CHAPTER - 2

PROPERTIES OF GRAPHS HAVING
SQUARE ROOTS
1. **GRAPHS**

In this section we present a number of properties satisfied by graphs having square roots. These properties enable us to construct a large class of graphs having no square roots. Besides we study properties possessed by graphs having square roots of particular types such as tree square roots, cyclic square roots etc. Lower and upper bounds on the number of edges in a graph with square root are given. We can also decide to which square roots are minimal.

By definition, a square root of a graph is a spanning subgraph of it. It is obvious that a square root of a connected graph remains connected. Without loss of generality we can assume that a graph is connected in the study of roots of graphs. Moreover, if $H$ is a square root of a connected graph $G$, then every path of length $r$ in $H$ is a path of length $r$ in $G$. This is because of $E(H) \subseteq E(G)$.

However, the converse of these results do not hold. For example, in Figure 2.1.1 $H$ is a connected spanning subgraph of $G$ and every path of length two in $H$ is a path of length two in $G$ but $H$ is not a square root of $G$.

![Figure 2.1.1](image)
It is worth observing that a path in a graph $G$ need not be a path in its square root. A graph that has a square root can have no cut vertices. This is the content of Theorem below.

**THEOREM 2.1.1** If a graph has a square root, then it has no cut-vertices, i.e. it is a block.

**PROOF:** Let $H$ be a square root of a graph $G$ and $u$ be any vertex of $G$. Suppose that $u$ is a cut-vertex of $G$. Then $u$ cannot be pendant and there must exist at least one pair say $(v, w)$ of non adjacent vertices both adjacent to $u$ in $G$. But $u$ is a cut-vertex of $H$ as well because $H$ is a spanning connected subgraph of $G$. Hence $(v, w)$ is a pair of non-adjacent vertices both adjacent to $u$ in $H$ also. Thus $v - u - w$ is a path in $H$. This means that $vw \in E(H^2) = E(G)$. Which is a contradiction to the choice of the pair $(v, w)$ in $G$. □

Converse of Theorem 2.1.1 is not true as shown by the example of a cycle.

**COROLLARY 2.1.2** If a graph $G$ has a square root then $2 \leq K(G) \leq \lambda(G) \leq \delta(G) \leq p-1$. Where $K(G)$, $\lambda(G)$ and $\delta(G)$ denotes vertex connectivity, line connectivity and minimum degree of $G$ respectively.

**THEOREM 2.1.3** Let $G$ be a graph with more than three vertices. If $G$ has a square root, then every edge of $G$ lies in some triangle of $G$.

**PROOF:** Let $H$ be a square root of $G$ and $uv$ be any edge of $G$. Then $d(u, v) = 1$ or $d(u, v) = 2$ in $H$. 19
CASE 1: \( d(u,v) = 2 \) in \( H \).

If \( u \rightarrow w \rightarrow v \) is a path of length 2 in \( H \), then it is a path in \( G \) as well. Therefore, \( uv \) is in a triangle of \( G \).

CASE 2: \( d(u,v) = 1 \) in \( H \).

In this case \( uv \in E(H) \) and \( H \) is a connected graph such that \( |V(H)| \geq 3 \). Hence there would exist \( w \in V(H) \) which is adjacent to \( u \) or \( v \) or both. Thus \( w \rightarrow u \rightarrow v \) or \( u \rightarrow v \rightarrow w \) or \( u \rightarrow w \rightarrow v \) is a path of length 2 in \( H \). Therefore \( uv \) is in a triangle of \( G \). Thus in any case, \( uv \in E(G) \rightarrow uv \) lies in some triangle of \( G \). 

**COROLLARY 2.1.4** If \( G \) has a square root and \( |V(G)| \geq 3 \) then the following hold

(i) No edge of \( G \) is a bridge.

(ii) No vertex of \( G \) is a pendent vertex.

(iii) Girth of \( G \) is 3.

(iv) \( G \) is not bipartite.

(v) \( G \) is orientable.

**PROOF:** (i) is immediate from Theorem 2.1.3 and 1.1.4. (ii) and (iii) follow from Theorem 2.1.3. (iv) is a consequence of Theorem 2.1.3 and 1.1.5. While, (v) follows from (i) and Theorem 1.2.1. 

If \( |V(G)| = 2 \) then Corollary 2.1.4 is not true as is shown by \( K_2 \).

The converse of Theorem 2.1.3 and Corollary 2.1.4 do not hold. The graph of Figure 2.1.2 serves as a counter example.
Following theorem gives bounds on the number of edges of a graph having a square root.

THEOREM 2.1.5 If a graph $G$ with $p$ vertices have a square root then

$$\left( \frac{p^2}{4} \right) \leq |E(G)| \leq \frac{p(p-1)}{2}.$$ 

PROOF: The upper bound is obvious since every complete graph has a square root. On the otherhand Turan's Theorem (Theorem 1.1.1) and Theorem 2.1.3 yield the other inequality. 

