CHAPTER - 2

LITERATURE SURVEY
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This chapter consists of the historical development of the work done in solving different types of integral and series equations and mixed boundary value problems of elasticity.

2.1 INTEGRAL EQUATIONS

2.1.1 Dual Integral Equations

In the following lines we describe the dual integral equations with different kernels.

(i) Bessel Function Kernel

After the publication of the book "An introduction to the Fourier Integrals" by Titchmarsh [153], research workers actually took interest on dual integral equations. In this text, the following dual integral equations involving Bessel functions were considered

\[
\int_{0}^{\infty} u^{\alpha}[1 + H(u)] \psi(u) J_{\nu}(xu) \, du = f(x), \quad 0 < x < 1, \tag{2.1.1}
\]

\[
\int_{0}^{\infty} \psi(u) J_{\nu}(xu) \, du = g(x), \quad 1 < x < \infty, \tag{2.1.2}
\]

with \( H(u) = g(x) = 0 \) and \( \alpha = -1 \).

The range of the solution of the above equations was extended by Miss
Busbridge [7]. The solution of the above equations by the integral equation method was given by Tranter [154]. He assumed the suitable integral representation for the unknown function in terms of another unknown function and obtained the solution in the form of Fredholm integral equation which can be solved numerically. Gorden [70] obtained the solution of the above equations by using certain Sonine's discontinuous integrals. Peters [119] solved the more general equations

\[ \int_0^\infty t^a \phi(t) J_v(xt) \, dt = f(x), \quad 0 < x < 1, \quad (2.1.3) \]

\[ \int_k^\infty (t^2 - k^2)^\beta \phi(t) J_v(xt) \, dt = g(x), \quad 1 < x < \infty, \quad k > 0, \quad (2.1.4) \]

The multiplying factor method for solving the above equations was given by Noble [109]. Dwivedi [21] obtained the solution of some equations by applying the same method.

Some dual integral equations and simultaneous equations were considered by Fan [68] and solved them by transferring the equation to general system of functional equations in the complex domain. He also mentioned the applications of these equations to solid and fluid mechanics.


Srivastav and Parihar [150] applied generalized function concept to
obtain the solution of certain dual integral equations. Further Srivastava [151] considered dual integral equations with trigonometric kernels and tempered distribution. He also introduced $L_2$- theory for solving trigonometric equations.

Tranter [154] gave the solution of dual integral equations with Bessel function of zeroth order. An elegant method for solving the dual integral equations with Bessel function of zeroth order was considered by Sneddon [137] and reduced them to an Abel-Schlömilch type of integral equation whose solution was well known.

The dual integral equations connected with Fourier transforms were considered by Nguyen Van Ngok and Popov [106] and solved them by applying the method of successive approximations.

Chakrabarti [8] obtained the three different sets of dual integral equations involving Bessel functions of first kind and of order one.

A method to reduce the dual integral equations into an infinite algebraic system was given by Aizikovich [5]. The kernels of these equations are the eigen functions of a Sturm-Liouville problem for a second order equation in terms of a parameter, which is naturally small.

Cherskii [10] has considered a multidimensional dual equations of convolution type
\[ u(x) = \int_{\Omega^+ \cup \Omega^-} k_1(x-s) u(s) \, ds = g(x), \quad (x \in \Omega_+) \]  \hspace{1cm} (2.1.5)
\[ u(x) = \int_{\Omega^+ \cup \Omega^-} k_2(x-s) u(s) \, ds = g(x), \quad (x \in \Omega_-) \]  \hspace{1cm} (2.1.6)

where \( k \) is defined on the space \( L(M^n_m) \) of complex valued functions defined on \( M^n_m \) with finite norms and \( M^n_m (0 \leq m \leq n) \) is the set of points \( (x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n) \) whose first \( m \) co-ordinates are real, while the others are integers. Using the factorization method the system is solved by quadratures. Moreover, \( k \) is considered as an absolutely integrable function, vanishing in the exterior of a set \( W \).

An elementary procedure based on Sonine's integrals is used by Mandal [91] to reduce dual integral equations with Bessel functions of different orders as kernels and an arbitrary weight function, into a Fredholm integral equation of the second kind. The result obtained here is more general in nature and includes many results, concerning dual integral equations with Bessel functions as kernels, known in the literature.

In the consideration of the three dimensional problems of the theory of elasticity with mixed boundary condition with circular lines of the boundary conditions, dual integral equations arise. Such types of dual integral equations considered by Abramyan [1], are

\[ \int_0^\infty \beta^\alpha \psi(\beta) J_m(\beta r) \, d\beta = f(r), \quad 0 < r < a \]  \hspace{1cm} (2.1.7)
\[ \int_0^\infty \psi(\beta) J_m(\beta r) \, d\beta = g(r), \quad r > a \]  \hspace{1cm} (2.1.8)
where \( \alpha = \pm 1 \), \( f(r) \) and \( g(r) \) are known functions in the prescribed intervals. \( \psi(\beta) \) is the function to be determined and \( m = 0, 1, \ldots \). He solved these equations by the method of orthogonalisation of equations in a shorter way.

Rahman [123] found an effective polynomial solution to a class of dual integral equations which arise in many mixed boundary value problems in the theory of elasticity. The dual integral equations are first transformed into a Fredholm integral equation of the second kind via an auxiliary function, which is next reduced to an infinite system of linear algebraic equations by representing the unknown auxiliary function in the form of an infinite series of Jacobi polynomials. The approximate solution of this infinite system of equations can be obtained by a suitable truncation. It is shown that the unknown function involving the dual integral equations can also be expressed in the form of an infinite series of Jacobi polynomials with the same expansion coefficient with no numerical integration involved. The main advantage of the present approach is that the solution of the dual integral equations thus obtained is numerically more stable than that obtained by reducing them directly to an infinite system of equations, insofar as the expansion coefficients are determined essentially by solving a second kind integral equation.

(ii) Trigonometric Kernels

The general form of dual integral equation with trigonometric kernels is as below:
\begin{align}
\int_0^\infty u^\alpha \psi(u) \sin(xu) \, du &= f(x), \quad 0 < x < 1 \tag{2.1.9} \\
\int_0^\infty \psi(u) \cos(xu) \, du &= g(x), \quad x > 1 \tag{2.1.10}
\end{align}

where \(\psi(u)\) is unknown function and \(f(x)\) and \(g(x)\) are prescribed functions.

Titchmarsh [153] solved above equations by taking \(\alpha = \pm 1/2\). These equations are also solved by Sneddon [137] by elementary method.

The solution of above equations was obtained by Dwivedi [20] for general values of \(\alpha\) with various possible combinations of trigonometric functions.

Singh and Dhaliwal [135] considered the following set of dual integral equations with trigonometric kernels

\begin{align}
\int_0^\infty \left[1 - \frac{2\xi (1 - \xi \delta) + 1 - e^{2\xi \delta}}{2\xi \delta + \sinh 2\xi \delta}\right] \xi A(\xi) \cos \xi x \, d\xi &= f(x), \quad 0 < x < a \tag{2.1.11} \\
\int_0^\infty A(\xi) \cos \xi x \, d\xi &= 0, \quad x > a \tag{2.1.12}
\end{align}


(iii) **Inverse Mellin Transforms**

The dual integral equations involving inverse Mellin transforms of the type
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tan \alpha s \psi(s) \rho^{-s} ds = f(\rho), \quad 0 < \rho < 1 \quad (2.1.13)
\]

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s \psi(s) \rho^{-s} ds = g(\rho), \quad \rho > 1 \quad (2.1.14)
\]

were considered by Srivastav and Parihar [150].

