CHAPTER - 6

GENERALIZED SEVEN SERIES EQUATIONS
6.1 INTRODUCTION

It is a matter of fact that series solution have always been a useful tool to solve the mixed boundary value problems of mathematical physics like electrostatics, elasticity, fluid mechanics etc. In the past few years many researchers have taken considerable interest in it. There has been continuous efforts for developing and investigating new classes of series equations which arise in various mixed boundary value problems. In the present chapter, we have considered new class of seven series equations involving different kinds of polynomials.

Till now there has been no work on seven series equations [79]. This chapter contains two sets of seven series equations involving different polynomials.

(i) Seven Series Equations Involving Generalised Bateman K- Functions

Here we consider the sets of seven series equations involving Bateman K-functions. Srivastava [141] for the first time solved dual series equations involving Bateman K-functions. Later on Dwivedi [24], Dwivedi & Trivedi [28, 30] obtained the solutions of dual, triple and quadruple series which were in more general form from those considered previously. After that Dwivedi & Pandey [42, 44], Dwivedi & Singh [45] and Dwivedi & Chandel [51] considered some five series and six series
equation. Present problem is extensions of corresponding six series equations and solution is reduced to Fredholm integral equation of the second kind.

(ii) **Seven Series Equations Involving Generalised Laguerre Polynomials**

Seven series equations involving generalised Laguerre polynomials considered here, are generalisations of corresponding six series equations considered by Dwivedi & Singh [46, 48] and solution is reduced to Fredholm integral equations of the second kind. Initially, Lowndes considered the sets of dual & triple series equations involving. Laguerre polynomials. Later on some other sets of dual, triple, quadruple and five series equations involving Laguerre polynomials has been considered by srivastava [139, 140] Dwivedi & Trivedi [26, 27, 29] Dwivedi & Singh [48]. In the present problem, we have given the solution of new class of seven series equations and method used here is similar to that used previously.

6.2 **SEVEN SERIES EQUATION INVOLVING GENERALISED BATEMAN K-FUNCTIONS**

Solutions of seven series equations involving generalised Bateman K - functions has been obtained in this problem by reducing them to simultaneous Fredholm integral equations of second kind.

6.3 **THE EQUATIONS**

We shall solve the following sets of seven series equations involving generalised Bateman K – function.
\[ \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\beta + \sigma + n + 1)} K_{2(\alpha+\sigma)}^{(\beta+\sigma)}(x) = \begin{cases} f_1(x), & 0 < x < a \\ f_2(x), & a < x < b \\ f_3(x), & b < x < c \\ f_4(x), & c < x < d \\ f_5(x), & d < x < e \\ f_6(x), & e < x < f \\ f_7(x), & f < x < \infty \end{cases} \] (6.3.1)

where \( K_{2(\alpha+\sigma)}^{(\beta+\sigma)}(x) \) is the generalised Bateman K-function, \( \{A_n\} \) is unknown coefficients and \( f_i(x), i = 1,2,3 \ldots 7 \) are known functions. It is assumed that the series (6.3.1) to (6.3.7) are uniformly convergent and the known function \( f_i(x), (i = 1,2,3 \ldots 7) \) and their derivatives are continuous and integrable in the interval of their definitions.

6.4 PRELIMINARY RESULTS

In the course of the analysis we shall used the following results:

(i) The Orthogonality Relation for Generalised Bateman K-function is

\[ \int_0^{\infty} x^{-2\alpha - 2\sigma - 1} K_{2(\alpha+\sigma)}^{(\alpha+\sigma)}(x) K_{2(\alpha+\sigma)}^{(\beta+\sigma)}(x) \, dx = \frac{2^{2\alpha+2\sigma} \Gamma(n-\sigma)}{\Gamma(2\alpha + \sigma + n + 1)} \delta_{mn} \] (6.4.1)

where \( \alpha + \sigma + 1 > 0, \sigma + 1 \leq 0 \) and \( \delta_{mn} \) is the kronecker delta.

(ii) The Series

\[ S(r, x) = \sum_{n=0}^{\infty} \frac{\Gamma(2\nu + \sigma + n + 1)}{2^{2\beta+2\sigma} \Gamma(n-\sigma)} K_{2(\alpha+\sigma)}^{(\beta+\sigma)}(r) K_{2(\alpha+\sigma)}^{(\beta+\sigma)}(x) \] (6.4.2)
\[ S(r, x) = \frac{e^{-x} 2^{2\alpha-2\nu}}{\Gamma(2\alpha-2\nu) \Gamma(2\beta-2\nu)} \int_0^t E(\xi)(x - \xi)^{2\alpha - 2\nu - 1} (r - \xi)^{2\beta - 2\nu - 1} d\xi \]  