The example of Figure 2.1.2 shows that the converse of Theorem 2.1.5 does not hold.

The following theorem gives the relation between the diameter of a graph and its square root.

THEOREM 2.1.6 If a graph $G$ has a square root $H$, then

$$d(G) = \frac{d(H)}{2} \text{ if } d(H) \text{ is even}$$

$$= \frac{d(H)+1}{2} \text{ if } d(H) \text{ is odd}.$$ 

PROOF: Let $d(H) = n$ and $x$ and $y$ be vertices in $H$ such that $d(x,y) = n$. Then there would exist in $H$ a shortest path of maximum length $n$ between the vertices $x$ and $y$. Say,
\[ x = u_0 \rightarrow u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_{n-4} \rightarrow u_{n-3} \rightarrow u_{n-2} \rightarrow u_{n-1} \rightarrow u_n = y. \]

But every subpath of length two in \( H \) becomes a path of length one in \( G \). Now \( x = u_0 \rightarrow u_2 \rightarrow u_4 \rightarrow \ldots \rightarrow u_{n-4} \rightarrow u_{n-2} \rightarrow u_n = y \) is a shortest path of maximum length \( \frac{n}{2} \) in \( G \), if \( n \) is even and \( x = u_0 \rightarrow u_2 \rightarrow u_4 \rightarrow \ldots \rightarrow u_{n-3} \rightarrow u_{n-1} \rightarrow u_n = y \) is a shortest path of maximum length \( \frac{n+1}{2} \) in \( G \), if \( n \) is odd. Hence the theorem. \( \blacksquare \)

The converse of Theorem 2.1.8 do not hold. The graphs of Figure 2.1.3 serves as a counter-example.

To investigate the properties of graphs having tree square roots we recall that a graph is said to be a triangulated graph if each cycle of length \( > 3 \) possesses a chord. Every tree is trivially a triangulated graph.

**Lemma 2.1.7** Square of a triangulated graph is triangulated.

**Proof:** Let \( H \) be a triangulated graph. In forming \( H^2 \) only those pairs of non-adjacent vertices, which are at a distance 2 in \( H \) are joined by an edge. Therefore every cycle of length \( > 3 \) in \( H^2 \) has also a chord. \( \blacksquare \)
THEOREM 2.1.8 If a graph $G$ has a tree square root then $G$ is triangulated.

PROOF: Let $T$ be a tree square root of $G$. By Lemma 2.1.7, square of a tree is triangulated. Therefore $G$ is triangulated. \[ \square \]

Following corollary is an immediate consequence of Theorem 2.1.8 and 1.1.8.

COROLLARY 2.1.9 If a graph $G$ has a tree square root then the following hold:

(i) Every minimal cutset of $G$ is a complete subgraph of $G$.

(ii) $G$ has a complete subgraph which meets every maximal stable set.

(iii) $\alpha(G) = \omega(G)$.

(iv) $G$ has a stable set which meets every clique of $G$.

(v) $\gamma(G) = \omega(G)$.

(vi) Every subgraph of $G$ has a simplicial vertex.

(vii) There are two non-adjacent vertices of $G$ which each belong to only one clique of $G$.

(viii) There exists a circuit-free simple digraph $D$ which can be obtained from $G$ by orienting each edge such that each neighbourhood of a vertex $x$ is a complete subgraph and every clique is of this form.

The converse of Corollary 2.1.9 do not hold. The graph of Figure 2.1.2 serves as a counter example.

Now we show that cyclic square roots, bipartite square roots and tree square roots are always minimal. We require the following theorem.
THEOREM 2.1.10 Let $H$ be a square root of a graph $G$. If $H$ does not contain any triangle, then $H$ is a minimal square root of $G$.

PROOF: Suppose that $H$ is not a minimal square root of $G$. Then we should be able to remove at least one edge from $H$ which is restored and removal of this edge does not change the distances between the vertices of $H$ except that the end vertices of the removed edge and $H$ remains connected. The removed edge with the restoring chain forms a triangle in $H$. Which is a contradiction.

COROLLARY 2.1.11 Bipartite square roots, in particular tree square roots and cyclic square roots $C_p$ ($p \geq 4$) are minimal.

The converse of Theorem 2.1.10 is however not true as shown by graphs of Figure 2.1.4.

Figure 2.1.4.

2. DIGRAPHS

In this section we present some properties satisfied by digraphs having square roots. These properties enable us to construct a large class of digraphs having no square roots. Besides, we study properties possessed by digraphs having roots of particular types. We can also decide as to which square roots are minimal.
By definition, a square root of a digraph $D$ is a spanning subdigraph of $D$. It is obvious that a square root of a connected digraph remains connected. We assume therefore that a digraph is connected while studying the properties of roots of digraphs. Moreover, if $K$ is a square root of a connected digraph $D$ then every dipath of length $r$ in $K$ is a dipath of length $r$ in $D$. This is because $A(K) \subseteq A(D)$.