Erdélyi [64] solved the following set of dual equations

\[
M^{-1} \left[ \frac{\Gamma(1 + \eta - s/\sigma)}{\Gamma(1 + \eta + \alpha - s/\sigma)} \phi(s); x \right] = f(x), \quad 0 < x < a \quad (2.1.15)
\]

\[
M^{-1} \left[ \frac{\Gamma(\xi + s/\delta)}{\Gamma(\xi + \beta + s/\delta)} \phi(s); x \right] = g(x), \quad a < x < \infty \quad (2.1.16)
\]

Cooke obtained the solution of the above equations with \( f(x) = h(x) = 0 \) and later extended the method for \( f(x) \neq 0 \).

Tweed [160] solved the dual integral equations involving the inverse of certain Mellin type transforms. Trivedi and Pandey [158] have also obtained the solution of dual integral equation involving Naylor’s Mellin type transforms by employing Tweed’s method of solution but the equations considered by them were quite different from those investigated by Tweed himself. Later on Trivedi and Pandey [159] also solved the two pairs of dual integral equations involving the inverse of Naylor’s Mellin type transforms.

(v) H-Functions

Dual integral equations involving H-functions as kernels were first
introduced by Fox [69]. Later on Saxena [126] and Saxena and Kumbhat [128] extended Fox’s result to more general dual equations.

Later on Mehra and Ahuja [94] considered a pair of integral equations of the type

\[ r \int_{y}^{x} H(x, y) f(y) \prod_{i=1}^{r} (dy_{i}) = U(x), \quad 0 \leq (x) < 1 \quad (2.1.17) \]

where \( H(x, y) \) is a kernel called a multivariable H-function. The second integral equation, defined with other boundary conditions, \( (x) > 1 \), has another H-function as kernel with parameters different from that given above and involves another function \( V(x) \) on the right hand side while the prefix \( (r) \) before the infinite integral signals shows presence of \( r \) such operations, the first boundary condition \( 0 \leq (x) \leq 1 \) stands for two sided inequalities \( 0 \leq x_{1} < 1, \ldots, 0 \leq x_{r} < 1. \)

The method of solution is Fox’s method involving the Laplace’s operators \( L \) and \( L^{-1} \) to solve very general integral equation.

(iv) Legendre Function

Dhaliwal and Singh [19] considered the following pair of dual integral equations and obtained a closed from solution
\[ \int_{0}^{\infty} \tau^{-1} A(\alpha) P_{-(1/2)}(\cosh \alpha) \tan \alpha f(\tau) \, d\tau = F(\alpha), \quad 0 < \alpha < a, \quad (2.1.18) \]

\[ \int_{0}^{\infty} A(\tau) P_{-(1/2)}(\cosh \alpha) \, d\tau = 0, a < \alpha, \quad (2.1.19) \]

where the function \( A(\tau) \) is to be determined, \( P_{-(1/2)}(\cosh \alpha) \) is a Legendre function of complex index, \( f(\tau) \) is a real positive constant and \( F(\alpha) \) is a prescribed function of \( \alpha \).

Recently Mandal [92] considered certain dual integral equations involving generalized associated Legendre functions of the first kind and trigonometric functions as kernels. These equations are solved by using properties of the generalized associated Legendre functions and the inversion formula for the generalized Mehler-Fock transform involving generalized associated Legendre functions of the first kind.

\( \text{(vi)} \quad \text{Hankel Kernel of Order Zero} \)

Tranter [154] considered the dual integral equations involving Hankel Kernel of order zero and solved them, which were not derived by Titchmarsh and Busbridge's solution. These equations are

\[ \int_{0}^{\infty} U^\alpha \{1 + H(u)\} \psi(u) J_0(xu) \, du = f(x), \quad 0 < x < 1 \quad (2.1.20) \]

\[ \int_{0}^{\infty} \psi(u) J_0(xu) \, du = 0, \quad x > 1, \quad (2.1.21) \]
Labedev and Uflyand [75a] gave the solution in the form of Fredholm integral equation with symmetric kernel of the equations

\[ \int_0^\infty \{1-H(t)\} \phi(t) J_0(\alpha t) \, dt = f(x), \quad 0 < x < a \tag{2.1.22} \]

\[ \int_0^\infty \phi(t) J_0(\alpha t) \, dt = 0, \quad x > a \tag{2.1.23} \]

Later on Nasim [102] has shown that the dual integral equations with Hankel kernel

\[ \int_0^\infty t^{-2\alpha} J_\nu(\alpha t) [1+\omega(t)] \phi(t) \, dt = f(x), \quad 0 < x < 1 \tag{2.1.24} \]

\[ \int_0^\infty t^{-2\beta} J_\mu(\alpha t) \phi(t) \, dt = g(x), \quad x > 1 \tag{2.1.25} \]

where \( \omega \) is an arbitrary weight function, can be reduced to a singular Fredholm integral equation of the first kind.

Ahuja and Mehra [4] solved certain fractional integrals and a Hankel type transformation on certain spaces of test functions and of distributions. They have shown that the classical relations between fractional integrals and the Hankel transformation persist in these spaces, and thus a classical solution of a certain dual integral equation persists also.

**(vii) Fourier Transforms**

Nguyen and Popov [106] studied dual integral equations on a system of intervals
\[ \frac{d}{dx} \int_0^\infty A(\xi) \sin (\xi x) \, d\xi = f_n(x), \quad x \in I_n, \quad n = 1, \ldots, N \]  
\[ \int_0^\infty A(\xi) \cos (\xi x) \, d\xi = 0, \quad x \in \mathbb{R}^+ \setminus \bigcup_{n=1}^N I_n \]  

where \( A(\xi) \) is an unknown function, \( f_n \) are given functions and \( I_n = (a_n, b_n) \) are certain finite non-intersecting intervals contained in \( \mathbb{R}^+ \), where \( a_1 < a_2 < \ldots < a_n \).

The above dual integral equations reduce into an equivalent system of integral equation of the first kind and the author gives the criteria for it to have a unique solution. This solution can be obtained by the method of successive approximations. The authors gave the complete mathematical justification of the formal constructions used and constructions in the appropriate function spaces. Nguyen [107] investigated very elaborately the existence and uniqueness of the problems for certain dual integral equations involving the Fourier transforms of generalized functions. The following dual integral equations are considered

\[ p F^{-1} [K(t) \tilde{u}(t)](x) = f(x), \quad x \in \omega, \]  
\[ p' F^{-1} [\tilde{u}(t)](x) = g(x), \quad x \in \omega', \]  

where \( \omega' = \mathbb{R}/\omega \), \( u = F^{-1} [\tilde{u}(t)] \in s' \) is a function to be determined, \( K(t) \) is the non-negative function, \( f \in D'(\omega) \) and \( g \in D'(\omega') \) are given distributions on \( \omega \) and \( \omega' \) respectively and \( p, p' \) are restriction operators to \( \omega \) and \( \omega' \) respectively.
Pathak [117] extended the reciprocal formula \( f = g \hat{^k} h \) for the Watson transform of the function \( f \), defined by \( g = f \hat{^k} k \) on the generalised function space \( M_{a,b}' \) provided the functions \( k(x) \) and \( h(x) \) satisfy the functional equation \( K(s)H(1-s) = 1 \), where \( \hat{\wedge} \) denotes the Mellin convolution and \( K(s) \) and \( H(s) \) are Mellin transforms of \( k(x) \) and \( h(x) \) respectively. Finally, this theory was applied by him to obtained solution of a pair of dual integral equations.