(6.4.3)

\[ S(r, x) = \frac{e^{-x} 2^{2\alpha-2\nu}}{\Gamma(2\alpha-2\nu) \Gamma(2\beta-2\nu)} S_t(r, x) \]  

(6.4.4)

where

\[ E(\xi) = e^{2\xi \xi^2 + 2\sigma + 1}, \quad t = \min (r, x) \]

(iii) If \( f(x) \) and \( f'(x) \) are continuous in \( a < x < b \) and if \( 0 < \sigma < 1 \), then the solutions of the Abel integral equations

\[ f(x) = \int_a^x \frac{F(y)}{(x - y)^\sigma} dy \]  

(6.4.5)

and

\[ f(x) = \int_x^b \frac{F(y)}{(y - x)^\sigma} dy \]  

(6.4.6)

are given by

\[ F(y) = \frac{\sin \pi \sigma}{\pi} \int_y^b \frac{f(x)}{(y - x)^{1-\sigma}} dx \]  

(6.4.7)

\[ F(y) = -\frac{\sin \pi \sigma}{\pi} \int_y^b \frac{f(x)}{(x - y)^{1-\sigma}} dx \]  

(6.4.8)

respectively.
6.5 **THE SOLUTION**

Let us suppose

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(2\beta + \sigma + n + 1)} K_{2(n+\sigma)}^{2(\alpha+\sigma)}(x) = \begin{cases} 
\phi_1(x), & a < x < b \\
\phi_2(x), & c < x < d \\
\phi_3(x), & e < x < f 
\end{cases} \tag{6.5.1}
\]

Using orthogonality relation in equations (6.3.1) to (6.3.4) and (6.5.1) to (6.5.3), we get

\[
A_n = \frac{\Gamma(2\beta + \sigma + n + 1)\Gamma(2\alpha + \sigma + n + 1)}{2^{2\alpha+2\sigma} \Gamma(n-\sigma)} \left[ \int_{a}^{b} f_1(x) + \int_{a}^{b} \phi_1(x) + \int_{b}^{c} f_3(x) 
+ \int_{c}^{d} \phi_2(x) + \int_{d}^{e} f_5(x) + \int_{e}^{f} \phi_3(x) + \int_{f}^{\infty} f_7(x) \right] \times x^{-2\alpha-2\sigma-1} K_{2(n-\sigma)}^{2(\alpha+\sigma)}(x) dx \tag{6.5.4}
\]

Substituting this expression for \(A_n\) in equations (6.3.5) to (6.3.7), we obtain

\[
\sum_{n=0}^{\infty} \frac{K_{2(n+\beta)}^{2(\alpha+\sigma)}(x)\Gamma(2\beta + \sigma + n + 1)\Gamma(2\alpha + \sigma + n + 1)}{\Gamma(2\nu + \sigma + n + 1)\Gamma(n-\sigma)2^{2\alpha+2\sigma}} \left[ \int_{a}^{b} f_1(x) + \int_{a}^{b} \phi_1(x) + \int_{b}^{c} f_3(x) + \int_{c}^{d} \phi_2(x) + \int_{d}^{e} f_5(x) + \int_{e}^{f} \phi_3(x) + \int_{f}^{\infty} f_7(x) \right] \times x^{-2\alpha-2\sigma-1} K_{2(n+\sigma)}^{2(\alpha+\sigma)}(x) dx = \begin{cases} 
\phi_2(x), & a < x < b \\
\phi_4(x), & c < x < d \\
\phi_6(x), & e < x < f 
\end{cases} \tag{6.5.5}
\]

Interchanging the order of integration and summation, and we using get (6.4.2)

\[
\int_{a}^{b} \phi_1(r) r^{-2\alpha-2\sigma-1} S(r,x) dr + \int_{c}^{d} \phi_2(r) r^{-2\alpha-2\sigma-1} S(r,x) dr \\
+ \int_{e}^{f} \phi_3(r) r^{-2\alpha-2\sigma-1} S(r,x) dr = \begin{cases} 
F(x), & a < x < b \\
G(x), & c < x < d \\
H(x), & e < x < f 
\end{cases} \tag{6.5.6}
\]
where
\[
F(x) = \frac{\Gamma(2\nu + \sigma + n + 1)^2}{\Gamma(2\beta + \sigma + n + 1)\Gamma(2\alpha + \sigma + n + 1)} f_2(x) \\
- \left[ \int_0^x f_1(r) + \int_0^x f_3(r) + \int_0^x f_5(r) + \int_0^x f_7(r) \right] r^{-2\alpha-2\sigma-1} S(r, x) dr, \tag{6.5.8}
\]

