CHAPTER - 3

GENERALIZED DUAL INTEGRAL EQUATIONS WITH SPECIAL FUNCTION KERNELS
3.1 INTRODUCTION

In the solution of certain mixed boundary-value problems of mathematical physics, in particular, of boundary-value problems for piecewise-nonhomogeneous media [122], dual and triple integral equations etc. with principally different kernels arise.

In the present chapter, we consider the case of dual integral equations when one of the equations contains the generalized Legendre function of first kind \( P_{-1/2+i\tau}^{m,n} (\cosh \alpha) \) as kernel and the second equation contains the trigonometric function \( \cos \alpha \tau \) or \( \sin \alpha \tau \) as kernel. The method used is a generalization of that of Tranter [156a].

3.2 THE PROBLEM

we solve the dual integral equations

\[
\int_{0}^{\infty} f(\tau) P_{-1/2+i\tau}^{m,n} (\cosh \alpha) d\tau = \psi_1 (\alpha), \quad 0 \leq \alpha < a, \quad (3.2.1)
\]

\[
\int_{0}^{\infty} f(\tau) \cos \alpha \tau d\tau = \psi_2 (\alpha), \quad a < \alpha < +\infty, \quad (3.2.2)
\]
Here $f(\tau)$ is the unknown function, $\psi_1(\alpha)$ and $\psi_2(\alpha)$ are known functions such that $\psi_1(\alpha) = 0(1/\alpha q)$ for $\alpha \to 0$, $q < 3/2$, $|m| < \frac{1}{2}$, $m < n < 3/2$, and $P_k^{m,n}(z)$ is the generalized associated Legendre function of first kind, i.e., one of the two linearly independent solutions of the generalized Legendre equation.

\[
(1-z^2)\frac{d^2 u}{dz^2} - 2z \frac{du}{dz} + \left[ k(k+1) - \frac{m^2}{2(1-z)} - \frac{n^2}{2(1+z)} \right] u = 0.
\]

### 3.3 RESULTS REQUIRED

Certain auxiliary results are required to solve above dual integral equations (3.3.1) and (3.3.2). Using the formula [61]

\[
F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \int_0^1 t^{c-1}(1-t)^{a-1} (1-tz)^b (1-tz)^{c-a-b} \ dx dt
\]

\[
F(a',b';c'-\lambda;z) = \frac{\Gamma(c')}{\Gamma(a')\Gamma(b')\Gamma(c'-\lambda-a')\Gamma(c'-\lambda-b')} \int_0^1 t^{c'\lambda-1}(1-t)^{a'-\lambda} (1-tz)^b (1-tz)^{c'-\lambda-b'} \ dx dt,
\]

\[
Re \ c > Re \ \lambda > 0, |arg(1-z)| < \pi,
\]

We establish the following integral representation for $P_k^{m,n}(\cosh \alpha)$:

\[
P_k^{m,n}(\cosh \alpha) = \frac{2^{(n-m+1)/2} \sinh^m \alpha \cosh^n \alpha}{\sqrt{\alpha \Gamma(1/2-m)}} \int_0^\infty \frac{\cosh(k+1/2)\varphi}{(\cosh \alpha - \cosh \varphi)^{m+1/2}} \ F\left(\frac{n-m}{2}, -\frac{m+n}{2}; -\frac{1}{2} - m, -\frac{1}{1+\cosh \alpha} \right) \ d\varphi
\]

(3.3.2)

In particular, when $m = n = \mu$ and $k = \nu$, we get the well-known formula for the Legendre function.

For $k = -1/2 + it$, from (3.3.2) we have
\[ P_{m,n}^{\alpha}(\cosh \alpha) = \frac{2^{(n-m+1)/2}}{\sqrt{\pi \Gamma(1/2-m)}} \frac{\sinh^{m} \alpha \cos \tau \varphi}{(\cosh \alpha - \cosh \varphi)^{m+1/2}} \int_{0}^{\infty} \left( \frac{\cosh \alpha - \cosh \varphi}{1 + \cosh \alpha} \right)^{m+1/2} d\varphi \] (3.3.3)

To find \( \cos \tau \varphi \) from (3.3.3), we use the solution of an integral equation of the form

\[ \int_{a}^{x} \phi(t)[\omega(x) - \phi(t)]^{-\ell} F\left( p, q, r; \frac{\omega(x) - \psi(t)}{\omega(x) + d} \right) dt = \Phi(x) \] (3.3.4)

for \( \ell = m+1/2, p = (n-m)/2, r = \frac{1}{2} - m, a = 0, d = 1, \omega(x) = \cosh x, \psi(t) = \cosh t: \)

\[ \phi(x) = \frac{\Gamma^{-1}(1/2-m)}{\Gamma(1/2+m)} \frac{d}{dx} \left( \frac{[\omega(x) + d]^{(n-m)/2}}{\int_{a}^{x} [\omega(x) - \psi(t)]^{m+1/2}} \right) \left[ \frac{\omega(x) + d}{m-n/2} \right]^{(m-n)/2} \]

\[ \left( \frac{m-n}{2}, \frac{1+m-n}{2} ; \frac{1}{2} + m; \frac{\omega(x) - \psi(t)}{\omega(x) + d} \right) \Phi(t) \omega'(t) dt \] (3.3.5)

