $v$. This means that there exist exactly one $u,v$-path in $T$. Thus $T$ is a directed tree such that $T^n = D$. Therefore $D$ has a directed tree $n$-th root.

ILLUSTRATION 4.2.6

A digraph $D$ and its carrier-complete subdigraphs $K_1 - K_p$ are shown in Figure 4.2.1. The conditions of Theorem 4.2.5 are satisfied by $K_1 - K_p$. $T$ is a directed tree $n$-th root of $D$ for $n \geq 3$. 

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REGULAR n-TH ROOTS

The n-th power of a regular digraph may or may not be a regular digraph. For example the digraph $H$ of Figure 3.2.9 is regular but $H^2$ is not regular. Thus the irregular digraph may have regular digraph as an n-th root. $n \geq 2$. The following theorem gives a set of necessary and sufficient conditions for the existence of a regular n-th root for a digraph.

THEOREM 4.2.7 A digraph $D$ with $p$ labelled vertices $u_1, u_2, \ldots, u_p$ has a r-regular digraph as an n-th root iff $D$ contains a collection $\Omega = \{K_1, K_2, \ldots, K_p\}$ of $p$ carrier-complete subdigraphs $K_i = K(S_i, u_i, T_i)$ such that

(i) $u_i \in T_j$ iff $u_j \in S_i$ for every $i \neq j$.
(ii) \((u,v) \in \text{ACD}\) iff there exists a \(u,v\)-carrier linking of length \(\leq n\).

(iii) \(|V(K_i)| = r+1\) for every \(i\).

**Proof:** Let \(H\) be a \(r\)-regular \(n\)-th root of a digraph \(D\). For each \(u_i\), let \(S_i\) be the in-neighbourhood of \(u_i\) in \(H\) and \(T_i\) be the out-neighbourhood of \(u_i\) in \(H\). Therefore \(u_i \in S_i \cup T_i\) and 

\[ \text{K}_i = K(S_i, u_i, T_i) \]

is a carrier-complete digraph defined for each \(i = 1, 2, \ldots, p\). It is obvious that the \(K_i\)'s are subdigraphs of \(H^n = D\) and \(u_i \in V(K_i)\) for every \(i\). The condition (i) is immediate. Condition (iii) is satisfied because \(|V(K_i)| = |S_i \cup \{u_i\} \cup T_i| = r+1\), for every \(i\), since \(H\) is a \(r\)-regular digraph. For (ii), 

\((u,v) \in \text{ACD} = \text{ACH}^n\) iff there exists a directed path \(u = u_i_0 \rightarrow u_i_1 \rightarrow u_i_2 \rightarrow \ldots \rightarrow u_i_k = v\) in \(H\) of length \(k \leq n\). But this is true iff \(K_{i_0}, K_{i_1}, \ldots, K_{i_k}\) forms a \(u,v\)-carrier linking of length \(k \leq n\).

Conversely, suppose \(\Omega = (K_1, K_2, \ldots, K_p)\) is a collection of carrier complete subdigraphs of \(D\) satisfying (i), (ii) and (iii). We define a digraph \(H\) as \(V(H) = V(D) = \{u_{i_0}, u_{i_1}, \ldots, u_{i_p}\}\) and

\(\text{ACH} = \{(u_i, u_j) \in \text{ACD} \mid u_i \in S_i \text{ or } u_j \in T_j\}\), where \(K_i = K(S_i, u_i, T_i)\), for \(i = 1, \ldots, p\). Now \((u,v) \in \text{ACH}^n\) iff there exists a directed path \(u = u_{i_0} \rightarrow u_{i_1} \rightarrow u_{i_2} \rightarrow \ldots \rightarrow u_{i_k} = v\) in \(H\) of length \(k \leq n\). But this means that \(u = u_{i_0} \in S_{i_0}, u_{i_1} \in S_{i_1}, u_{i_{k-1}} \in S_{i_{k-1}}, u_{i_{k}} = v\) and \(k \leq n\). In other words, \((u,v) \in \text{ACH}^n\) iff \(L = (K_{i_0}, K_{i_1}, \ldots, K_{i_k})\) forms a \(u,v\)-carrier linking of length \(k \leq n\), w.r.t \(\Omega\). By condition (ii), this is equivalent to \((u,v) \in \text{ACD}\). Therefore \(\text{ACH}^n = \text{ACD}\).