\[
G(x) = \frac{\Gamma(2\nu + \sigma + n + 1)^2}{\Gamma(2\beta + \sigma + n + 1)\Gamma(2\alpha + \sigma + n + 1)} f_4(x) \\
- \left[ \int_0^x f_1(r) + \int_0^x f_3(r) + \int_0^x f_5(r) + \int_0^x f_7(r) \right] r^{-2\alpha-2\sigma-1} S(r, x) dr, \tag{6.5.9}
\]

and

\[
H(x) = \frac{\Gamma(2\nu + \sigma + n + 1)^2}{\Gamma(2\beta + \sigma + n + 1)\Gamma(2\alpha + \sigma + n + 1)} f_6(x) \\
- \left[ \int_0^x f_1(r) + \int_0^x f_3(r) + \int_0^x f_5(r) + \int_0^x f_7(r) \right] r^{-2\alpha-2\sigma-1} S(r, x) dr, \tag{6.5.10}
\]

Now starting with equation (6.5.5), and with the help of equation (6.4.4),
\[
\int_a^x \phi_1(r) r^{-2\alpha-2\sigma-1} S_r(r, x) dr + \int_a^b \phi_1(r) r^{-2\alpha-2\sigma-1} S_r(r, x) dr \\
+ \int_a^x \phi_2(r) r^{-2\alpha-2\sigma-1} S_x(r, x) dr + \int_a^b \phi_2(r) r^{-2\alpha-2\sigma-1} S_x(r, x) dr \\
= \frac{\Gamma(2\alpha - 2\nu)\Gamma(2\beta - 2\nu)}{2^{\alpha - \nu}} e^xF(x), \quad a < x < b \tag{6.5.11}
\]

With the help of the result (6.4.3), equation (6.5.11) becomes
\[
\int_{\xi}^{r} \phi_1(r)^{-2a-2\alpha-1} dr \int_{0}^{\xi} E(\xi)(x - \xi)^{2\alpha-2\nu-1} (r - \xi)^{2\beta-2\nu-1} d\xi \\
+ \int_{\xi}^{r} \phi_1(r)^{-2a-2\alpha-1} dr \int_{0}^{\xi} E(\xi)(x - \xi)^{2\alpha-2\nu-1} (r - \xi)^{2\beta-2\nu-1} d\xi \\
+ \int_{\varepsilon}^{\xi} \phi_2(r)^{-2a-2\alpha-1} dr \int_{0}^{\xi} E(\xi)(x - \xi)^{2\alpha-2\nu-1} (r - \xi)^{2\beta-2\nu-1} d\xi \\
+ \int_{\varepsilon}^{\xi} \phi_3(r)^{-2a-2\alpha-1} dr \int_{0}^{\xi} E(\xi)(x - \xi)^{2\alpha-2\nu-1} (r - \xi)^{2\beta-2\nu-1} d\xi \\
= \frac{\Gamma(2\alpha - 2\nu)\Gamma(2\beta - 2\nu)}{2^{2\alpha - 2\nu}} e^{xF(x)}, \quad a < x < b
\]

Inverting the order of integration in above equation, we get

\[
\int_{0}^{r} \frac{E(\xi) d\xi}{(x - \xi)^{2a+2\nu}} \int_{0}^{s} \frac{\phi_1(r)^{-2a-2\alpha-1}}{(r - \xi)^{2\beta+2\nu}} dr = \frac{\Gamma(2\alpha - 2\nu)\Gamma(2\beta - 2\nu)}{2^{2\alpha - 2\nu}} e^{xF(x)}
\]

\[
- \int_{0}^{r} \frac{E(\xi) d\xi}{(x - \xi)^{2a+2\nu}} \int_{s}^{r} \frac{\phi_1(r)^{-2a-2\alpha-1}}{(r - \xi)^{2\beta+2\nu}} dr \\
- \int_{0}^{r} \frac{E(\xi) d\xi}{(x - \xi)^{2a+2\nu}} \int_{s}^{r} \frac{\phi_2(r)^{-2a-2\alpha-1}}{(r - \xi)^{2\beta+2\nu}} dr \\
- \int_{0}^{r} \frac{E(\xi) d\xi}{(x - \xi)^{2a+2\nu}} \int_{s}^{r} \frac{\phi_3(r)^{-2a-2\alpha-1}}{(r - \xi)^{2\beta+2\nu}} dr
\]

(6.5.12)

Let us assume

\[
\int_{\varepsilon}^{s} \frac{\phi_1(r)^{-2a-2\nu-1}}{(r - \xi)^{2\beta+2\nu}} dr = \phi_1(\xi)
\]

(6.5.13)

\[
\int_{\varepsilon}^{s} \frac{\phi_2(r)^{-2a-2\nu-1}}{(r - \xi)^{2\beta+2\nu}} dr = \phi_2(\xi)
\]