(xi) Weber-Orr Transforms

Recently Nasim [103] has obtained the solution of some dual integral equations involving Weber-Orr transforms

\[
W_{\nu}^{-1} \left[ \xi^{-2\alpha} \Psi (\xi) ; \rho \right] = f(\rho), \quad a \leq \rho \leq c \quad (2.1.30)
\]

\[
W_{\nu}^{-1} \left[ \xi^{-2\beta} \Psi (\xi) ; \rho \right] = -g(\rho), \quad c < \rho < \infty \quad (2.1.31)
\]

where \( k = 0, 1, 2, \ldots, \nu > -1 \) and \( \Psi \) is an unknown function. Elementary methods are used to construct a general solution that contains many known results among them the Beltrami-Michel equations for torsion.

(x) Cauchy, Abel and Titchmarsh Type Kernels

Estrada and Kanwal [66] considered the following singular dual integral equations and gave distributional solutions
Cauchy Type

\[
\alpha_1(x) f(x) + \beta_1(x) \int_{a_1}^{b_1} \frac{f(t)}{(t-x)} \, dt + \gamma_1(x) \int_{a_2}^{b_2} \frac{f(t)}{(t-x)} \, dt
\]

\[= g(x), \quad a_1 < x < b_1 \quad (2.1.32)\]

\[
\alpha_2(x) f(x) + \beta_2(x) \int_{a_1}^{b_1} \frac{f(t)}{(t-x)} \, dt + \gamma_2(x) \int_{a_2}^{b_2} \frac{f(t)}{(t-x)} \, dt
\]

\[= g(x), \quad a_2 < x < b_2 \quad (2.1.33)\]

where \(\alpha_1(x), \alpha_2(x), \beta_1(x), \beta_2(x), \gamma_1(x), \gamma_2(x)\) and \(g(x)\) are known distributions and \(f(t)\) is to be determined.

Abel Type

\[
\alpha_1(x) \int_{a_1}^{x} \frac{f(t)}{(t-x)^{\alpha_1}} \, dt + \beta_1(x) \int_{x}^{b_1} \frac{f(t)}{(t-x)^{\alpha_1}} \, dt + \gamma_1(x) \int_{a_2}^{b_2} \frac{f(t)}{(t-x)^{\alpha_1}} \, dt
\]

\[= g(x), \quad a_1 < x < b_1 \quad (2.1.34)\]

\[
\alpha_2(x) \int_{a_2}^{x} \frac{f(t)}{(t-x)^{\alpha_2}} \, dt + \beta_2(x) \int_{x}^{b_2} \frac{f(t)}{(t-x)^{\alpha_2}} \, dt + \gamma_2(x) \int_{a_1}^{b_1} \frac{f(t)}{(t-x)^{\alpha_1}} \, dt
\]

\[= g(x), \quad a_2 < x < b_2 \quad (2.1.35)\]

Titchmarsh Type

\[
\int_{0}^{\infty} t^{-2a} f(t) J_n(tx) \, dt = A(x), \quad 0 < x < 1 \quad (2.1.36)
\]

\[
\int_{0}^{\infty} t^{-2b} f(t) J_n(tx) \, dt = B(x), \quad 1 < x < \infty \quad (2.1.37)
\]

where the \(J_n(tx)\) and \(J_m(tx)\) are Bessel functions of order \(n\) and \(m\), \(m, n \geq -1/2\).
a and b are real constants; A(x) and B(x) are prescribed generalized functions.

Later on Eswaran [67] obtained the solution of dual integral equations that are classical for diffraction theory

\[
\int_{-\infty}^{\infty} \sqrt{u^4 - k^2} \ A(u) \ e^{iux} \ du = f(x), \ |x| < 1 \quad (2.1.38)
\]

\[
\int_{-\infty}^{\infty} A(u) \ e^{iux} \ du = 0, \ |x| > 1 \quad (2.1.39)
\]

by reducing them to an infinite system of linear algebraic equations.

After that more general pair of the form

\[
\int_{-\infty}^{\infty} F(v_1, v_2, \ldots, u) \ A(u) \ e^{iux} \ du = f(x), \ |x| < 1 \quad (2.1.40)
\]

\[
\int_{-\infty}^{\infty} A(u) \ e^{iux} \ du = 0, \ |x| > 1 \quad (2.1.41)
\]

where \( F \) is some given function with a number of specified properties.

\[
v_{1,2}(u) = \sqrt{(u^2 - k_{1,2}^2)}, \ k_{1,2} \geq 0, \ k_1 + k_2 > 0
\]

undergoes a similar treatment.

A formal solution of dual integral equations of two variables with two weight functions using the fractional integral operators has been obtained by Prabha [121] by reducing them to Fredholm integral equations of the second kind.
Veliev and Shestopalov [164] found a general method for solving dual integral equations. He considered the integral equations of the type

\[- \int_{-\infty}^{\infty} h(\alpha) k(\alpha) e^{i \xi \alpha} \, d\alpha = f(\eta), \quad |\eta| < 1 \]

\[\int_{-\infty}^{\infty} h(\alpha) e^{\pm i \xi \alpha} \, d\alpha = 0, \quad |\eta| > 1 \]

These equations are in the class of functions \( h(\eta) \) for which \( h(\eta), |\eta|^{1/2} h(\eta) \in L_2(k) \) under certain assumptions on \( k(\eta) \) and \( f(\eta) \) and proved the unique solvability of this equation.

2.1.2 Simultaneous Dual Integral Equations

Erdogen and Bahar [65] obtained for the first time the solution of the simultaneous dual integral equations involving Bessel functions.

These equations are as follows

\[\int_{0}^{\infty} \sum_{n=1}^{\infty} C_{ij} (x) \psi_j (x) J_{\mu_i} (xy) \, dx = P_i(y), \quad y \in I_1 \]

\[\int_{0}^{\infty} \psi_j (x) J_{\mu_j} (xy) \, dx = 0, \quad y \in I_2 \]

where \( i = 1, 2, \ldots, n \). A particular case of above equations for \( n = 2 \) is solved by Westmann [165], Dwivedi [22] and Dwivedi and Singh [33] also considered some other sets of simultaneous dual integral equations.

Later on Narain and Lal [99] considered simultaneous dual integral equations involving Meijer's G-functions of \( n \)-variables.
2.1.3 Triple Integral Equations

In the following lines we describe the triple integral equations with different kernels.