By definition of \(\text{ACH}\) and condition (iii), degree of each vertex in \(H\) is \(r\). Therefore \(H\) is a \(r\)-regular \(n\)-th root of \(D\).

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A digraph $D$ and its carrier-complete subdigraphs $K_1$ - $K_8$ are shown in Figure 4.2.2. The conditions of Theorem 4.2.7 are satisfied by $K_1$ - $K_8$. $D$ itself is a 3-regular $n$-th root of $D$ for $n \geq 2$.

**HAMILTONIAN n-TH ROOTS OF DIGRAPHS**

Recall that a digraph is hamiltonian if it has a directed cycle containing all its vertices. It is obvious that the $n$-th power of a hamiltonian digraph is hamiltonian. But converse is not true. For example, in the Figure 3.2.11, the digraph $H^2$ is hamiltonian but $H$ is not hamiltonian. J.J. Karaganis [10] proved that the cube of every connected undirected graph with $p \geq 3$ vertices is hamiltonian. However this result is not true for digraphs. The proof of the following Theorem is straightforward.
THEOREM 4.2.9

(i) Let $D$ be a digraph. If there exists a vertex $u \in V(D)$ such that $\delta^-(u) = 0$ (or $\delta^+(u) = 0$) in $D$ then $\delta^-(u) = 0$ (or $\delta^+(u) = 0$) in $D^n$, for every $n \geq 2$.

(ii) An $n$-th power of a connected digraph $D$ is not Hamiltonian if there exists at least one vertex $u \in V(D)$ such that either $\delta^-(u) = 0$ or $\delta^+(u) = 0$ in $D$, $n \geq 2$.

As every directed tree has at least two vertices whose indegree or outdegree is zero, by Theorem 4.2.9, the $n$-th power of no ditree is hamiltonian for $n \geq 2$. Hence no hamiltonian digraph has a directed tree as $n$-th root, for $n \geq 2$.

As an $n$-th power of a hamiltonian digraph is hamiltonian only hamiltonian digraphs may have hamiltonian $n$-th roots. Also not every hamiltonian digraph may have a hamiltonian $n$-th root. Indeed no directed cycle has an $n$-th root, for $n \geq 2$. However, we have the following theorem.

THEOREM 4.2.11 A hamiltonian digraph $D$ with $p$ vertices labeled as $u_1, u_2, \ldots, u_p$ has a hamiltonian $n$-th root iff there exists a collection $\Omega = \{K_1, K_2, \ldots, K_p\}$ of carrier-complete subdigraphs $K_i \subseteq K(S_i, u_i, T_i)$ of $D$, where the $K_i$'s are associated in a one-to-one manner with vertices $u_i$'s of $D$ such that

(i) $u_i \in T_j$ iff $u_j \in S_i$, for every $i \neq j$,

(ii) $(u, v) \in \overline{A(D)}$ iff there exists a $u, v$-carrier linking of length $\leq n$, w.r.t. $\Omega$,

(iii) $\Omega = \{K_1, K_2, \ldots, K_p\}$ forms a closed carrier-linking of length $p$. 

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PROOF: Suppose $D$ has a hamiltonian $n$-th root $H$ and $u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_p \rightarrow u_1$ is a hamiltonian circuit in $H$. For each $u_i \in V(H)$, let $S_i$ be the in-neighbourhood of $u_i$ and $T_i$ be the out-neighbourhood of $u_i$ in $H$. Then $u_i \in S_i \cup T_i$ so that $K_i = K(S_i, u_i, T_i)$ is a carrier-complete digraph defined for each $i = 1, 2, \ldots, p$. It is obvious that the $K_i$'s are subdigraphs of $H^n = D$ and $u_i \in V(K_i)$ for every $i$. Thus we have a collection $\Omega = \{K_1, K_2, \ldots, K_p\}$ of carrier-complete subdigraphs of $D$. The proofs of conditions (i) and (ii) being as usual. We prove (iii). By the definition of $S_i$'s, $u_1 \in S_2 u_2 \in S_3 \ldots u_{p-1} \in S_p$ and $u_p \in S_1$. Therefore $\Omega = \{K_1, K_2, \ldots, K_p\}$ forms a closed carrier-linking of length $p$.