(6.5.14)

\[
\int_{\varepsilon}^{s} \frac{\phi_3(r)^{-2a-2\nu-1}}{(r - \xi)^{2\beta+2\nu}} dr = \phi_3(\xi)
\]

(6.5.15)

Using (6.5.13) in equation (6.5.12), we obtain
\[
\int_0^\infty \frac{\phi_1(x)E(x)}{(x-\xi)^{2\alpha+2\nu}} \, dx = \frac{\Gamma(2\alpha-2\nu)\Gamma(2\beta-2\nu)}{2^{2\alpha-2\nu}} e^x F(x)
\]

This is an Abel type of integral equation, and hence its solution is given by

\[
E(\xi) = \frac{\sin(1-2\alpha+2\nu)\pi}{\Gamma(2\alpha-2\nu)\Gamma(2\beta-2\nu)} \int_0^\infty \frac{e^x F(x)}{(x-\xi)^{2\alpha-2\nu}} \, dx
\]

Let

\[
F_1(\xi) = \frac{\sin(1-2\alpha+2\nu)\pi}{\Gamma(2\alpha-2\nu)\Gamma(2\beta-2\nu)} \int_0^\infty \frac{e^x F(x)}{(x-\xi)^{2\alpha-2\nu}} \, dx
\]

It can be easily shown that
\[
\frac{d}{d\xi} \int_0^\infty \frac{dx}{(\xi-x)^{2\alpha-2v}(x-y)^{1-2\alpha+2v}} = \frac{(a-y)^{2\alpha-2v}}{(\xi-y)(\xi-a)^{2\alpha-2v}}
\]  

(6.5.18)

and

\[
\int_0^\infty \frac{dx}{y(\xi-x)^{2\alpha-2v}(x-y)^{1-2\alpha+2v}} = \frac{\pi}{\sin(1-2\alpha+2v)\pi}
\]  

(6.5.19)

Using the results given by (6.5.18), (6.5.19) in equation (6.5.16), we get

\[
E(\xi)\phi_1(\xi) = F_1(\xi) = \frac{\sin(1-2\alpha+2v)\pi}{\pi} \times \left\{ \int_0^\infty \frac{E(y)(a-y)^{2\alpha-2v}}{(\xi-y)(\xi-a)^{2\alpha-2v}} dy \right. \\
\times \int_0^a \frac{\phi_1(r)^{2\alpha-2\alpha-1}}{(r-\xi)^{2\beta+2v}} dr + \int_0^\xi \frac{E(y)(a-y)^{2\alpha-2v}}{(\xi-y)(\xi-a)^{2\alpha-2v}} dy \\
\times \int_a^\infty \frac{\phi_2(r)^{2\alpha-2\alpha-1}}{(r-\xi)^{2\beta+2v}} dr + \int_\xi^\infty \frac{E(y)(a-y)^{2\alpha-2v}}{(\xi-y)(\xi-a)^{2\alpha-2v}} dy \\
\times \left. \int_0^\xi \frac{\phi_3(r)^{2\alpha-2\alpha-1}}{(r-\xi)^{2\beta+2v}} dr \right\} - \frac{d}{d\xi} \int_0^\xi E(y) dy \\
\times \left\{ \int_0^a \frac{d\phi_2(r)^{2\alpha-2\alpha-1}}{(r-\xi)^{2\beta+2v}} dr + \int_a^\infty \frac{\phi_3(r)^{2\alpha-2\alpha-1}}{(r-\xi)^{2\beta+2v}} dr \right\}
\]  

(6.5.20)

Equation (6.5.13) is an Abel type integral equation, hence its solution is given by

\[
\phi_1(\xi)^{-2\alpha-2\alpha-1} = -\frac{\sin(1-2\beta+2v)\pi}{\pi} \frac{d}{d\xi} \int_0^\xi \frac{\phi_1(\xi)}{(\xi-r)^{2\beta+2v}} d\xi
\]  

(6.5.21)

Similarly the solutions of equations (6.5.14) and (6.5.15) are given as

\[
\phi_2(\xi)^{2\alpha-2\alpha-1} = -\frac{\sin(1-2\beta+2v)\pi}{\pi} \frac{d}{d\xi} \int_0^\xi \frac{\phi_2(\xi)}{(\xi-r)^{2\beta+2v}} d\xi
\]  

(6.5.22)
and

\[
\phi_3(r) r^{-2\alpha - 2\alpha - 1} = -\frac{\sin(1 - 2\beta + 2\nu)\pi}{\pi} \int \frac{\phi_3(\xi)}{(\xi - r)^{2\beta - 2\nu}} d\xi
\]  

(6.5.23)