(i) **Bessel Functions Kernel**

Tranter [156] was the first person who extended dual integral equations to triple integral equations. He considered the equations

\[ \int_0^\infty \psi(u) \, J_\nu(xu) \, du = \begin{cases} f(x), & 0 < x < a \\ h(x), & b < x < \infty \end{cases} \quad (2.1.46) \]

\[ \int_0^\infty u^\alpha \, \psi(u) \, \{1 + H(u)\} \, J_\nu(xu) \, du = g(x), \quad a < x < b \quad (2.1.47) \]

with \( H(u) = 0 \) and more general equations were considered by Cooke [14]. In fact Cooke [15, 16] put the solutions of triple integral equations involving Bessel functions on sound footings. He made heavy use of Erde'lyi–Köber operators and introduced new operators to obtain the solutions.

Srivastav [147] considered the following two sets of triple integral equations involving Bessel function as kernel

I Set

\[ \int_0^\infty s \, A(s) \, J_\nu(\rho s) \, ds = \begin{cases} 0, & 0 < \rho \leq a \\ 0, & 1 \leq \rho < \infty \end{cases} \quad (2.1.48) \]

\[ \int_0^\infty s^2 A(s) \, J_\nu(\rho s) \, ds = -f(\rho), \quad a \leq \rho < 1 \quad (2.1.49) \]
\[ \int_0^\infty (\lambda s + \mu s^2) A(s) J_\nu (\rho s) \, ds = \begin{cases} 0, & 0 < \rho \leq a \\ 0, & 1 \leq \rho < \infty \end{cases} \quad (2.1.50) \]

\[ \int_0^\infty A(s) J_\nu (\rho s) \, ds \quad = \quad f(\rho), \quad a \leq \rho < 1 \quad (2.1.51) \]

The triple integral equations are first reduced to ordinary differential equation. The solution of this ordinary differential equation together with the inversion of Abel integral equation yields Fredholm integral equation of the second kind.

(ii) **Trigonometric Kernels**

Srivastav and Lowengrub [145] solved first time the triple integral equations with trigonometric kernels by applying finite Hilbert transform technique. Later on Singh [133] considered the different set of equations

\[ \int_0^\infty A(t) \cos (xt) \, dt = \begin{cases} 0, & 0 < x < a \\ 0, & x > b \end{cases} \quad (2.1.52) \]

\[ \int_0^\infty A(t) \tanh (xt) \cos (xt) \, dt = p(x), \quad a < x < b \quad (2.1.53) \]

and solved them.

\[ \frac{1}{2\mu i} \int_{C_{-ix}}^{C_{+ix}} s\psi(s) \rho^{-s} \, ds = \begin{cases} f_1(\rho), & 0 < \rho < a \\ f_2(\rho), & a < \rho < 1 \end{cases} \quad (2.1.54) \]

\[ \frac{1}{2\mu i} \int_{C_{-ix}}^{C_{+ix}} s\psi(s) \tan \alpha \rho^{-s} \, ds = \frac{f_2(\rho)}{\tan \alpha}, \quad a < \rho < 1 \quad (2.1.55) \]
(iii) Inverse Mellin Transforms

Srivastav and Parihar [150] considered the triple integral equations of the type

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s \psi(s) \rho^{-s} ds = \begin{cases} 
  f_1(\rho), & 0 < \rho < a \\
  f_3(\rho), & \rho > 1 
\end{cases} \tag{2.1.54}
\]

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s \psi(s) \tan \alpha \rho^{-s} ds = f_2(\rho), \ a < \rho < 1 \tag{2.1.55}
\]

They reduced the equations into dual cosine series equations and then finally wrote the solutions by using Tranter's method.

Lowndes [88] extended the operators of fractional integration, considered earlier by Cooke, and obtained the solution of the following set of triple integral equations

\[
M^{-1} \left[ \frac{\Gamma(\xi + s/\delta)}{\Gamma(\xi + \beta + s/\delta)} \phi(s); \right] = \begin{cases} 
  0, & 0 < x < a \\
  f_1(x), & b < x < \infty 
\end{cases} \tag{2.1.56}
\]

\[
M^{-1} \left[ \frac{\Gamma(1 + \eta - s/\sigma)}{\Gamma(1 + \eta + \alpha - s/\sigma)} \phi(s); \right] = f_2(x), \ a < x < b \tag{2.1.57}
\]

where \(\alpha, \beta, \xi, \eta, \delta > 0, \sigma > 0\) are real parameters.

Singh [130] solved triple integral equations of inverse Mellin type transforms. Tweed [161], Pandey and Trivedi [113] considered triple integral equations involving finite inverse Mellin transforms. Recently Dwivedi and
Chandel [56] obtained the solution and application of certain triple integral equations involving inverse Mellin transforms. The equations are solved by the method of reducing the triple integral equations to a singular integral equation.

We have also considered the two different sets of triple integral equations in this thesis and solved them by reducing the equations to singular integral equation and also given the application of second set of such equations in crack problems.

(iv) **H-Function Kernel**

The solution of triple integral equations involving H-functions was obtained by Saxena and Kumbhat [129] for the first time. Mehra and Prabha [95] obtained the formal solution of certain triple integral equations containing H-functions of $n$ - variables.

(v) **Legendre Functions of Imaginary Argument As Kernel**

The triple integral equations containing Legendre function as kernel were considered by Srivastava [142]. Dwivedi considered more general triple equations and gave better solution than those given by the previous authors.

(vi) **Mehler-Fock Transform**

Srivastava [148] considered the following two sets of triple integral equations involving Mehler-Fock transforms
I Set:

\[ \int_{0}^{\infty} \tau m(\tau) A(\tau) P_{-(1/2)+it}^{\beta}(\cos hx) \, d\tau = 0, \quad 0 \leq x \leq a \]  \hspace{1cm} (2.1.58)

\[ \int_{0}^{\infty} \left[ \frac{1}{4} + \tau^2 \right] A(\tau) P_{-(1/2)+it}^{\beta}(\cos hx) \, d\tau = -f(x), \quad a \leq x \leq 1 \]  \hspace{1cm} (2.1.59)

\[ \int_{0}^{\infty} \tau m(\tau) A(\tau) P_{-(1/2)+it}^{\beta}(\cos hx) \, d\tau = 0, \quad 1 \leq x \leq \infty \]  \hspace{1cm} (2.1.60)

II Set:

\[ \int_{0}^{\infty} \left[ \frac{1}{4} + \tau^2 \right] m(\tau) A(\tau) P_{-(1/2)+it}^{\beta}(\cos hx) \, d\tau = -f(x), \quad a \leq x \leq 1 \]  \hspace{1cm} (2.1.62)

\[ \int_{0}^{\infty} \tau A(\tau) P_{-(1/2)+it}^{\beta}(\cos hx) \, d\tau = 0, \quad 1 \leq x \leq \infty \]  \hspace{1cm} (2.1.63)

\[ \int_{0}^{\infty} \tau A(\tau) P_{-(1/2)+it}^{\beta}(\cos hx) \, d\tau = 0, \quad 0 \leq x \leq a \]  \hspace{1cm} (2.1.61)

The above triple integral equations are first reduced to the problem of solving an ordinary differential equation. The differential equation is solved together with inversion theorem for some variants of Abel integral equation of the second kind. Then the problem of solving triple integral equations is reduced to that of solving Fredholm integral equations of the second kind.

