CHAPTER - 4

GENERALIZED CERTAIN
TRIPLE INTEGRAL
EQUATIONS INVOLVING
INVERSE MELLIN
TRANSFORM
GENERALIZED CERTAIN TRIPLE INTEGRAL EQUATIONS INVOLVING INVERSE MELLIN TRANSFORM

4.1 INTRODUCTION

In 1963, D. Naylor introduced the following Mellin type transforms defined by the relations

\[ N_R[f(x); s] = \int_0^R [x^{s-1} - R^{2s}x^{-s-1}] f(x) \, dx \] \hspace{1cm} (4.1.1)

\[ M_R[f(x); s] = \int_0^R [x^{s-1} + R^{2s}x^{-s-1}] f(x) \, dx \] \hspace{1cm} (4.1.2)

\[ G_R[f(x); s] = \int_R^\infty [x^{s-1} - R^{2s}x^{-s-1}] f(x) \, dx \] \hspace{1cm} (4.1.3)

\[ H_R[f(x); s] = \int_R^\infty [x^{s-1} + R^{2s}x^{-s-1}] f(x) \, dx \] \hspace{1cm} (4.1.4)

These transforms are suited to regions bounded by the natural co-ordinate surfaces of a cylindrical or spherical co-ordinate system and apply to finite regions or to infinite regions bounded internally.

Transforms (4.1.1) and (4.1.3) will be effective only in those instances in which the function is prescribed on the internal boundary, while transforms (4.1.2) and (4.1.4) are applicable, if the derivative is given.

Naylor, in his paper [104, 105] and later on, Tweed [160], have proved
the theorem in which they found the inversion formulae of the transforms defined above.

4.1.1 Theorem 1

Let \( y^{\sigma-1} f(y) \in L(0, R) \) for every real number \( \sigma \) such that \( |\sigma| < \sigma \) and let \( f(y) \) be of bounded variation in the neighbourhood of the point \( y = x \in (0, R) \). Let

\[
\overline{f_1}(s) = N_R[f(x); s] = \int_0^R \{ x^{s-1} - R^{2s} x^{-s-1} \} f(x) \, dx,
\]

\( (s = \sigma + ib) \) \hspace{1cm} (4.1.5)

and

\[
\overline{f_2}(s) = M_R[f(x); s] = \int_0^R \{ x^{s-1} + R^{2s} x^{-s-1} \} f(x) \, dx,
\]

\( (s = \sigma + ib) \) \hspace{1cm} (4.1.6)

then

\[
\frac{1}{2} [f(x + 0) + f(x - 0)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \overline{f_1}(s) x^{-s} \, ds, \quad (|\sigma| < \sigma) \hspace{1cm} (4.1.7)
\]

and

\[
\frac{1}{2} [f(x + 0) + f(x - 0)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \overline{f_2}(s) x^{-s} \, ds, \quad (|\sigma| < \sigma) \hspace{1cm} (4.1.8)
\]

respectively are the inversion formula for transforms for transform (4.1.1) and
(4.1.2), if \( f(x) \) satisfies the condition of the theorem.

4.1.2 Theorem 2

Let \( y^{c-1} \ f(y) \in L(R, \infty) \) for every real number \( c \) such that \(|c| < \sigma\) and let \( f(y) \) be of bounded variation in the neighbourhood of the point \( y = x \in (R, \infty) \). Let

\[
\overline{f_3}(s) = G_R[f(x); s] = \int_R^\infty \{x^{s-1} - R^{2s}x^{-s-1}\} f(x) \, dx,
\]

\( (s = c + ib) \) \quad (4.1.9)

and

\[
\overline{f_4}(s) = G_R[f(x); s] = \int_R^\infty \{x^{s-1} - R^{2s}x^{-s-1}\} f(x) \, dx,
\]

\( (s = c + ib) \) \quad (4.1.10)

Then:

\[
\frac{1}{2} \left[ f(x + 0) + f(x - 0) \right] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \overline{f_4}(s) x^{-s} \, ds, \quad (|c| < \sigma)
\]