For \( \cos \tau \varphi \), we have

\[ \cos \tau \varphi = \frac{\Gamma(1/2-m) \cos \pi m}{\sqrt{\pi \Gamma(1/2-m+1/2)}} \frac{d}{d\varphi} \left( \frac{(\cosh \varphi + 1)^{(n-m)/2}}{\int_{0}^{\varphi} (\cosh \varphi - \cosh \alpha)^{m+1/2}} \right) \\
(\cosh \alpha + 1)^{(m-n)/2} \left( \frac{m-n}{2}, \frac{1+m-n}{2} ; \frac{1}{2} + m; \frac{(\cosh \varphi - \cosh \alpha)}{1 + \cosh \varphi} \right) \] (3.3.6)

\[ P_{m,n}^{\cosh \alpha}(\cosh \alpha \sinh \alpha^{1-m} \alpha d\alpha} \]

### 3.4 THE SOLUTION

We now find the solution of dual integral equations (3.2.1), (3.2.2).

After multiplying (3.2.1) by
\[
\frac{2^{(m-n)/2}}{\pi} \Gamma \left( \frac{1}{2} - m \right) \cos \pi m \sinh^{1-m} \alpha (\cosh \varphi - \cosh \alpha)^{m-1/2} \\
(cosh \varphi + 1)^{(n-m)/2} (cosh \alpha + 1)^{(m-n)/2} \Gamma \left( \frac{m-n}{2}, \frac{1 + m - n}{2}, \frac{1}{2}, \frac{1}{1 + \cosh \varphi} \right)
\]

integrating with respect to \( \alpha \) from 0 to \( \varphi \), and differentiating with respect to \( \varphi \), we get

\[
2^{(m-n)/2} \Gamma \left( \frac{1}{2} - m \right) \cos \pi m \int_0^\infty f(t) dt \int_0^\varphi \frac{\sinh^{1-m} \alpha \psi_1(\varphi) (cosh \alpha + 1)^{(m-n)/2}}{(cosh \varphi - \cosh \alpha)^{1/2-m}} \right) d\alpha = \psi_1(\varphi), \quad 0 \leq \varphi \leq a
\]

where

\[
\psi_1(\varphi) = \frac{2^{(m-n)/2} \Gamma \left( \frac{1}{2} - m \right)}{\pi} \cos \pi m \frac{d}{d\varphi} \left( (cosh \varphi + 1)^{(n-m)/2} \right) \\
\int_0^\varphi \frac{\sinh^{1-m} \alpha \psi_1(\alpha)}{(cosh \varphi - \cosh \alpha)^{1/2-m}} (cosh \alpha + 1)^{(m-n)/2} \right) d\alpha
\]

By virtue of (3.3.6), we can rewrite (3.4.2) in the form

\[
F_c[f(t)] = \psi_1(\varphi), \quad 0 \leq \varphi \leq a
\]

Equation (3.2.2) can be rewritten in the form
where $\psi_2(\phi) = \sqrt{\frac{2}{\pi}} \psi_2$ and $F_c$ is the Fourier cosine transform. Consequently, we have

$$F_c[f(\tau)] = \begin{cases} 
\psi_1(\phi), & 0 \leq \phi \leq a, \\
\psi_2(\phi), & \phi > a.
\end{cases}$$

Using the inversion formula for the Fourier cosine transform, we get the solution of the dual integral equations (3.2.1), (3.2.2)

$$f(\tau) = \sqrt{\frac{2}{\pi}} \left[ \int_0^a \psi_1(\phi) \cos \tau \phi d\phi + \int_a^\infty \psi_2(\phi) \cos \tau \phi d\phi \right]$$

(3.4.6)

### 3.5 Further Equations

Next we consider the dual integral equations

$$\int_0^\infty f(\tau) P_{-1/2+it}^{m,n}(\cosh \alpha) d\tau = h_1(\alpha) \quad 0 \leq \alpha \leq a,$$

(3.5.1)

$$\int_0^\infty f(\tau) \sin \tau \alpha d\tau = h_2(\alpha) \quad \alpha > a.$$  

(16)

Multiplying (3.5.1) by the expression (3.4.1) and integrating with respect to $\alpha$ from 0 to $\phi$, we get
\[
\frac{2^{(m-n)/2}}{\pi} \Gamma \left( \frac{1}{2} - m \right) \cos \pi m \int_0^\infty \tau f(\tau)(\cosh \varphi + 1)^{(n-m)/2} d\tau
\]
\[
\int_0^{\varphi} \frac{h_1(\alpha) \sinh^{1-m} \alpha (\cosh \varphi - \cosh \alpha)^{m-1/2} (\cosh \alpha + 1)^{(m-n)/2}}{1 + \cosh \varphi} d\alpha
\]
\[
\mathcal{F}\left( \frac{m-n}{2}, \frac{1+m-n}{2}, \frac{1}{2} + m, \frac{\cosh \varphi - \cosh \alpha}{1 + \cosh \varphi} \right) d\alpha = M_1(\varphi) \quad (3.5.3)
\]