With the help of equation (6.5.21), we obtain

\[
\int_a^b \frac{\phi_1(r) r^{-2\alpha - 2\alpha - 1}}{(r - y)^{1 - 2\beta + 2\nu}} dr = \frac{\sin(1 - 2\beta + 2\nu)\pi}{\pi(a - y)^{2\beta + 2\nu}} \int_b^a \frac{\phi_1(\xi) d\xi}{(\xi - y)(\xi - a)^{2\beta - 2\nu}}
\]  

(6.5.24)

Similarly, by equation (6.5.22) and (6.5.23), we obtain

\[
\int_0^c \frac{\phi_2(r) r^{-2\alpha - 2\alpha - 1}}{(r - y)^{1 - 2\beta + 2\nu}} dr = \frac{\sin(1 - 2\beta + 2\nu)\pi}{\pi(c - y)^{2\beta + 2\nu}} \int_a^b \frac{\phi_3(\xi) d\xi}{(\xi - y)(\xi - c)^{2\beta - 2\nu}}
\]  

(6.5.25)

and

\[
\int_0^d \frac{\phi_3(r) r^{-2\alpha - 2\alpha - 1}}{(r - y)^{1 - 2\beta + 2\nu}} dr = \frac{\sin(1 - 2\beta + 2\nu)\pi}{\pi(c - y)^{2\beta + 2\nu}} \int_b^a \frac{\phi_3(\xi) d\xi}{(\xi - y)(\xi - c)^{2\beta - 2\nu}}
\]  

(6.5.26)

respectively.

Now using the results (6.5.24), (6.5.25) and (6.5.26) in equation (6.5.20), we obtain

\[
E(\xi) \overline{F_1(\xi)} = F_1(\xi) - \frac{\sin(1 - 2\alpha + 2\nu)\pi}{\pi^2 (\xi - a)^{-2\beta + 2\nu}} \times \left[ \int_0^a E(y) (a - y)^{2\alpha + 2\beta - 4\nu} \frac{\phi_1(t)}{(\xi - y)(t - a)^{2\beta - 2\nu}} dt \frac{d\xi}{\xi - y} \right]
\]

\[
+ \int_0^a E(y) (a - y)^{2\alpha - 2\nu} \frac{\phi_2(t)}{(\xi - y)(c - y)^{2\beta + 2\nu}} d\xi \int_c^d \frac{\phi_3(t)}{(t - y)(t - c)^{2\beta - 2\nu}} dt
\]
\[\begin{align*}
+ \int_{0}^{a} \frac{E(y)(a - y)^{2\alpha - 2\nu}}{(\xi - y)(e - y)^{2\beta + 2\nu}} dy & \int_{0}^{t} \frac{\phi_3(t)}{(t - y)(t - e)^{2\beta - 2\nu}} dt \\
- \frac{\sin((1 - 2\alpha + 2\nu)\pi}{\pi} & \int_{0}^{\xi} E(y)(c - y)^{2\beta - 2\nu} dy \\
\times \int_{c}^{t} \frac{\phi_2(t)}{(t - y)(t - e)^{2\beta - 2\nu}} dt + \frac{d}{dx} & \int_{e}^{\xi} E(y)(e - y)^{2\beta - 2\nu} dy \\
\times \int_{e}^{t} \frac{\phi_3(t)}{(t - y)(t - e)^{2\beta - 2\nu}} dt \quad & a < \xi < b
\end{align*}\] (6.5.27)

Now equation (6.5.27) can be rewritten as

\[\begin{align*}
E(\xi)\phi_1(\xi) + \int_{a}^{b} \phi_1(t)L(t, \xi) dt &= F_1(\xi) \\
- \int_{e}^{\xi} \phi_2(t)M(t, \xi) dt - \int_{e}^{t} \phi_3(t)N(t, \xi) dt, \quad & a < \xi < b
\end{align*}\] (6.5.28)

where

\[L(t, \xi) = \frac{\sin((1 - 2\alpha + 2\nu)\pi\sin(1 - 2\beta + 2\nu)\pi}{\pi^2(\xi - a)^{2\alpha - 2\nu}} \]

\[\frac{1}{(t - a)^{2\beta - 2\nu}} \int_{0}^{a} \frac{E(y)(a - y)^{2\alpha - 2\beta - 4\nu}}{(\xi - y)(t - y)} dy \] (6.5.29)

\[M(t, \xi) = \frac{\sin((1 - 2\alpha + 2\nu)\pi\sin(1 - 2\beta + 2\nu)\pi}{\pi^2(\xi - a)^{2\alpha - 2\nu}} \]

\[\int_{0}^{c} \frac{E(y)(a - y)^{2\alpha - 2\nu}(c - y)^{2\beta - 2\nu}}{(\xi - y)(t - y)} dy + \frac{\sin(1 - 2\beta + 2\nu)\pi}{\pi} \]