\( (4.1.11) \)

and

\[
\frac{1}{2} \left[ f(x + 0) + f(x - 0) \right] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \overline{f_4}(s) x^{-s} \, ds, \quad (|c| < \sigma)
\]

\( (4.1.12) \)

respectively are the inversion formulae for transforms (4.1.3) and (4.1.4), provided \( f(x) \) satisfies the condition of the theorem.
In 1972 Tweed [160] found the solution of some dual integral equations involving the inverses of certain Mellin type transforms $M_R$ and $N_R$. Again, the same author [161, 162] solved some triple integral equations involving the inverses of finite Mellin transforms $M_R$ and $H_R$, and illustrated their application in the solution of certain crack problems in the theory of elasticity.

Trivedi and Pandey [157, 158, 159] obtained the solution of dual integral equations involving the inverse of Naylor's Mellin type transforms $M_R$, $N_R$, $H_R$ and $G_R$ different from those investigated by Tweed. Recently, Dwivedi & Chandel [56] gave the solution of triple integral equations involving the inverse Mellin transforms $N_R$ and $G_R$ and also the application of $G_R$ in the solution of crack problems. In the present chapter we are solving two sets of triple integral equations in which first set is involving inverse finite Mellin transform $N_R$ and second set is involving inverse Mellin transforms $G_R$ different from others.

Tweed [162] found the solution of triple equations involving inverse of Mellin type transforms.

$$H_R[f(x), s] = \int_R^\infty \left[ x^{s-1} + R^{2s} x^{-s-1} \right] f(x) \, dx \quad (4.1.13)$$

and gave their application in theory of elasticity. Here we follow his technique to solve our parallel but different problems.
4.2 TRIPLE INTEGRAL EQUATIONS INVOLVING $G_R^{-1}$

4.2.1 The Equations

In this section we shall consider the following set of triple integral equations:

\[ G_R^{-1} [s^{-1}A(s) ; x] = 0, \quad R < x < a \]  \hspace{1cm} (4.2.1)

\[ G_R^{-1} [A(s) \cot \frac{\pi s}{n} ; x] = f(x), \quad a < x < b \] \hspace{1cm} (4.2.2)

\[ G_R^{-1} [s^{-1}A(s) ; x] = 0, \quad b < x < \infty \] \hspace{1cm} (4.2.3)

where $|\text{Re}(s)| < n$ and $n$ is a positive integer.

4.2.2 The Solution

In order to solve these equations let us assume

\[ A(s) = \int_a^b t^{n-s-1} P(t^n) (t^s - R^s)^2 dt \] \hspace{1cm} (4.2.4)

where

\[ \int_a^b P(t^n) t^{n-1} dt = 0 \] \hspace{1cm} (4.2.5)

with this choice of $A(s)$ and making use of the result

\[ G_R[H(t-x); s] = \frac{t^{-s}}{s} \left[ t^s - R^s \right]^2, \quad R < t < \infty, \quad |\text{Re}(s)| < \infty \] \hspace{1cm} (4.2.6)

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We find that

\[ G_R^{-1}[s^{-1} A(s); x] = \int_a^b t^{n-1} P(t^n) G_R^{-1}[s^{-1} t^{-s}(t^s - R^s)^2; x] \, dt \]

\[ = \int_a^b t^{n-1} P(t^n) H(t - x) \, dt \]

\[ = \begin{cases} \int_a^b t^{n-1} P(t^n) \, dt, & a < x < b \\ 0, & \text{otherwise} \end{cases} \quad (4.2.7) \]

(4.2.8)