Conversely, suppose $\Omega = \{K_1, K_2, \ldots, K_p\}$ is a collection of carrier complete subdigraphs of $D$ satisfying (i) - (iii). We define a digraph $H$ as $V(H) = V(D) = \{u_1, u_2, \ldots, u_p\}$ and $A(H) = \{(u_i, u_j) \in A(D) \cap u_i \in S_j \text{ or } u_j \in T_i\}$ where $K_i = K(S_i, u_i, T_i)$, for $i = 1, \ldots, p$. The proof that $H$ is an $n$-th root of $D$ is on the same lines as the proof of Theorem 4.2.7. Clearly $H$ is connected and by condition (iii) and definition of $A(H)$, $H$ is hamiltonian. Thus $H$ is a hamiltonian $n$-th root of $D$. ■

ILLUSTRATION 4.2.12

A digraph $D$ and its carrier-complete subdigraphs $K_1 - K_7$ are shown in Figure 4.2.3. The conditions of Theorem 4.2.11 are satisfied by $K_1 - K_7$. $H$ is a hamiltonian cube root of $D$. 

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EULERIAN n-TH ROOTS OF DIGRAPHS

This section is devoted to the question of existence of an eulerian n-th roots of digraphs. The n-th power of an eulerian digraph may not be eulerian therefore non-eulerian digraphs may have an eulerian n-th roots. The following theorem gives necessary and sufficient conditions for the existence of eulerian n-th roots of digraphs.

THEOREM 4.2.13 A digraph D with p vertices \( u_1, u_2, \ldots, u_p \) has an eulerian n-th root iff there exists a collection

\[ \Omega = \{K_1, K_2, \ldots, K_p\} \]

of p carrier-complete subdigraphs

\[ K_i = K(S, u_i, T) \]

of D such that

1. \( u_i \in T_i \) iff \( u_j \in S_i \) for every \( i \neq j \);
(ii) \((u,v) \in A(D)\) iff there exists a \(u,v\)-carrier linking of length \(\leq n\), w.r.t. \(\Omega\);

(iii) \(|T_i| = |S_i|\) for every \(i\).

PROOF: Let \(H\) be an eulerian \(n\)-th root of a digraph \(D\). The carrier-complete digraph \(K_i = K(S_i,u_i,T_i)\) are defined as in the proof of Theorem 4.2.7. Thus we have a collection \(\Omega = \{K_1,K_2,...,K_p\}\) of \(p\) carrier-complete subdigraphs of \(D\). Now it remains to show that conditions (i) - (iii) are satisfied.

Proof of (i) and (ii) are as usual.

To prove (iii): As \(H\) is an eulerian digraph, in-degree of \(u_i = \) out-degree of \(u_i\) for every \(i\). Hence by definition of \(S_i's\) and \(T_i's\), \(|T_i| = |S_i|\) for every \(i\). Hence condition (iii) is satisfied.

Conversely, suppose \(\Omega = \{K_1,K_2,...,K_p\}\) is a collection of \(p\) carrier-complete subdigraph of \(D\) satisfying (i) - (iii). We define a digraph \(H\) as \(V(H) = V(D) = \{u_1,u_2,...,u_p\}\) and \(A(H) = \{((u_i,u_j) \in A(D)/ u_i \in S_i or u_j \in T_j\}, \) where \(K_i = K(S_i,u_i,T_i),\) for \(i = 1,2,...,p.\) As \(V(H) = V(H^p) = V(D)\) to show that \(H^p = D,\) we have only to show that \(A(H^p) = A(D)\) and whose proof is on the same line as the proof of Theorem 4.2.7. Clearly \(H\) is connected and by condition (iii) and definition of \(A(H)\), \(H\) is eulerian. Thus \(H\) is an eulerian \(n\)-th root of \(D\).