\[\int_{e}^{t} \frac{d}{dx} \frac{E(y)(c - y)^{2\beta - 2\nu}}{(t - y)} dy \] (6.5.30)
\[ N(t, \xi) = \frac{\sin(1 - 2\alpha + 2\nu) \pi \sin(1 - 2\beta + 2\nu) \pi}{\pi^2 (t - a)^{2\alpha - 2\nu}} \int_0^1 \frac{E(y)(a - y)^{2\alpha - 2\nu}(e - y)^{2\beta - 2\nu}}{(\xi - y)(t - y)} dy + \frac{\sin(1 - 2\beta + 2\nu) \pi}{\pi} \int_0^1 \frac{d}{(t - e)^{2\beta - 2\nu}} \int_0^e \frac{E(y)(e - y)^{2\beta - 2\nu}}{(t - y)} dy \] (6.5.31)

Again starting from equation (6.5.6), and using the result (6.4.4) we have

\[ \int_c^b \phi_1(\xi)^{-2\alpha - 2\sigma - 1} S_r(r, x) dr + \int_c^a \phi_2(\xi)^{-2\alpha - 2\sigma - 1} S_r(r, x) dr \\
+ \int_c^d \phi_2(\xi)^{-2\alpha - 2\sigma - 1} S_x(r, x) dr + \int_c^e \phi_3(\xi)^{-2\alpha - 2\sigma - 1} S_x(r, x) dr \\
= \frac{\Gamma(2\alpha - 2\nu)\Gamma(2\beta - 2\nu)}{2^{2\alpha - 2\nu}} e^x G(x), \quad c < x < d \] (6.5.32)

In view of the result (6.4.3), equation (6.5.32) becomes

\[ \int_c^b \phi_1(\xi)^{-2\alpha - 2\sigma - 1} d \int_0^e E(\xi)(x - \xi)^{2\alpha - 2\nu - 1} (r - \xi)^{2\beta - 2\nu - 1} d \xi \\
+ \int_c^a \phi_2(\xi)^{-2\alpha - 2\sigma - 1} d \int_0^e E(\xi)(x - \xi)^{2\alpha - 2\nu - 1} (r - \xi)^{2\beta - 2\nu - 1} d \xi \\
+ \int_c^d \phi_2(\xi)^{-2\alpha - 2\sigma - 1} d \int_0^e E(\xi)(x - \xi)^{2\alpha - 2\nu - 1} (r - \xi)^{2\beta - 2\nu - 1} d \xi \\
+ \int_c^e \phi_3(\xi)^{-2\alpha - 2\sigma - 1} d \int_0^e E(\xi)(x - \xi)^{2\alpha - 2\nu - 1} (r - \xi)^{2\beta - 2\nu - 1} d \xi \\
= \frac{\Gamma(2\alpha - 2\nu)\Gamma(2\beta - 2\nu)}{2^{2\alpha - 2\nu}} e^x G(x), \quad c < x < d \] (6.5.33)

Changing the order of integration and using (6.5.14) in equation (6.5.33), we get

\[ \int_c^x \frac{E(\xi)\phi_2(\xi)}{(x - \xi)^{-2\alpha + 2\nu}} d\xi = \frac{\Gamma(2\alpha - 2\nu)\Gamma(2\beta - 2\nu)}{2^{2\alpha - 2\nu}} e^x G(x) \]
Solution of this Abel-type integral equation is given as

\[
E(\xi, \eta) = \frac{\sin(1 - 2\alpha + 2\nu)\pi}{\pi} \left[ \frac{\Gamma(2\alpha - 2\nu)\Gamma(2\beta - 2\nu)}{2^{2\alpha - 2\nu}} \right] \frac{d}{d\xi} \int_0^{\xi} \frac{e^x G(x) dx}{(\xi - x)^{2\alpha - 2\nu}} \]

\[
- \int_0^a E(\eta) \frac{d}{d\eta} \int_0^\eta \frac{dx}{(\xi - x)^{2\alpha - 2\nu}} (x - y)^{1 - 2\alpha + 2\nu} \frac{b\phi_1(r) r^{-2\alpha - 2\nu - 1}}{(r - \xi)^{-2\beta + 2\nu}} dr
\]

\[
- \int_a^b E(\eta) \frac{d}{d\eta} \int_0^\eta \frac{dx}{(\xi - x)^{2\alpha - 2\nu}} (x - y)^{1 - 2\alpha + 2\nu} \frac{b\phi_1(r) r^{-2\alpha - 2\nu - 1}}{(r - \xi)^{-2\beta + 2\nu}} dr
\]