Equation (4.2.8) shows that equations (4.2.1) and (4.2.3) are satisfied automatically. Similarly on making use of the result

\[ G_R \left[ \frac{R^{2n}}{R^{2n} - x^n t^n} + \frac{t^n}{t^n - x^n}; s \right] = \frac{\pi}{n} \left( R^{2s} t^{-s} + t^s \right) \cot \frac{\pi s}{n}, \]

\[ R < t < \infty, \quad (4.2.9) \]

we get

\[ G_R^{-1}[A(s) \cot \frac{\pi s}{n}, x] = G_R^{-1}\left[ \int_a^b t^{n-1} P(t^n) t^{-s}(t^s - R^s)^2 \, dt \cot \frac{\pi s}{n}; x \right] \]

\[ = \int_a^b t^{n-1} P(t^n) \left[ t^{-s}(t^s - R^s)^2 \cot \frac{\pi s}{n}; x \right] \, dt \]

\[ = \int_a^b t^{n-1} P(t^n) \left[ (t^s - R^{2s} + t^{-s} R^{2s}) \cot \frac{\pi s}{n}; x \right] \, dt \]
\[
\int_a^b t^{n-1} P(t^n) G_R^{-1} \left[ (t^s + t^{-s} R^{2s}) \cot \frac{\pi s}{n}; x \right] dt \\
= \frac{n}{\pi} \int_a^b t^{n-1} P(t^n) \left[ \frac{R^{2n}}{R^{2n} - x^n t^n} + \frac{t^n}{t^n - x^n} \right] dt \\
= \frac{n}{\pi} \int_a^b t^{n-1} P(t^n) \left[ \frac{1}{1 - \left( \frac{x t}{R^2} \right)^n} + \frac{(t/R)^n}{(t/R)^n - (x/R)^n} \right] dt 
\]

So that the equation (4.2.2) will be satisfied if

\[
\frac{n}{\pi} \int_a^b t^{n-1} P(t^n) \left[ \frac{1}{1 - \left( \frac{x t}{R^2} \right)^n} + \frac{(t/R)^n}{(t/R)^n - (x/R)^n} \right] dt = f(x), 
\]

\[a < x < b \quad (4.2.11)\]

Let us assume \( \tau = (t/R)^n, \rho = (x/R)^n \) and also let \( \alpha = (a/R)^n, \beta = (b/R)^n \), then equation (4.2.11) takes the form

\[
\frac{R^n}{\pi} \int_\alpha^\beta P(R^n \tau) \left[ \frac{1}{1 - \rho \tau - \rho} \right] d\tau = f(R \rho^{1/n}), \quad \alpha < \rho < \beta \quad (4.2.12) 
\]

\[
\frac{R^n}{\pi} \int_\alpha^\beta P(R^n \tau) \left[ \frac{1 - \rho \tau + \rho \tau}{1 - \rho \tau + \rho \tau} + \frac{\tau - \rho + \rho}{\tau - \rho} \right] d\tau = f(R \rho^{1/n}) 
\]

\[
\frac{R^n}{\pi} \int_\alpha^\beta P(R^n \tau) \left[ 1 + \frac{\rho \tau}{1 - \rho \tau} + 1 + \frac{\rho}{\tau - \rho} \right] d\tau = f(R \rho^{1/n}) 
\]
\[
\frac{R^n}{\pi} \int_{a}^{\beta} P(R^n \tau) \left[ 2 + \frac{\rho \tau}{1 - \rho \tau} + \frac{\rho}{\tau - \rho} \right] d\tau = f(R \rho^{1/n})
\]

\[
\frac{2}{\pi} R^n \int_{a}^{\beta} P(R^n \tau) d\tau + \frac{R^n}{\pi} \int_{a}^{\beta} P(R^n \tau) \left[ \frac{\rho \tau}{1 - \rho \tau} + \frac{\rho}{\tau - \rho} \right] d\tau = f(R \rho^{1/n})
\]

Now using condition (4.2.5)

\[
\int_{a}^{\beta} P(R^n \tau) d\tau = 0
\]