ILLUSTRATION 4.2.14

A digraph \(D\) and its carrier-complete subdigraphs \(K_1-K_8\) are shown in Figure 4.2.4. The conditions of Theorem 4.2.13 are satisfied by \(K_1-K_8,\) \(H\) is an eulerian cube root of \(D\).
BIPARTITE n-TH ROOTS OF DIGRAPHS

The n-th power of a bipartite digraph may or may not be a bipartite digraph. Also the n-th power of a complete bipartite digraph may or may not be a complete digraph. Here we give necessary and sufficient conditions for the existence of bipartite n-th roots of digraphs.

THEOREM 4.2.15 A digraph D with p vertices \( u_1, u_2, \ldots, u_p \) has a bipartite n-th root iff there exists a collection \( \Omega = (K_1, K_2, \ldots, K_p) \) of p carrier-complete subdigraphs

\[ K_i = K(S_i, u_i, T_i), \quad i = 1, 2, \ldots, p \]

and two disjoint subsets X and Y of \( V(D) \) such that

(i) \( V(D) = X \cup Y \);

(ii) \( X \cap Y = \emptyset \);

(iii) \( X \cap S_i = \emptyset \) for all \( i = 1, 2, \ldots, p \);

(iv) \( Y \cap T_i = \emptyset \) for all \( i = 1, 2, \ldots, p \);

(v) \( X \cap T_i = \emptyset \) for all \( i = 1, 2, \ldots, p \);

(vi) \( Y \cap S_i = \emptyset \) for all \( i = 1, 2, \ldots, p \).

Figure 4.2.4.
(ii) $u_j \in S_i \cap X$ iff $u_j \in T_j \cap Y$ and $u_j \in S_i \cap Y$ iff $u_j \in T_j \cap X$, for every $i \neq j$;

(iii) $(u,v) \in A(D)$ iff there exists a $u,v$-carrier linking of length $\leq n$, w.r.t. $Q$.

**PROOF:** Suppose $D$ has a bipartite $n$-th root $H$. Therefore there exist two disjoint subsets $X$ and $Y$ of $V(H)$ such that $V(H) = X \cup Y$ and no two vertices in the same subsets are adjacent. Let $V(H) = \{u_1, u_2, \ldots, u_p\}$. The carrier-complete subdigraph $K_i$'s are defined in the usual manner as in the proof of Theorem 4.2.11. Thus we have a collection of $p$ carrier-complete subdigraphs of $D$ and two disjoint subsets $X$ and $Y$ of $V(D)$. It remains to verify that the conditions (i) - (iii) are satisfied. Condition (i) follows from the fact that $V(H) = V(D)$.

To prove (ii) : $u_j \in S_i \cap X$

$\iff u_j \in S_i$ and $u_j \in X$.

$\iff (u_j, u_j) \in A(H)$ and $u_j \in X$.

$\iff u_j \in T_j$ and $u_j \in Y$ (since $H$ is bipartite)

$\iff u_j \in T_j \cap Y$.

Similarly, $u_j \in S_i \cap Y$ iff $u_j \in T_j \cap X$. Thus condition (ii) is satisfied. Proof of (iii) is on the same line as in the proof of Theorem 4.2.7.

Conversely, suppose $\Omega = (K_1, K_2, \ldots, K_p)$ is a collection of carrier-complete subdigraph of $D$ and two disjoint subsets $X$ and $Y$ of $V(D)$ satisfying (i) - (iii). We define a digraph $H$ as $V(H) = V(D) = \{u_1, u_2, \ldots, u_p\}$ and

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\( \text{ACH} = \{ (u_i, u_j) \in \text{ACH})/ u_i \in S_j \cap X \text{ or } u_i \in S_j \cap Y \}, \text{ where} \)

\[ K_i = K(S_i, u_i, T_i), \text{ for } i = 1, 2, \ldots, p. \]