\[
- \int_b^{c} E(\eta) \frac{d}{d\eta} \int_0^\eta \frac{dx}{(\xi - x)^{2\alpha - 2\nu}} (x - y)^{1 - 2\alpha + 2\nu} \frac{d\phi_2(r) r^{-2\alpha - 2\nu - 1}}{(r - \xi)^{-2\beta + 2\nu}} dr
\]

\[
- \int_c^{\infty} E(\eta) \frac{d}{d\eta} \int_0^\eta \frac{dx}{(\xi - x)^{2\alpha - 2\nu}} (x - y)^{1 - 2\alpha + 2\nu} \frac{d\phi_2(r) r^{-2\alpha - 2\nu - 1}}{(r - \xi)^{-2\beta + 2\nu}} dr
\]

\[
- \int_0^\xi E(\eta) \frac{d}{d\eta} \int_0^\eta \frac{dx}{(\xi - x)^{2\alpha - 2\nu}} (x - y)^{1 - 2\alpha + 2\nu} \frac{d\phi_3(r) r^{-2\alpha - 2\nu - 1}}{(r - \xi)^{-2\beta + 2\nu}} dr
\]

\[
- \int_0^{\xi} E(\eta) \frac{d}{d\eta} \int_0^\eta \frac{dx}{(\xi - x)^{2\alpha - 2\nu}} (x - y)^{1 - 2\alpha + 2\nu} \frac{d\phi_3(r) r^{-2\alpha - 2\nu - 1}}{(r - \xi)^{-2\beta + 2\nu}} dr
\]

\[
(6.5.35)
\]

Let

\[
G_1(\xi) = \frac{\sin(1 - 2\alpha + 2\nu)\pi}{\pi} \frac{\Gamma(2\alpha - 2\nu)\Gamma(2\beta - 2\nu)}{2^{2\alpha - 2\nu}} \frac{d}{d\xi} \int_0^{\xi} \frac{e^x G(x) dx}{(\xi - x)^{2\alpha - 2\nu}}
\]

\[
(6.5.36)
\]
It can easily be shown that

\[
\frac{d}{d\xi} \int_{0}^{y} \frac{dx}{(\xi - x)^{2\alpha - 2\nu}(x - y)^{1 - 2\alpha + 2\nu}} = \frac{(c - y)^{2\alpha - 2\nu}}{(\xi - y)(\xi - c)^{2\alpha - 2\nu}} \quad (6.5.37)
\]

using the expressions from equation (6.5.19), (6.5.36) and (6.5.37) in equation (6.5.35), we get

\[
E(\xi)\Phi_2(\xi) = G_1(\xi) - \frac{\sin(1-2\alpha + 2\nu)\pi}{\pi} \left\{ \int_{0}^{a} \frac{E(y)(c - y)^{2\alpha - 2\nu}}{(\xi - y)(\xi - c)^{2\alpha - 2\nu}} dy \right. \\
+ \int_{b}^{c} \frac{E(y)(c - y)^{2\alpha - 2\nu}}{(r - \xi)^{1 - 2\alpha + 2\nu}} dr + \int_{c}^{d} \frac{E(y)(c - y)^{2\alpha - 2\nu}}{(r - \xi)^{1 - 2\alpha + 2\nu}} dr \\
- \int_{0}^{a} \frac{\Phi_1(r)^{2\alpha - 2\sigma - 1}}{(r - \xi)^{1 - 2\beta + 2\nu}} dr + \int_{b}^{c} \frac{\Phi_2(r)^{2\alpha - 2\sigma - 1}}{(r - \xi)^{1 - 2\beta + 2\nu}} dr \left. \right\}, \ c < \xi < d \quad (6.5.38)
\]

Now using the results given by equations (6.5.24) to (6.5.26) in equation (4.3.48), we get

\[
E(\xi)\Phi_2(\xi) = G_1(\xi) - \frac{\sin(1-2\alpha + 2\nu)\pi \sin(1-2\beta + 2\nu)\pi}{\pi^2(\xi - c)^{2\alpha - 2\nu}} \\
\times \left\{ \int_{0}^{a} \frac{E(y)(c - y)^{2\alpha - 2\nu}}{(\xi - y)(a - y)^{-2\beta + 2\nu}} dy \int_{a}^{b} \frac{\Phi_1(t)}{(t - y)(t - a)^{2\beta - 2\nu}} dt \\
+ \int_{0}^{b} \frac{E(y)(c - y)^{2\alpha + 2\beta - 4\nu}}{(\xi - y)^{1 - 2\alpha + 2\nu}} dy \int_{c}^{d} \frac{\Phi_2(t)}{(t - y)(t - c)^{2\beta - 2\nu}} dt \\
+ \int_{0}^{c} \frac{E(y)(c - y)^{2\alpha - 2\nu}}{(\xi - y)(c - y)^{-2\beta + 2\nu}} dy \int_{c}^{d} \frac{\Phi_3(t)}{(t - y)(t - e)^{2\beta - 2\nu}} dt \right\} 
\]
\[- \frac{\sin(1-2\beta+2\nu)}{\pi} \left\{ \frac{d}{d\xi} \int_{c}^{d} E(y)(a-y)^{2\beta-2\nu} dy \right\}^{b} \frac{\phi_{1}(t) \phi_{2}(t) \phi_{3}(t)}{(t-y)(t-a)^{2\beta-2\nu}} \]