Therefore

\[
\frac{\rho R^n}{\pi} \int_{a}^{\beta} P(R^n \tau) \left[ \frac{\tau}{1 - \rho \tau} + \frac{1}{\tau - \rho} \right] d\tau = f(R \rho^{1/n})
\]

\[
\frac{R^n}{\pi} \int_{a}^{\beta} P(R^n \tau) \left[ \frac{\tau}{1 - \rho \tau} + \frac{1}{\tau - \rho} \right] d\tau = \rho^{-1} f(R \rho^{1/n})
\] (4.2.13)

Again by taking \( \rho = \rho^{-1} \), the equation (4.2.12) implies that

\[
\frac{R^n}{\pi} \int_{a}^{\beta} P(R^n \tau) \left[ \frac{1}{1 - \rho^{-1} \tau} + \frac{\tau}{\tau - \rho^{-1}} \right] d\tau = f(R \rho^{-1/n})
\]

\[
\frac{R^n}{\pi} \int_{a}^{\beta} P(R^n \tau) \left[ \frac{\rho}{\rho - \tau} + \frac{\rho \tau}{\rho \tau - 1} \right] d\tau = f(R \rho^{-1/n})
\]

\[
- \frac{\rho R^n}{\pi} \int_{a}^{\beta} P(R^n \tau) \left[ \frac{1}{\tau - \rho} + \frac{\tau}{1 - \rho \tau} \right] d\tau = f(R \rho^{-1/n})
\]

\[
\frac{R^n}{\pi} \int_{a}^{\beta} P(R^n \tau) \left[ \frac{\tau}{1 - \rho \tau} + \frac{1}{\tau - \rho} \right] d\tau = -\rho^{-1/n} f(R \rho^{-1/n})
\] (4.2.14)

Therefore equations (4.2.13) and (4.2.14) can be written in the form
\[
\frac{R^n}{\pi} \int_\alpha^\beta P(R^n \tau) \left[ \frac{\tau}{1 - \rho \tau} + \frac{1}{\tau - \rho} \right] d\tau = g(\rho) \tag{4.2.15}
\]

where

\[
g(\rho) = \begin{cases} 
\rho^{-1} f(R\rho^{1/n}) & \alpha < \rho < \beta \\
-\rho^{-1} f(R\rho^{-1/n}) & \beta^{-1} < \rho < \alpha^{-1}
\end{cases} \tag{4.2.16}
\]

By equation (4.2.15)

\[
\frac{R^n}{\pi} \int_\alpha^\beta P(R^n \tau) \frac{1}{1 - \rho \tau} d\tau + \frac{R^n}{\pi} \int_\alpha^\beta P(R^n \tau) \frac{1}{\tau - \rho} d\tau = g(\rho) \tag{4.2.17}
\]

Putting \(\tau = 1/\tau\) in the first integral of the above equation

\[
\frac{R^n}{\pi} \int_{\beta^{-1}}^{\alpha^{-1}} \frac{\tau^{-2}}{\tau - \rho} P(R^n \tau^{-1}) d\tau + \int_\alpha^\beta P(R^n \tau) \frac{1}{\tau - \rho} d\tau = g(\rho),
\]

\((\beta^{-1} < \rho < \alpha^{-1}) \cup (\alpha < \rho < \beta) \quad \tag{4.2.18}\)

If we now let

\[
h(\tau) = \begin{cases} 
R^n \tau^{-2} P(R^n \tau), & \beta^{-1} < \tau < \alpha^{-1} \\
R^n P(R^n \tau), & \alpha < \tau < \beta
\end{cases} \tag{4.2.19}
\]

we find that equation (4.2.18) may be written in the form
\[ \frac{1}{\pi} \int_{\beta^{-1}}^{\beta} \frac{h(\tau)}{(\tau - \rho)} \, d\tau = g(\rho), \]

\[ (\beta^{-1} < \rho < \alpha^{-1}) \cup (\alpha < \rho < \beta) \]  \hspace{1cm} (4.2.20)

where the bar in the symbol \( \int_{\beta^{-1}}^{\beta} \) means that the interval \((\alpha^{-1}, \alpha)\) is excluded from the range of integration.