However, the converse of these results do not hold. For example, in Figure 2.2.1, $K$ is a connected spanning subdigraph of $D$ and every dipath of length two in $K$ is a dipath of length two in $D$ but $K$ is not a square root of $D$.

It is worth observing that a dipath in a digraph $D$ need not be a dipath in its square root. But there is a $u,v$-dipath in a digraph $D$ iff there is a $u,v$-dipath in its square root (if it exists).

Theorem 2.1.1, 2.1.3 and Corollary 2.1.4 are no longer valid for digraphs. The digraph $K$ of Figure 2.2.2 serves as a counter example in each case.

![Figure 2.2.1](image1)

![Figure 2.2.2](image2)
Following theorem is similar to Corollary 2.1.2.

**THEOREM 2.2.1**  If a digraph $D$ has a square root then $1 \leq k(D) \leq \kappa(D) \leq \lambda(D) \leq \delta(D) \leq 2^{p-2}$, where $k(D)$, $\kappa(D)$, $\lambda(D)$ and $\delta(D)$ denote vertex connectivity, line (arc) connectivity and minimum degree of $D$, respectively.

Similar to Theorem 2.1.6, the following theorem gives the relation between the diameter of a digraph and its square root.

**THEOREM 2.2.2**  If a digraph $D$ has a square root $K$ and $d(K) < \infty$ then $d(D) = \frac{d(K)}{2}$ if $d(K)$ is even

$$= \frac{d(K)+1}{2}$$ if $d(K)$ is odd.

**PROOF:** Suppose, $d(K) = n$ and $u$ and $v$ are vertices in $K$ such that $d(u,v) = n$ in $K$. Then there would exist in $K$ a shortest dipath of maximum length $n$ between the vertices $u$ and $v$. Say,

$$u = u_0 \rightarrow u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_{n-4} \rightarrow u_{n-3} \rightarrow u_{n-2} \rightarrow u_{n-1} \rightarrow u_n = v.$$  

Now $u = u_0 \rightarrow u_2 \rightarrow u_4 \rightarrow \ldots \rightarrow u_{n-4} \rightarrow u_{n-2} \rightarrow u_n = v$ is a shortest dipath of maximum length $\frac{n}{2}$ in $D$, if $n$ is even and $u = u_0 \rightarrow u_2 \rightarrow u_4 \rightarrow \ldots \rightarrow u_{n-4} \rightarrow u_{n-1} \rightarrow u_n = v$ is a shortest dipath of maximum length $\frac{n+1}{2}$ in $D$, if $n$ is odd. Hence

$$d(D) = \frac{n}{2}$$ if $n$ is even

$$= \frac{n+1}{2}$$ if $n$ is odd.

Hence the theorem. ■

The converse of Theorem 2.2.2 does not hold as shown by the digraphs of Figure 2.2.3.
As a square root of a (connected) digraph appears as a spanning subdigraph it is possible to search for it by removing those arcs of the initial digraph which do not belong to a maximal root. But for this we must take care that we do not lose connectedness at any stage. The following theorem gives an upper bound to the number of arcs that can be removed from a connected digraph \( D = (V,A) \) so that the resulting spanning subdigraph \( H = (V,A') \) remains connected.

**Theorem 2.2.3** A spanning subdigraph \( H = (V,A') \) of a connected digraph \( D = (V,A) \) is not its square root if \( |A-A'| > |A|-|V|+1 \).

**Proof:** We note that \( |A-A'| > |A|-|V|+1 \) iff \( |A| - |A'| > |A| - |V|+1 \) which is true iff \( |A'| < |V| - 1 \). Now if \( H \) is a square root of a connected digraph \( D \) then \( H \) is a connected spanning subdigraph of \( D \). Hence \( H \) will contain a spanning ditree of \( D \). Therefore \( |A'| \geq |V| - 1 \) which is a contradiction.

A set \( A^* \subseteq A \) of a digraph \( D = (V,A) \) is said to be forbidden if there exists an arc \( (u,v) \in A^* \) such that in the spanning subdigraph \( (V,A-A^*) \) the distance from \( u \) to \( v \) is strictly greater than two.
For example, in Figure 2.2.4 the sets \((u_4, u_5), (u_3, u_5)\)
\((u_1, u_2)\), and \((u_9, u_2), (u_4, u_5)\) are forbidden where as
\((u_1, u_2), (u_4, u_5)\) is not forbidden.