2.1.4 Simultaneous Triple Integral Equations

Dwivedi and Sharma [36] have considered the simultaneous triple integral equations involving H and G-functions of two variables. Dwivedi and Singh [35] generalised the problems for H-functions of n-variables. Later on
Paliwal and Mishra [112] have also considered the formal solution of the simultaneous triple integral equations which are very useful for some crack problems in the mathematical theory of elasticity.

2.1.5 Quadruple Integral Equations

In the following lines we describe the quadruple integral equations with different kernels.

(i) Bessel Function Kernel

The obvious extension of triple integral equations to quadruple integral equations was done by Ahmad [2] and those integral equations are as follows:

\[
\int_{0}^{\infty} u^{-2\alpha} \phi(t) J_{v}(xt) \, dt = \begin{cases} 
F_{1}(x), & 0 < x < a \\
F_{3}(x), & b < x < c 
\end{cases} 
\quad (2.1.64)
\]

\[
\int_{0}^{\infty} u^{-2\beta} \phi(t) J_{v}(xt) \, dt = \begin{cases} 
F_{2}(x), & a < x < b \\
F_{4}(x), & x > c 
\end{cases} 
\quad (2.1.65)
\]

He obtained the solution of the above set of integral equations by using Erde\'lyi-Köber operators and reduced them in the form of Fredholm integral equations of the second kind.

Some more general operators to find the solution of the above equations with general weight function were considered by Cooke [16]. Dwivedi and Trivedi [37] generalised the results of Cooke [16]. Gupta and Chaturvedi [73], Saxena and Sethi [127] have also solved quadruple integral
equations with Bessel function Kernels.

Prabha [120] considered certain quadruple integral equations involving Bessel functions as kernels and solved them by the application of generalised operators of the Hankel transform and Erde'lyi-Köber operators of two variables.

(ii) Trigonometric Kernels

Singh and Jain [134] considered quadruple integral equations involving trigonometric kernels, which are as follows:

\[
\int_0^\infty t A(t) \cos \alpha t \, dt = \begin{cases} 
 f_1(\alpha), & 0 < \alpha < a \\
 f_3(\alpha), & b < \alpha < c 
\end{cases} 
\]  
\tag{2.1.66}

\[
\int_0^\infty A(t) \cos \alpha t \cot\beta t \, dt = \begin{cases} 
 f_2(\alpha), & a < \alpha < b \\
 0, & c < \alpha < \infty 
\end{cases} 
\]  
\tag{2.1.67}

These integral equations arise in two dimensional steady state heat conduction problems. Singh [134] gave the solution of integral equations and its applications to electrostatics.

(iii) Inverse Mellin Transforms

Dwivedi, Kushwaha and Trivedi [37] have considered the following set of quadruple integral equations involving inverse Mellin transforms
\[ M^{-1}\left[ \frac{\Gamma(1+n-s/\sigma)}{\Gamma(1+n+\alpha-s/\sigma)} \phi(s) ; x \right] = \begin{cases} f_1(x), & 0 < x < a \\ f_3(x), & b < x < c \end{cases} \]  
(2.1.68)

\[ M^{-1}\left[ \frac{\Gamma(\xi+s/\delta)}{\Gamma(\xi+\beta+s/\delta)} \phi(s) ; x \right] = \begin{cases} g_2(x), & a < x < b \\ g_4(x), & c < x < \infty \end{cases} \]  
(2.1.69)

Other authors also considered certain quadruple integral equations of inverse Mellin transform type.

(iii) **H-Functions**

Saxena and Sethi [127] introduced us the quadruple integral equations with H-functions as kernels. These quadruple integral equations are the extensions of the dual and triple integral equations involving same kernels.

(v) **Legendre Functions of Imaginary Argument as Kernel**

Dange and Singh [17] extended the paper of Srivastava [142] for triple integral equations with Legendre functions of imaginary argument as kernel to the following set of quadruple integral equations.

\[ \int_0^\infty A(v) P_{-(1/2)+iv} (\cos h\alpha) \, dv = \begin{cases} f_1(\alpha), & 0 < \alpha < a \\ f_3(\alpha), & b < \alpha < c \end{cases} \]  
(2.1.70)

\[ \int_0^\infty \nu \tan (h\pi\nu) A(v) P_{(1/2)+iv} (\cos h\alpha) \, dv = \begin{cases} f_2(\alpha), & a < \alpha < b \\ f_4(\alpha), & c < \alpha < \infty \end{cases} \]  
(2.1.71)

where \[ A(v) \] is to be determined.
2.1.6 5-Tuple Integral Equations

Dwivedi and Singh [49] solved n-integral equations involving Bessel functions from where the solution of five integral equations can be obtained as a particular case. However in one of Dwivedi, Chandel and Bajpai [59] we have derived the solution and application of 5-integral equations involving inverse Mellin transforms in chapter four.

2.1.7 6-Tuple Integral Equations

In the following lines we describe the 6-tuple integral equations with different kernels.

(i) Bessel Function Kernel

Ahuja [3] obtained the following set of six integral equations involving Bessel functions as kernel

\[
\int_0^\infty \xi^p \psi(\xi) J_\nu(\xi x) \, d\xi = \begin{cases} 
  f_1(x), & 0 < x < a_1 \\
  f_2(x), & a_2 < x < a_3 \\
  f_3(x), & a_4 < x < a_5 
\end{cases} \quad (2.1.72)
\]

\[
\int_0^\infty \xi^{-q} \psi(\xi) J_\nu(\xi x) \, d\xi = \begin{cases} 
  g_1(x), & a_1 < x < a_2 \\
  g_2(x), & a_3 < x < a_4 \\
  g_3(x), & a_5 < x < \infty 
\end{cases} \quad (2.1.73)
\]

where \( \xi^p \) and \( \xi^q \) are weight functions, \( \pm p \neq \pm q \), the values of \( p \) and \( q \) are taken to be 0, 1, -1. The solution of this set is in the form of simultaneous Fredholm
integral equations.

(ii) Legendre Functions

Dwivedi, Kushwaha and Gupta [38] obtained the solution of the following set of six integral equations involving associated Legendre functions

\[
\int_0^\infty \phi (\tau) \mathbf{P}_\nu (1/2) \pm (\cosh \alpha) \, d\tau = \begin{cases} f_1 (\alpha), & 0 < \alpha < a_1 \\ f_3 (\alpha), & a_2 < \alpha < a_3 \\ f_5 (\alpha), & a_4 < \alpha < a_5 \end{cases} \quad (2.1.74)
\]

\[
\int_0^\infty \frac{\tau}{\pi} \sinh(\pi \tau) \Gamma(1/2 + \beta - i\tau) \Gamma(1/2 + \beta - i\tau) \mathbf{P}_\nu (1/2) \pm (\cosh \alpha) \phi (\tau) \, d\tau = \begin{cases} g_2 (\alpha), & a_1 < \alpha < a_2 \\ g_4 (\alpha), & a_3 < \alpha < a_4 \\ g_6 (\alpha), & a_4 < \alpha < a_6 \end{cases} \quad (2.1.75)
\]

where \( f_1 (\alpha), f_3 (\alpha), f_5 (\alpha), g_2 (\alpha), g_4 (\alpha) \) and \( g_6 (\alpha) \) are the prescribed functions and \( \phi (\tau) \) is an unknown, to be determined. The final solution is in the form of a Fredholm integral equation.