The equation (4.2.20) is well known Air foil equation and its solution is given by

\[ h(\tau) = \frac{\text{sgn}(2\alpha \tau - 1 - \alpha^2)}{\Delta(\alpha, \beta, \tau)} \left\{ C_0 + C_1 \tau \right\} - \frac{1}{\pi} \int_{\beta^{-1}}^{\beta} \frac{\Delta(\alpha, \beta, \tau) g(r) \, dr}{\text{sgn}(2\alpha r - 1 - \alpha^2) (r - \tau)} \]  \hspace{1cm} (4.2.21)

where \( C_0 \) and \( C_1 \) are arbitrary constants and

\[ \Delta(\alpha, \beta, \tau) = \left[ (\beta - \tau)(\tau - \alpha)(\tau - \alpha^{-1})(\tau - \beta^{-1}) \right]^{1/2} \]  \hspace{1cm} (4.2.22)

since \( h(\tau^{-1}) = \tau^2 h(\tau) \) and \( g(\rho^{-1}) = -\rho^2 g(\rho) \), we see that

\[ h(\tau^{-1}) = \frac{\text{sgn}(2\alpha \tau^{-1} - 1 - \alpha^2)}{\Delta(\alpha, \beta, \tau^{-1})} \left\{ C_0 + C_1 \tau^{-1} \right\} - \frac{1}{\pi} \int_{\beta^{-1}}^{\beta} \frac{\Delta(\alpha, \beta, \tau) g(r) \, dr}{\text{sgn}(2\alpha r - 1 - \alpha^2) (r - \tau^{-1})} \]  \hspace{1cm} (4.2.23)

Taking \( \text{sgn}(2\alpha \tau^{-1} - 1 - \alpha^2) = -\text{sgn}(2\alpha \tau^{-1} - 1 - \alpha^2) \)
\[ \Delta (\alpha, \beta, \tau^{-1}) = \left[ (\beta - \tau^{-1}) (\tau^{-1} - \alpha) (\tau^{-1} - \alpha^{-1}) (\tau^{-1} - \beta^{-1}) \right]^{1/2} \]

\[ = \frac{1}{\tau^2} \left[ (\tau - \beta^{-1}) (\alpha^{-1} - \tau) (\alpha - \tau) (\beta - \tau) \right]^{1/2} \]

\[ = \frac{1}{\tau^2} \left[ (\beta - \tau) (\tau - \alpha) (\tau - \alpha^{-1}) (\tau - \beta^{-1}) \right]^{1/2} \]

\[ = \frac{\Delta (\alpha, \beta, \tau)}{\tau^2} \quad (4.2.24) \]

and putting \( r = 1/r \) in the integral of the equation (4.2.22).

\[ \tau^2 h(\tau) = -\frac{\text{sgn} (2\alpha \tau - 1 - \alpha^2)}{\Delta (\alpha, \beta, \tau)} \left\{ C_0 + C_1 \tau^{-1} \right. \]

\[ + \frac{1}{\tau^2} \left. \int_{\beta^{-1}}^{\beta} \frac{\Delta (\alpha, \beta, r) g(r^{-1}) \, dr}{\text{sgn} (2\alpha r^{-1} - 1 - \alpha^2) (r^{-1} - \tau^{-1})} \right\} \]

or

\[ \left\{ C_0 + C_1 \tau^{-1} - \frac{1}{\pi} \int_{\beta^{-1}}^{\beta} \frac{\Delta (\alpha, \beta, r) g(r) \, dr}{\text{sgn} (2\alpha r - 1 - \alpha^2) (r - \tau)} \right\} \]

\[ - \left\{ C_0 + C_1 \tau^{-1} + \frac{\pi}{\tau} \int_{\beta^{-1}}^{\beta} \frac{\Delta (\alpha, \beta, r) g(r) \, dr}{\text{sgn} (2\alpha r - 1 - \alpha^2) (r - \tau) r} \right\} \]