We observe that super set of a forbidden set is forbidden and
every arc-cutset is forbidden but not every forbidden set is an
arc-cutset. An arc \((u,v)\) of a digraph \(D\) is said to be restored if
after its removal the distance from \(u\) to \(v\) is two and said to be
forbidden otherwise.

**THEOREM 2.2.4** Let \(H = (V, A')\) be a spanning subdigraph of \(D = (V, A)\)
such that \(H^2 \subseteq D\). Then \(H\) is not a square root of \(D\) iff \(A'\) does not
meet at least one forbidden set of arcs of \(D\).

**PROOF:** Let \(H^2 = (V, A'')\) and \(H\) is not a square root of \(D\). Then \(A''\)
is properly contained in \(A\). Therefore there exists an arc \((u,v) \in A\)
which is not in \(A''\), i.e. \(d(u,v) > 2\) in \(H\), so that \((u,v) \in A'\) and
there is no dipath of length two from \(u\) to \(v\) in \(H\). Hence \(A'\) does
not contain an arc \((u,v)\) and the arcs of the dipaths with whose
help it can be restored. But the arc \((u,v)\) and the arcs of each
dipath with whose help it can be restored form a forbidden set in
\(D\). This means that \(A'\) does not meet at least one forbidden set of
arcs of \(D\).
Conversely, suppose $F$ is a forbidden set of arcs of $D$ such that $A' \cap F = \emptyset$ and $H^2 \subseteq D$. As $F$ is a forbidden set in $D$, there exists $(u,v) \in F$ which is not restored in $(V,A-F)$ and $(u,v) \notin A'$. Therefore $(u,v) \notin A''$ and $(u,v) \in F \subseteq A$. Hence $H^2 \neq D$. Thus $H$ is not a square root of $D$. □

**Theorem 2.2.5** If a spanning subdiagraph $H$ of a digraph $D$ contains at least one arc from each of its minimal forbidden set then $D$ is a spanning subdigraph of $H^2$.

**Proof:** Suppose, $D$ is not a spanning subdigraph of $H^2$. There exists an arc $(u,v) \in A(D)$ such that $(u,v) \notin A(CH^2)$. Therefore $(u,v) \notin A(CH)$ and $d(u,v) > 2$ in $H$. Let $F$ be a minimal forbidden set formed by the arc $(u,v)$ and arcs of each dipath of length two from $u$ to $v$ in $A(D) - A(CH)$. Then $A(CH) \cap F = \emptyset$ which is a contradiction. □

Following theorem concerns transitivity of a digraph with a tree square root.

**Theorem 2.2.6** If a digraph $D$ has a tree square root, then every triangle in $D$ is transitive.

**Proof:** Suppose $D = T^2$ for some tree $T$.

**Case 1:** If there is no dipath of length $\geq 2$ in $T$, then $T^2 = T = D$ and trivially every triangle in $D$ is transitive.

**Case 2:** Suppose there exists a dipath of length two in $T$. Say, $a \rightarrow b \rightarrow c$, then $(a,b), (b,c) \in A(T)$. Therefore $(a,b), (b,c)$ and $(a,c) \in A(T^2) = A(D)$ and these form a transitive triangle in $D$. Thus every dipath of length two in $T$ gives rise to a transitive triangle in $D$, i.e. every dipath of length $\geq 2$ in $T$ gives rise to
only transitive triangles in $D$. In any case triangle in $D$ is transitive. 

The converse of Theorem 2.2.6 does not hold. The digraph of Figure 2.2.5 is a counter example.

![Figure 2.2.5](image)

**THEOREM 2.2.7** Let $D$ be a digraph having a square root $K$. Every directed 3-cycle in $K$ is symmetric in $D$.

**PROOF:** Let the triangle $abc$ be a directed cycle say, $a \rightarrow b \rightarrow c \rightarrow a$ in $K$. Then $D$ contains the arcs $(a,b)$, $(b,c)$, $(c,a)$, $(b,a)$, $(c,b)$ and $(a,c)$, i.e. the triangle $abc$ is symmetric in $D$. 

The converse of Theorem 2.2.7 is not true, as shown by Figure 2.2.6.

![Figure 2.2.6](image)

Directed cyclic square roots and tree square roots are always minimal. For this we require the following theorem.
THEOREM 2.2.8 Let $K$ be a square root of a digraph $D$. If $K$ does not contain any transitive triangle then $K$ is minimal.

PROOF: Suppose, $K$ is not a minimal square root of $D$. Then we should be able to remove at least one arc $(u,v)$ from $K$ which is restored by a dipath of length two say $u \rightarrow w \rightarrow v$ in $K$. Then $K$ contains a transitive triangle viz. $uvw$ which is a contradiction. ■

COROLLARY 2.2.9 Tree and directed cyclic square roots are minimal.

Converse of the Theorem 2.2.8 is however not true, as shown by the example of Figure 2.2.7.

![Diagrams](image)