Dwivedi and Gupta [39] also considered six integral equations with generalised Legendre functions as kernel. Those equations are as follows:

\[
\int_0^\infty \psi (\tau) \mathbf{P}_\nu (1/2) \pm (\cosh \alpha) \, d\tau = f_1 (\alpha), \quad a_{j-1} < \alpha < a_j \\
\quad \quad \quad \quad \quad j = 1, 3, 5 \text{ and } a_0 = 0 \quad (2.1.76)
\]
\[ \int_0^\infty \psi(\tau) \Gamma(\frac{1}{2} - \mu + i\tau) \Gamma(\frac{1}{2} - \mu - i\tau) \tau \sinh \pi \tau \, \mathbb{P}^\mu_{-(1/2)+i\tau} (\cosh \alpha) \, d\tau = g_j(\alpha), \]

where \( f_j \) and \( g_j \) are prescribed functions and \( \psi(\tau) \) is to be determined. The final solution is in the form of Fredholm integral equation of the second kind.

Estrada and Kanwal [66] present various techniques of treating related equations with successively more complicated logarithmic kernels. Both classical and distributional methods are used.

\[ \int_a^b \log |x - y| \, g(y) \, dy = f(x), \quad a < x < b \]  

They started with this equation.

2.1.8 7-Tuple Integral Equations (with Inverse Mellin Transforms)

On reviewing the literature we find that none author has written a paper on seven integral equations involving kernel of any kind. However in one of my paper we have obtained the solution and application of seven integral equations involving inverse Mellin transforms in chapter four.

2.1.9 n-Tuple Integral Equations

In the following lines we describe the dual integral equations with different kernels.
\[ \int_0^\infty \psi(\tau) \Gamma\left(\frac{1}{2} - \mu + i\tau\right) \Gamma\left(\frac{1}{2} - \mu - i\tau\right) \tau \sinh \pi \tau P_{\alpha}^\mu (\cosh \alpha) \, d\tau = g_j(\alpha), \]

\[ a_{j-1} < \alpha < a_j, \quad j = 2, 4, 6 \quad \text{and} \quad a_6 = \infty \quad (2.1.77) \]

where \( f_j \) and \( g_j \) are prescribed functions and \( \psi(\tau) \) is to be determined. The final solution is in the form of Fredholm integral equation of the second kind.

Estrada and Kanwal [66] present various techniques of treating related equations with successively more complicated logarithmic kernels. Both classical and distributional methods are used.

\[ \int_a^b \log |x - y| \, g(y) \, dy = f(x), \quad a < x < b \quad (2.1.78) \]

They started with this equation.

2.1.8 7-Tuple Integral Equations (with Inverse Mellin Transforms)

On reviewing the literature we find that none author has written a paper on seven integral equations involving kernel of any kind. However in one of my paper we have obtained the solution and application of seven integral equations involving inverse Mellin transforms in chapter four.

2.1.9 n-Tuple Integral Equations

In the following lines we describe the dual integral equations with different kernels.
\[ A_0 + \sum_{n=1}^{\infty} \lambda_n^2 \, n \, A_n \, J_\nu (\lambda_n \rho) = F(\rho), \quad 0 \leq \rho < c, \quad (2.2.3) \]

\[ \sum_{n=1}^{\infty} A_n J_\nu (\lambda_n \rho) = G(\rho), \quad c < \rho \leq 1 \quad (2.2.4) \]

(i) **Bessel Function Kernel**

Dwivedi and Singh [49] obtained the solution of n-integral equations involving Bessel functions

\[ \int_{0}^{\infty} \xi^p \psi (\xi) J_\nu (\xi x) \, d\xi = f_j (x), \quad a_{j-1} < x < a_j, \quad j = 1, 3, 5, \ldots, (n-1) \text{ and } a_0 = 0 \quad (2.1.79) \]

\[ \int_{0}^{\infty} \xi^q \psi (\xi) J_\nu (\xi x) \, d\xi = f_j (x), \quad a_{j-1} < x < a_j, \quad j = 2, 4, 6, \ldots, n \text{ and } a_n = \infty \quad (2.1.80) \]

where \( \xi^p \) & \( \xi^q \) are the weight functions, \( \pm p \neq \pm q \) the values of \( p \) and \( q \) are taken to be 0, 1, \(-1\), \( \psi(\xi) \) is some unknown function and \( f_j (x), j = 1,2,3, \ldots (n-1) \), \( n \) are the prescribed functions.

**2.2 RELATED SERIES EQUATIONS**

**2.2.1 Dual Series Equations**

In the following lines we describe the dual series equations with different kernels.
First time Cooke and Tranter [13] studied the following set of dual series equations involving Bessel functions

\[ \sum_{n=0}^{\infty} \lambda_n^{-2p} A_n J_\nu (\rho \lambda_n) = F(\rho), \quad 0 \leq \rho \leq 1 \]  \hspace{1cm} (2.2.1)

\[ \sum_{n=0}^{\infty} A_n J_\nu (\rho \lambda_n) = 0, \quad 1 < \rho < a, \]  \hspace{1cm} (2.2.2)

They reduced the above equations to a system of algebraic equations, which can be easily solved by numerical methods.

Sneddon and Srivastava [136] considered many other types of series equations.

Srivastava [149] introduced the equations

\[ A_0 + \sum_{n=1}^{\infty} \lambda_n^{2p} A_n J_\nu (\lambda_n \rho) = F(\rho), \quad 0 \leq \rho < c, \]  \hspace{1cm} (2.2.3)

\[ \sum_{n=1}^{\infty} A_n J_\nu (\lambda_n \rho) = G(\rho), \quad c < \rho \leq 1 \]  \hspace{1cm} (2.2.4)

where \( \{\lambda_n\} \) is the sequence of positive roots of the transcendental equation

\[ \lambda J_\nu (\lambda) + H J_\nu (\lambda) = 0 \]

\( H \) and \( \nu \) being the real coefficients with \( \nu \geq -1/2 \) and \( H + \nu \geq 0 \).
Noble [108] gave the following set of dual series equations

\[
\sum_{n=0}^{\infty} P_n (\nu, \beta) A_n J_n (\alpha, \beta, x) = f(x), \quad 0 \leq x < a, \tag{2.2.5}
\]

\[
\sum_{n=0}^{\infty} A_n J_n (\alpha, \beta, x) = g(x), \quad a < x < 1 \tag{2.2.6}
\]

where \( J_n (\alpha, \beta, x) \) denotes the Jacobi polynomial and \( P_n (\nu, \beta) \) is a constant defined by

\[
P_n (\nu, \beta) = \frac{\Gamma (\nu + n) \Gamma (1 + \alpha - \beta + n)}{\Gamma (1 + \alpha - \nu + n) \Gamma (\beta + n)} \tag{2.2.7}
\]

He solved the above equations by developing multiplying factor method.