where

\[
M_1(\varphi) = \frac{2^{(m-n)/2}}{\pi} \Gamma \left( \frac{1}{2} - m \right) \cos \pi m (\cosh \varphi + 1)^{(n-m)/2}
\]
\[
\int_0^\varphi h_1(\alpha) \sinh^{1-m} \alpha (\cosh \varphi - \cosh \alpha)^{m-1/2} (\cosh \alpha + 1)^{(m-n)/2} \quad (3.5.4)
\]
\[
\times \mathcal{F}\left( \frac{m-n}{2}, \frac{1+m-n}{2}, \frac{1}{2} + m, \frac{\cosh \varphi - \cosh \alpha}{1 + \cosh \varphi} \right) d\alpha
\]

Integrating (3.5.6) with respect to \( \varphi \), and using (3.5.3), we get

\[
\mathcal{F}_s[f(\tau)] = M_1(\varphi), \quad 0 \leq \varphi \leq a \quad (3.5.5)
\]

Here \( \mathcal{F}_s \) is the Fourier sine transform.

Equation (3.5.2) can be rewritten in the form

\[
\mathcal{F}_s[f(\tau)] = M_2(\varphi) \quad \varphi > a \quad (3.5.6)
\]

where \( M_2(\varphi) = (2/\pi)^{1/2} h_2 \).

Using the inversion formula for the Fourier sine transform, we write the final solution of the dual integral equations (3.5.1), (3.5.2) in the form
\[ f(\tau) = \left( \frac{2}{\pi} \right)^{1/2} \left[ \int_{0}^{a} M_1(\phi) \sin \tau \phi d\phi + \int_{a}^{\infty} M_2(\phi) \sin \tau \phi d\phi \right] \] (3.5.7)

### 3.6 THE MORE GENERAL EQUATIONS

Now we consider dual integral equations of the more general form

\[ \int_{0}^{\infty} f(\tau)[1 + G(\tau)] P_{-1/2+ir}^{m,n}(\cosh \alpha) d\tau = \psi_1(\alpha) \quad 0 \leq \alpha \leq a \] (3.6.1)

\[ \int_{0}^{\infty} f(\tau) \cos \tau \alpha d\tau = \psi_2(\alpha) \quad \alpha \leq a \] (3.6.2)

where the functions \( G(\tau), \psi_1(\alpha) \) and \( \psi_2(\alpha) \) are known and \( f(\tau) \) is, as before, the unknown function.

Using (3.4.6), the solution of eqs. (3.6.1), (3.6.2) can be reduced to that of a Fredholm equation of second kind. We rewrite eq. (3.6.1) in the form

\[ \int_{0}^{\infty} f(\tau) P_{-1/2+ir}^{m,n}(\cosh \alpha) d\tau = \psi_1(\alpha) - \int_{0}^{\infty} f(\tau) G(\tau) P_{-1/2+ir}^{m,n}(\cosh \alpha) d\tau \] (3.6.3)

Now we have obtained dual integral equations of form (3.2.1), (3.2.2).

We write down the solution by the formula (3.4.6). In this case,

\[ \psi_2 = (2/\pi)^{1/2} \psi_2 \] and
\[
\psi_1(\phi) = \frac{2^{(m-n)/2}}{\pi} \int \frac{1}{2} \frac{\cos \pi m}{d\phi} \left( \cosh \phi + 1 \right)^{(n-m)/2} \frac{\sinh^{1-m} \alpha}{\alpha} F_0^\infty \frac{(\cosh \alpha + 1)^{(m-n)/2}}{(\cosh \phi - \cosh \alpha)^{(1/2-m)}} \left( \frac{m-n}{2}, \frac{1+m-n}{2}, \frac{1}{2}, \cosh \phi - \cosh \alpha \right) \left[ \psi_1(\alpha) - \int_0^\infty f(\tau) G(\tau) \frac{\sinh^{1-m} \alpha}{\cosh \alpha} d\tau \right] d\alpha
\]

Thus, the solution of the dual integral equations (3.6.1), (3.6.2) has been reduced to the solution of a Fredholm integral equation of second kind for the function \(\psi_1(\phi)\), which can be written in the form

\[
\psi_1(\phi) + \int_0^a \psi_1(u) K(\phi, u) du = M(\phi) \quad 0 \leq \phi \leq a
\]

(3.6.5)

where

\[
K(\phi, u) = (2\pi)^{-1/2} \left\{ G_c(\phi + u) + G_c(\phi - u) \right\}, \quad G_c = F_c \left[ G(\tau) \right], \quad M(\phi) = \psi_3(\phi) - N_2(\phi),
\]

\[
N_2(\phi) = F_c \left[ N_1(\tau) G(\tau) \right] \quad \text{and} \quad N_1(\tau) = \sqrt{\frac{2}{\pi}} \int_0^\infty \psi_2(\phi) \cos \tau \phi d\phi.
\]