or

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\[ C_0 + C_0 + C_1 \tau + C_1 \tau^{-1} = \frac{1}{\pi} \left[ \int_{\beta^{-1}}^{\beta} \Delta(\alpha, \beta, r) g(r) \frac{\text{sgn}(2\alpha r - 1 - \alpha^2)(r - \tau)}{\tau - \frac{\tau}{r}} \right] \]

or

\[ C_0 = \frac{1}{2\pi} \int_{\beta^{-1}}^{\beta} \frac{\Delta(\alpha, \beta, r) g(r)dr}{\text{sgn}(2\alpha r - 1 - \alpha^2) r} - C_1 \frac{(\tau^2 + 1)}{2\tau} \quad (4.2.25) \]

Substituting the value of \( C_0 \) in equation (4.2.21) we get

\[ h(\tau) = \frac{\text{sgn}(2\alpha \tau - 1 - \alpha^2)}{\Delta(\alpha, \beta, \tau)} \left[ \frac{1}{2\pi} \int_{\beta^{-1}}^{\beta} \frac{\Delta(\alpha, \beta, r) g(r)dr}{\text{sgn}(2\alpha r - 1 - \alpha^2) r} \right. \]

\[ - C_1 \frac{(\tau^2 + 1)}{2\tau} + C_1 \tau - \frac{1}{\pi} \int_{\beta^{-1}}^{\beta} \frac{\Delta(\alpha, \beta, r) g(r)dr}{\text{sgn}(2\alpha r - 1 - \alpha^2)(r - \tau)} \left. \right] \]

\[ = \frac{\text{sgn}(2\alpha \tau - 1 - \alpha^2)}{\Delta(\alpha, \beta, \tau)} \left[ C_1 \left( \frac{\tau^2 - 1}{2\tau} \right) \right. \]

\[ - \frac{1}{2\pi} \int_{\beta^{-1}}^{\beta} \frac{\Delta(\alpha, \beta, r) g(r)(r + \tau)dr}{\text{sgn}(2\alpha r - 1 - \alpha^2)(r - \tau)} \right] \quad (4.2.26) \]

But condition (4.2.5) implies that

\[ \int_{\alpha}^{\beta} h(\tau) d\tau = 0 \quad (4.2.27) \]

Therefore.
\[ C_1 \int_{\alpha}^{\beta} \frac{\tau^2 - 1}{2} \Delta(\alpha, \beta, \tau) \, d\tau \]

\[ = \frac{1}{2\pi} \int_{\alpha}^{\beta} \frac{d\tau}{\Delta(\alpha, \beta, \tau)} \int_{\beta}^{\alpha} \frac{\Delta(\alpha, \beta, r) \, g(r) \, (r + \tau) \, dr}{\text{sgn}(2\alpha r - 1 - \alpha^2) \, r \, (r - \tau)} \]

\[ = \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{d\tau}{\Delta(\alpha, \beta, \tau)} \left[ \int_{\beta}^{\alpha} \frac{\Delta(\alpha, \beta, r) \, g(r) \, (r + \tau) \, dr}{\text{sgn}(2\alpha r - 1 - \alpha^2) \, r \, (r - \tau)} \right] \]

\[ + \int_{\alpha}^{\beta} \frac{\Delta(\alpha, \beta, r) \, g(r) \, (r + \tau) \, dr}{\text{sgn}(2\alpha r - 1 - \alpha^2) \, r \, (r - \tau)} \]

\[ = \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{d\tau}{\Delta(\alpha, \beta, \tau)} \left[ -\int_{\beta}^{\alpha} \frac{\Delta(\alpha, \beta, r) \, f(r^{-\frac{1}{n}} R) \, (r + \tau) \, dr}{(-r) \, r \, (r - \tau)} \right] \]

\[ + \int_{\alpha}^{\beta} \frac{\Delta(\alpha, \beta, r) \, f(r^\frac{1}{n} R) \, (r + \tau) \, dr}{r \, r \, (r - \tau)} \]