Srivastav [149] obtained the solution of another set of dual series equations

\[
\sum_{n=0}^{\infty} \frac{A_n P_n (\alpha, \beta) (\cos \theta)}{\Gamma (\alpha + n + 1) \Gamma (\beta + n + 3/2)} = F(\theta), \quad 0 \leq \theta \leq \phi \tag{2.2.8}
\]

\[
\sum_{n=0}^{\infty} \frac{A_n P_n (\alpha, \beta) (\cos \theta)}{\Gamma (\beta + n + 1) \Gamma (\alpha + n + 1/2)} = G(\theta), \quad \phi < \theta \tag{2.2.9}
\]

where \( \alpha > -1/2, \beta = -1 \) and \( P_n (\alpha, \beta) (\cos \theta) \) denotes the Jacobi polynomials.

He obtained closed form solution by using Abel integral equation method.
(iii) Trigonometrical Function


(iv) Generalised Laguerre Polynomials

Srivastava [140] gave the solution of the following dual series equations

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha + n + 1)} L_n^\nu(x) = f(x), \quad 0 \leq x < y \quad (2.2.10)
\]

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta + n + 1)} L_n^\alpha(x) = g(x), \quad y \leq x < \infty \quad (2.2.11)
\]

by applying Sneddon's [136] method.

(v) Legendre Polynomials

Collins [11] studied the following set of dual series equations

\[
\sum_{n=0}^{\infty} (1 + H_n) A_n T_{m+n}^{-m}(\cos \theta) = F(\theta), \quad 0 \leq \theta \leq \phi \quad (2.2.12)
\]

\[
\sum_{n=0}^{\infty} (2n + 2m + 1) A_n T_{m+n}^{-m}(\cos \theta) = G(\theta), \quad \phi \leq \theta \leq \pi \quad (2.2.13)
\]

where \( T_n^{-m}(\cos \theta) \) is the Legendre polynomial and the final solution of the
above equations is in the form of Fredholm integral equations of second kind.

(vi) **Heat Polynomials**

Pathak [116a] derived the closed form solution of the equations

\[
\sum_{n=0}^{\infty} \frac{A_n T^{-n} \rho^n}{\Gamma(\nu + \frac{1}{2} + n + \rho)} P_{n+\rho,\nu}(x,-t) = f(x,t), \quad 0 < x < y \quad (2.2.14)
\]

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\mu + \frac{1}{2} + n + \rho)} P_{n+\rho,\sigma}(x,-t) = g(x,t), \quad y < x < \infty \quad (2.2.15)
\]

where \( P_{n}(x,t) \) is a heat polynomial.

(vii) **Konhauser Biorthogonal Polynomials**

Patil [118] introduced the dual series equations

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\delta + 1 + K_n)} z_n^\delta(x,K) = f(x), \quad 0 < x < y \quad (2.2.16)
\]

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\delta + \beta + K_n)} z_n^\delta(x,K) = g(x), \quad y < x < \infty \quad (2.2.17)
\]

where \( z_n^\alpha(x,k) \) is the Konhauser biorthogonal polynomial.

(viii) **Generalised Bateman K-Functions**

Srivastava [141] considered the following set of dual series equations

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\beta + \sigma + n + 1)} K_2^{2(\alpha + \sigma)}(x) = f(x), \quad 0 \leq x < y \quad (2.2.18)
\]
\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\nu + \sigma + n + 1)} K_{2(n+\beta)}(x) = g(x), \quad y \leq x < \infty
\]  
(2.2.19)

where \(K_{\nu}(x)\) is the generalised Bateman K-function. Srivastava [141] obtained the solution of particular cases of the above equations.

2.2.2 Simultaneous Dual Series Equations

Dwivedi et al. [40] solved the simultaneous dual series equations involving the product of \(r\)-Laguerre polynomials. Dwivedi [23] also considered the simultaneous dual equations involving H-functions.

Dwivedi and Shukla [40] considered the simultaneous dual series equations. Recently Lal and Mathur [78] obtained the solution for a pair of simultaneous dual series equations involving Jacobi polynomials.

2.2.3 Triple Series Equations

Collins [12], for the first time, solved the triple series equations of the form

\[
\sum_{n=0}^{\infty} (2n+1) C_n P_n(\cos \theta) = \begin{cases} 
0, & 0 < \theta < \alpha \\
0, & \beta < \theta < \pi 
\end{cases}
\]  
(2.2.20)

\[
\sum_{n=0}^{\infty} (1 + H_n) C_n P_n(\cos \theta) = f(\theta), \quad \alpha < \theta < \beta
\]  
(2.2.21)

Which are equations of first kind and
\[
\sum_{n=0}^{\infty} \left(1 + H_n\right) C_n P_n (\cos \theta) = \begin{cases} f(\theta), & 0 < \theta < \alpha \\ f(\theta), & \beta < \theta < \pi \end{cases}
\] (2.2.22)

\[
\sum_{n=0}^{\infty} (2n + 1) C_n P_n (\cos \theta) = 0, \quad \alpha < \theta < \beta
\] (2.2.23)

are the equations of second kind. Williams [166] solved the triple series equations by reducing to triple integral equations. Srivastava [144] considered triple series equations involving series of Jacobi polynomials.

Lowndes [86] obtained the solution of certain triple series equations involving Jacobi polynomials. Dwivedi et. al. [27, 30, 31, 32] obtained the solution of triple series equations involving generalised Laguerre polynomials, generalised Bateman K-functions, Jacobi polynomials, Jacobi and Laguerre polynomials. Recently Dwivedi et. al. [84] and Narain et al. [100] considered triple series equations involving heat polynomials.

Melrose and Tweed [96] derived the solution of some triple trigonometrical series. Parihar [114] obtained the closed form solution of some triple trigonometrical series. Lowndes and Srviastava [89] have considered a class of triple series equations with Laguerre polynomial kernels and reduced them to triple integral equations with Bessel function kernels.

The operators in the integral equations are for a modified Hankel transformations. Hence, identities which connect Erde’lyi-Köber fractional integral operators with Hankel operator are used to reduce the problem to a single integral equation involving a modified Hankel operator. Inversion then
leads to an exact solution, and thus an exact solution to the series equations can be obtained.

2.2.4 Simultaneous Triple Series Equations


2.2.5 Quadruple Series Equations

Cooke [16] was the first researcher who obtained the solution of following quadruple series equations

$$\sum_{n=1}^{\infty} a_n \lambda_n^{-2\alpha} J_v(\lambda_n x) = \begin{cases} G_2(x), & a < x < b \\ 0, & c < x < d \end{cases}$$  \hspace{1cm} (2.2.24)

$$\sum_{n=1}^{\infty} a_n \lambda_n^{-2\alpha} J_v(\lambda_n x) = \begin{cases} F_1(x), & 0 < x < a \\ F_3(x), & b < x < c \end{cases}$$  \hspace{1cm} (2.2.25)


2.2.6 5- Tuple Series Equations

Dwivedi et. al. [41, 43, 42, 57] obtained the solution of various set of five series equations involving product of ‘t’ Jacobi polynomials, Jacobi and

In the present thesis we have obtained the solution of five series equations involving heat polynomials in chapter five.

- 2.2.7 6-Tuple Series Equations

Dwivedi et. al. [55, 50, 54] considered the different sets of six series equations involving Jacobi and Laguerre polynomials, generalised Bateman K-functions and Jacobi polynomials.