Now \((u, v) \in \text{ACH})

\[ (\Rightarrow) \text{ there exists a } u, v\text{-carrier linking } L = (K_i, K_2, \ldots, K_k) \]

\[ \text{of length } k \leq n \text{ w.r.t } \Omega \text{ (by condition (iii))}. \]

\[ (\Rightarrow) u = u_{i_0}, u_{i_1}, \ldots, u_{i_k} = v \text{ and} \]

\[ u = u_{i_0}, u_{i_1}, \ldots, u_{i_k} = v \in X \cup Y, k \leq n. \]

\[ (\Rightarrow) (u_{i_1}, u_{i_2}, \ldots, u_{i_k}, v) \in \text{ACH} \text{ (by definition of H)}. \]

\[ (\Rightarrow) u + u_{i_1} + u_{i_2} + \ldots + u_{i_k} = v \text{ is a dipath of length} \]

\[ k \leq n \text{ in } H. \]

\[ (\Rightarrow) (u, v) \in \text{ACH}^n. \]

Further, \(H\) is bipartite. For, if \((u_i, u_j) \in \text{ACH}^n\), then \(u_i \in S_j \cap X \)

or \(u_i \in X_j \cap Y\). Therefore \(u_i \in X \) and \(u_i \in S_j \cap X\) or \(u_i \in Y \) and \(u_i \)

\(\in S_j \cap Y\). By condition (ii), \(u_i \in X \) and \(u_j \in T_j \cap Y\) or \(u_i \in Y \) and \(u_j \)

\(\in T_j \cap X\). Hence \(u_i \in X \) and \(u_j \in Y\) or \(u_i \in Y \) and \(u_j \in X. \)

Therefore every arc in \(H\) joins a vertex of \(X\) to \(Y\) or \(Y\) to \(X\), i.e. \(H\) is bipartite. Thus \(H\) is a bipartite \(n\)-th root of \(D\). \(\square\)

ILLUSTRATION 4.2.15

A digraph \(D\) and its carrier-complete subdigraphs \(K_{1} - K_{8}\) are shown in Figure 4.2.5. The conditions of Theorem 4.2.15 are satisfied for \(X = (u_1, u_2, u_3, u_4)\), \(Y = (u_5, u_6, u_7, u_8)\) and \(K_{1} - K_{8}\). \(H\) is a bipartite \(n\)-th root of \(D\), for \(n \geq 3.\)
DIRECTED CYCLIC n-TH ROOTS OF DIGRAPHS

If C is a directed cycle on p vertices then diameter of C = p-1. Therefore C^n, n ≥ p-1 is always a symmetric, regular digraph of degree 2(p-1) and every symmetric, regular digraph of degree 2(p-1) on p vertices has a directed cyclic n-th root, for n ≥ p-1. Hence we study the existence of a directed cyclic n-th root for a regular, asymmetric, digraph of degree 2n on p vertices, for n < p-1. The following proposition is useful.

PROPOSITION 4.2.17 Let D be a digraph on p vertices and C be a directed cycle such that C^n = D, n < p-1, p ≥ 5. For a ∈ V(C), let K_{a} be the carrier-complete digraph K(I_{a}, a, O_{a}), where I_{a} and O_{a} are in and out-neighborhoods of a in C, respectively. Then the following properties hold.
(i) D is a regular of degree 2n.
(ii) If a ∈ V(K_2^b) then b ∈ V(K_a^a).
(iii) a is adjacent to b in C iff V(K_a^a) ∩ V(K_b^b) = \{a, b\}.
(iv) a is adjacent to b in C iff AC(K_a^a) ∩ AC(K_b^b) = \{(a, b)\} or \{(b, a)\}.
(v) The vertex c is adjacent to a and b in C iff V(K_a^a) ∩ V(K_b^b) = \{c\}.
(vi) V(K_a^a) ∩ V(K_b^b) = \{\} iff every dipath from a to b in C is of length ≥ 3.
(vii) AC(K_a^a) ∩ AC(K_b^b) = \{\} iff a is not adjacent to b in C.
(viii) For each complete -carrier K_i there exist exactly two complete-carriers K_j and K_k (say) such that AC(K_i^i) ∩ AC(K_i^j) = \{(u_j, u_i^i)\} and AC(K_i^i) ∩ AC(K_i^k) = \{(u_k, u_i^i)\}.