\[ = \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{d\tau}{\Delta(\alpha, \beta, \tau)} \left[ \int_{\alpha}^{\beta} \frac{\Delta(\alpha, \beta, r^{-1}) \, f(R r^{-\frac{1}{n}}) \, (r^{-1} + \tau) \, dr}{r^{-2} \, (r^{-1} - \tau) \, (-r^{-2})} \right] \]

\[ + \int_{\alpha}^{\beta} \frac{\Delta(\alpha, \beta, r) \, f(R r^\frac{1}{n}) \, (r + \tau) \, dr}{r^2 \, (r - \tau)} \]

\[ = \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{d\tau}{\Delta(\alpha, \beta, \tau)} \left[ \int_{\alpha}^{\beta} \frac{\Delta(\alpha, \beta, r) \, f(R r^\frac{1}{n}) \, (1 + \tau r) \, dr}{r^2 \, (1 - r \tau)} \right] \]

\[ + \int_{\alpha}^{\beta} \frac{\Delta(\alpha, \beta, r) \, f(R r^\frac{1}{n}) \, (r + \tau) \, dr}{r^2 \, (r - \tau)} \]
\[
C_1 = \frac{2}{\pi} \int_\alpha^\beta \frac{d\tau}{\Delta(\alpha, \beta, \tau)} \int_\alpha^\beta \frac{\Delta(\alpha, \beta, r) f(Rr^{1/n})}{r^2 (1-\tau^2) (r-\tau)} d\tau
\]

or

\[
C_1 = \frac{2}{\pi} \int_\alpha^\beta \frac{(1-\tau^2) d\tau}{\Delta(\alpha, \beta, \tau)} \int_\alpha^\beta \frac{\Delta(\alpha, \beta, r) f(Rr^{1/n})}{r (r-\tau) (1-r\tau)} d\tau
\]

Substituting the value of \(C_1\) in equation (4.2.26), we get

\[
h(\tau) = \frac{\text{sgn}(2\alpha \tau - 1 - \alpha^2)}{\Delta(\alpha, \beta, \tau)}
\]

\[
= \frac{2(\tau^2 - 1)}{2\pi \tau} \int_\alpha^\beta \frac{(1-\xi^2) d\xi}{\Delta(\alpha, \beta, \xi)} \int_\alpha^\beta \frac{\Delta(\alpha, \beta, r) f(Rr^{1/n})}{r (r-\xi) (1-r\xi)} d\xi
\]

\[
\int_\alpha^\beta \frac{\xi - \xi^{-1}}{\Delta(\alpha, \beta, \xi)} d\xi
\]
\[
- \frac{1}{2\pi} \int_{\beta^{-1}}^{\beta} \frac{\Delta (\alpha, \beta, r) g(r) (r + \tau) \, dr}{\text{sgn} (2\alpha r - 1 - \alpha^2) \, r (r - \tau)}
\]

or

\[
h(\tau) = -\frac{\text{sgn} (2\alpha \tau - 1 - \alpha^2)}{\pi \Delta (\alpha, \beta, \tau)}
\]

\[
\left[ \frac{(1 - \tau^2)}{\tau} \int_{\alpha}^{\beta} \frac{(1 - \xi^2)}{\Delta (\alpha, \beta, \xi)} \, d\xi \right] \int_{\alpha}^{\beta} \frac{\Delta (\alpha, \beta, r) f(Rr^{1/n}) \, dr}{r (r - \xi) (1 - r\xi)}
\]

\[
+ \int_{\alpha}^{\beta} \frac{(1 - \tau^2) \Delta (\alpha, \beta, r) f(Rr^{1/n}) \, dr}{r (r - \tau) (1 - r\tau)}
\]  \quad (4.2.29)