2.2.8 n-Tuple Series Equations

Dwivedi et. al. [51] solved the various set of n-series equations involving Jacobi and Laguerre polynomials, generalised Bateman K-functions and Jacobi polynomials.

2.3 APPLICATION OF INTEGRAL AND SERIES EQUATIONS IN THE FIELD OF ELASTICITY

Integral and series equations have been proved to be very useful in the theory of elasticity, electrostatics, elastostatics, diffraction theory and acoustics. Particularly, these equations are very much useful in finding the solution of crack problems of elasticity.

In crack analysis first attempt for theory of cracks was done by Inglis
[74] in the first quarter of the 20th century in which he presented the solution of a problem within the classical theory of elasticity concerning the equilibrium of an infinite body with an isolated cavity. Soon after that Muskhelishvili [98] in 1919 solved the same problem in much simpler way.

Griffith presented his first paper [71] in 1920 and second paper [72] in 1925 in which he rightly considered fundamentals for the theory of cracks.

Inglis [74] considered the crack in the shape of an elliptical hole. Griffith [72] made the minor axis of the ellipse to be zero and reduced the crack into a straight line, which is known as Griffith crack.

Li, Hu and Tang [84] studied the torsional problem for a circular cylinder containing an equilateral triangular opening and a line crack. The torsion solution is obtained by using the method of singular integral equations and the crack cutting technique, the advantage of which lies in preserving the stress singularities at the interior corners of the hole as well as those at the crack tips. Mode III stress intensity factors are computed at both ends of the crack, assuming that the singular behaviour of the stress near a notch tip and the intensities of the local stresses at corners of the hole are given.

Sneddon developed dual integral equations and applied first transform techniques in solving the mixed boundary value problems of the theory of elasticity. Muskhelishvili [98] developed a method based on the theory of Cauchy's integral for solving the mixed boundary value problems. These methods have played an important role in the development of potential theory.
and mathematical theory of elasticity. Sneddon [137] has dealt with different methods to solve mixed boundary value problems.

A 2-dimensional problem of an isotropic elastic strip having an infinite row of Griffith cracks was considered by Misra and Misra [97] by using dual integral equations approach. The problem analyzing analytically the stress intensity factor, the critical pressure and the energy required to open the crack, were studied. Numerical results were also derived.

Griffith’s 2-dimensional theory of rupture was extended to 3-dimensional by Sack [125]. He considered a disk shaped crack in the interior of an infinite elastic solid. It is usually known as “penny shaped” crack.

Lal and Jain [77] studied the problem of finding the distribution of stress and the displacement in an elastic half plane containing an external line crack perpendicular to the free surface of the plane. This problem has been formulated in the form of dual integral equations. Finally, the expressions for the stress intensity factor and the crack energy are derived numerically.

Palaiya and Majudar [111] solved the problem of 2-diffraction of harmonic antiplane shear waves by a pair of coplanar parallel rigid strips located at the interfaces by using Fourier cosine transforms. The problem was reduced to triple integral equations & solved by usual methods.

Srivastava and Dhawan [146] considered the problem of stress distribution due to Griffith crack at the interface of an elastic layer bonded to
half plane. They reduced this problem to a system of simultaneous dual integral equations involving trigonometric kernels. Some further crack problems have been solved by using simultaneous dual integral equations.

Lowengrub and Sneddon [85] considered the problem of stress distribution in a dissimilar medium when a penny-shaped crack is situated along the bounding plane. They converted this problem to simultaneous dual integral equations which were later solved by known methods.

Certain triple integral equations have been found very useful in the solution of three part boundary value problems. Lowengrub and Srivastava [82] considered certain problems where such equations arise. Several other research workers [47, 60, 62, 65] used triple integral equations to solve different crack problems.

Lowengrub [83] considered the problem of stress distribution for the two bonded dissimilar elastic half planes containing a pair of coplanar cracks at the interface line by using Fourier transforms. He reduced the problems to simultaneous triple integral equations and later to a Hilbert-problem.

Roy and Chatterjee [124] considered the effect of the free surface of the stress distribution of an elliptic crack aligned parallel to the free boundary and at depth $h$-below it. The title problem is posed a dual integral equation in cartesian co-ordinate system. By suitable transformation the dual integral equations are first reduced to an infinite system of dual integral in cylindrical coordinates. Then they are further reduced by a recently developed technique.
to an infinite system of Fredholm integral equations of the second kind.

The problem of determining the stress intensity factors in a semi-infinite orthotropic elastic medium containing two coplanar cracks parallel to the boundary was considered by Mahapatra and Parhi [90]. The above mixed boundary value problem was reduced to triple integral equations, which were solved by using infinite Hilbert-transform technique.

Quadruple integral equations have also been used to find the solution of certain crack problems. Singh [131] considered the problem of determining the stress distribution in the vicinity of a Griffith crack in an infinite elastic solid, when the crack is opened by two symmetrical rigid inclusions.

Lal [76] considered the problem of an elastic half-space under torsion by a flat annular rigid stamp in the linear micropolar elasticity. The problem is reduced to system of four Fredholm integral equations.

Recently Dwivedi and Chandel [56] used triple integral equations involving inverse Mellin transforms to solve a crack in an infinite elastic solid under longitudinal shear having circular hole with pair of line cracks.

Recently Wu, X. et. al. [167] found the solution of a Griffith crack moving at the interface of two bonded dissimilar elastic half planes.

Recently De., S.K. [18] has derived the stress distribution in the neighbourhood of a Griffith crack at the interface of an elastic half-plane and a rigid foundation. Dislocation layers have been utilized to solve the problem.
The solution leads to that of an integral equation of Carlerman type, which ultimately reduces to Hilbert problem.

A large number of problems are encountered in the theory of crack problems of elasticity. Some of these problems are being mentioned below where series equations have been found useful.

Parihar and Garg [115] derived the stress and displacement distributions when infinite Griffith cracks are presented along the bond line of two dissimilar elastic half planes. The problem is first reduced to the dual series equations and then to a Hilbert problem.

The problem of stress distribution in the presence of infinite row of interface cracks, located symmetrically on the central line in a composite strip was considered by Parihar and Garg [115]. This problem is first converted to simultaneous dual series equations and finally to Hilbert problem.

Parihar and Kushwaha [116] solved the problem of distribution of stress in an infinite strip containing two Griffith cracks under the action of body forces. They reduced the problem to a set of triple series equations and obtained the closed form solution.

Parihar and Kushwaha [63] also obtained the solution of a crack problem of a strip containing Barenblatt crack. They reduced the problem to dual series equations. By symmetry conditions the problem is equivalent to that of an infinite row of collinear Barenblatt cracks in an infinite elastic solid.
Tweed and Melrose [163] also considered 'The out of plane shear problem for an infinite sheet with a staggered array of pairs of crack'. They used a triple series technique to find closed form expressions for the mode III stress intensity factors of a staggered array of pairs of cracks in an infinite elastic solid.

A number of crack problems in the classical theory of elasticity have been solved with the help of integral and series equations. A complete Bibliography of the work done till 1966 can be found in a monograph by Lowengrub and Sneddon [84].

Due to lack of computer facility at the research centre we could not introduce here applications of Integral and Series equations to crack problems of elasticity.