PROOF : While (i) is obvious, the proof of (ii) - (viii) are on the same lines as in the proof of Proposition 3.2.14. •

THEOREM 4.2.18 A regular digraph D of degree 2n with p vertices u_1, u_2, ..., u_p has a directed cyclic n-th root, n < p-1 iff there exists a collection Ω = \{K_1, K_2, ..., K_p\} of p carrier-complete subdigraphs K_i=K(S_i,u_i,T_i) of D such that
(a) u_i ∈ V(K_i) and |V(K_i)| = 3 for every i,
(b) S_i = \{u_j\} iff T_j = \{u_i\} for every i ≠ j,
(c) \{(u, v)\} ∈ AC(D) iff there exists a u,v-carrier linking of length ≤ n, w.r.t. Ω,
(d) no two carrier-complete subdigraphs of Ω intersect in more than one arc,
(e) for each K_i, there exist exactly two complete-carriers such that K_i intersect with each of them in exactly one arc.

PROOF : Let C be a directed cyclic n-th root of a digraph D and V(C) = V(D) = \{u_1, u_2, ..., u_p\} = V(C^n). We define a collection
$\Omega = (K_1, K_2, \ldots, K_p)$ of carrier-complete subdigraphs of $D$ as in the proof of Theorem 4.2.7. Now it remains to show that conditions (a) - (e) are satisfied. The condition (a) is immediate from the definition of the $K_i$'s and $C$ is a directed cycle. (b) follows from the fact that in-degree = out-degree = 1 for each $u_i$ in $C$.

Proof of (c) is on the same line as in the proof of Theorem 4.2.7. Finally (d) and (e) follow from the Proposition 4.3.17.

Conversely, suppose $\Omega = (K_1, K_2, \ldots, K_p)$ is a collection of carrier-complete subdigraph of $D$ satisfying (a) - (e). We define a digraph $C$ as $V(C) = V(D) = (u_1, u_2, \ldots, u_p)$ and $A(C) = \{(u_i, u_j) \in A(D) / A(K_j) \cap A(K_i) = \{(u_i, u_j)\}, \text{ where}

K_i = K(S_i, u_i, T_i), \text{ for } i = 1, 2, \ldots, p.$

Now $(u, v) \in A(D)$

$\iff$ there exists a $u,v$-carrier linking $L = (K_{u_1}, K_{u_2}, \ldots, K_{u_k})$

of length $k \leq n$, w.r.t $\Omega$.

$\iff u_i \in S_{j+1}$, $j = 0, 1, \ldots, k-1$, $u = u_{i_0}$, $u_k = v$, $k \leq n$.

$\iff u_i \in S_{j+1}$ and $u_{i+1} \in T_{i_j}$, $j = 0, 1, \ldots, k-1$ (by condition (b)).

$\iff (u_{ij}, u_{ij+1}) \in A(K_i) \cap A(K_{i+1})$ and $(u_j, u_{ij+1}) \in A(K_j)$

$\iff j = 0, 1, \ldots, k-1$ (by definition of the $K_i$'s).

$\iff A(K_i) \cap A(K_{i+1}) = \{(u_j, u_{ij+1})\}$ (by condition (d))

$\iff j = 0, \ldots, k-1$.

$\iff (u_{ij}, u_{ij+1}) \in A(C)$ (by definition of $A(C)$).

$\iff u = u_{i_0} + u_{i_1} + u_{i_2} + \ldots + u_{i_k} = v$ is a dipath of length $k \leq n$ in $C$.

$\iff (u, v) \in A(C^n)$.

By conditions (b) and (e) $C$ is a directed cycle. Thus $C$ is a directed cyclic $n$-th root of $D$.  

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A digraph $D$ and its carrier-complete subdigraphs $K_1-K_8$ are shown in Figure 4.2.6. The conditions of Theorem 4.2.18 are satisfied by $K_1-K_8$. $C$ is a directed cyclic 4-th root of $D$. 

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