Let us assume \( \tau \) lies between \( \alpha \) and \( \beta \), therefore,

\[
P(R^\alpha \tau) = \frac{-1}{\pi \Delta (\alpha, \beta, r) R^\alpha}
\]

\[
\left[ \frac{(1 - \tau^2)}{\tau} \int_{\alpha}^{\beta} \frac{(1 - \xi^2)}{\Delta (\alpha, \beta, \xi)} \, d\xi \right] \int_{\alpha}^{\beta} \frac{\Delta (\alpha, \beta, r) f(Rr^{1/n}) \, dr}{r (r - \xi) (1 - r\xi)}
\]

\[
+ \int_{\alpha}^{\beta} \frac{(1 - \tau^2) \Delta (\alpha, \beta, r) f(Rr^{1/n}) \, dr}{r (r - \tau) (1 - r\tau)}
\]  \quad (4.2.30)

Now changing back to the original variables and taking
\[ D(t^n) = \left[(b^n - t^n)(t^n - a^n)(a^n t^n - R^{2n})(b^n t^n - R^{2n})\right]^{1/2} \quad (4.2.31) \]

We have

\[
P(t^n) = \frac{(a^n b^n)^{1/2} R^{2n}}{\pi D(t^n) R^n} \]

\[
\begin{bmatrix}
\frac{(R^{2n} - t^{2n}) R^n}{t^n R^n} \int_a^b R^{2n} (a^n b^n)^{1/2} n t^{n-1} dt \\
\int_a^b \frac{(R^{2n} - t^{2n}) R^n R^{2n} (a^n b^n)^{1/2} n t^{n-1} dt}{t^n R^{2n} D(t^n) R^n}
\end{bmatrix}
\]

\[
\int_a^b \frac{(R^{2n} - t^{2n}) D(x^n) f(x) n x^{n-1} R^{2n} R^{2n}}{R^{2n} R^{2n} (a^n b^n)^{1/2} R^{2n} (x^n - t^n) (R^{2n} - x^n t^n) x^n} dx
\]

\[
+ \int_a^b \frac{(R^{2n} - t^{2n}) D(x^n) f(x) n x^{n-1} R^{2n} R^{2n}}{R^{2n} R^{2n} (a^n b^n)^{1/2} R^{2n} (x^n - t^n) (R^{2n} - x^n t^n) x^n} dx
\]

or

\[
P(t^n) = \frac{1}{\pi D(t^n)}
\]

\[
\frac{R^{2n} - t^{2n}}{t^n} \int_a^b \frac{(R^{2n} - t^{2n}) t^{n-1} dt}{D(t^n)} \int_a^b \frac{D(x^n) f(x) dx}{(x^n - t^n) (R^{2n} - x^n t^n) x^n} \]

\[
\frac{1}{t} \int_a^b \frac{(R^{2n} - t^{2n}) dt}{D(t^n)}
\]

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\[ + \int_{a}^{b} \frac{(R^{2n} - t^{2n}) D(x^n) f(x) \, dx}{(x^n - t^n) \left( R^{2n} - x^n t^n \right)} \]

or

\[ P(t^n) = \frac{-n}{\pi D(t^n)} \left[ \frac{R^{2n} - t^{2n}}{t^n} \frac{I}{J} + K \right] \]

where

\[ I = \int_{a}^{b} \frac{t^{n-1}}{D(t^n)} \int_{a}^{b} \frac{(R^{2n} - t^{2n}) D(x^n) f(x) \, dx \, dt}{(x^n - t^n) \left( R^{2n} - x^n t^n \right)} \]

\[ J = \int_{a}^{b} \frac{(R^{2n} - t^{2n})}{t D(t^n)} \, dt \]

\[ K = \int_{a}^{b} \frac{(R^{2n} - t^{2n}) D(x^n) f(x) \, dx}{(x^n - t^n) \left( R^{2n} - x^n t^n \right)} \]

Equation (4.2.33) together with equation (4.2.4) yields \( A(s) \).

### 4.2.3 Particular Case

If we put \( a = R \) then the set of equations (4.2.1) to (4.2.3) reduce to dual integral equations considered by Trivedi and Pandey [158] and our solution (4.2.33) reduces to that given by them.