\[+ \frac{d}{d\xi} \int_{c}^{d} E(y)(e-y)^{2\beta-2\nu} dy \times \int_{e}^{f} \frac{\phi_{1}(t) \phi_{2}(t) \phi_{3}(t)}{(t-y)(t-e)^{2\beta-2\nu}} dt \]  

\[c < \xi < d \quad (6.5.39)\]

Now above equation can be rewritten as

\[E(\xi)\phi_{2}(\xi) + \int_{c}^{d} \phi_{2}(t) Q(t, \xi) dt = G_{1}(\xi) - \int_{e}^{f} P(t, \xi) \phi_{1}(t) dt - \int_{e}^{f} R(t, \xi) \phi_{3}(t) dt, \]

\[c < \xi < d \quad (6.5.40)\]

where

\[P(\xi, t) = \frac{\sin(1-2\alpha+2\nu)}{\pi^2 (\xi-c)^{2a-2\nu} (t-a)^{2\beta-2\nu}} \int_{c}^{d} E(y)(c-y)^{2a-2\nu} (a-y)^{2\beta-2\nu} (\xi-y)(t-y)^{2\beta-2\nu} dy \]

\[+ \frac{\sin(1-2\beta+2\nu)}{\pi} \int_{c}^{d} \frac{d}{d\xi} E(y)(a-y)^{2\beta-2\nu} dy \times \int_{e}^{f} \frac{\phi_{1}(t)}{(t-y)(t-a)^{2\beta-2\nu}} dt \quad (6.5.41)\]

\[P(\xi, t) = \frac{\sin(1-2\alpha+2\nu)}{\pi^2 (\xi-c)^{2a-2\nu} (t-c)^{2\beta-2\nu}} \int_{c}^{d} E(y)(c-y)^{2a-2\nu} (c-y)^{2\beta-2\nu} dy \quad (6.5.42)\]

\[R(\xi, t) = \frac{\sin(1-2\alpha+2\nu)}{\pi^2 (\xi-c)^{2a-2\nu} (t-c)^{2\beta-2\nu}} \int_{c}^{d} E(y)(c-y)^{2a-2\nu} (e-y)^{2\beta-2\nu} dy \]

\[+ \frac{\sin(1-2\beta+2\nu)}{\pi} \int_{c}^{d} E(y)(e-y)^{2\beta-2\nu} dy \times \int_{e}^{f} \frac{\phi_{1}(t)}{(t-y)(t-e)^{2\beta-2\nu}} dt \quad (6.5.43)\]

Again starting from equation (6.5.7) and using the result (6.4.4), we have
\[ \int_0^b \phi_1(r)e^{-2\alpha - 2\sigma - 1}Sr(r, x)dr + \int_0^b \phi_2(r)e^{-2\alpha - 2\sigma - 1}Sr(r, x)dr \\
+ \int_0^b \phi_3(r)e^{-2\alpha - 2\sigma - 1}S_\tau(r, x)dr + \int_0^b \phi_4(r)e^{-2\alpha - 2\sigma - 1}S_\tau(r, x)dr \\
= \frac{\Gamma(2\alpha - 2\nu)\Gamma(2\beta - 2\nu)}{2^{2\alpha - 2\nu}} e^xH(x), \quad \nu \leq x \leq f \]

Using the result (6.4.3) in above equation, we get

\[ \int_0^b \phi_1(r)e^{-2\alpha - 2\sigma - 1}dr \int_0^\infty E(\xi)(x - \xi)^{2\alpha - 2\nu - 1}(r - \xi)^{2\beta - 2\nu - 1}d\xi \\
+ \int_0^b \phi_2(r)e^{-2\alpha - 2\sigma - 1}dr \int_0^\infty E(\xi)(x - \xi)^{2\alpha - 2\nu - 1}(r - \xi)^{2\beta - 2\nu - 1}d\xi \\
+ \int_0^b \phi_3(r)e^{-2\alpha - 2\sigma - 1}dr \int_0^\infty E(\xi)(x - \xi)^{2\alpha - 2\nu - 1}(r - \xi)^{2\beta - 2\nu - 1}d\xi \\
+ \int_0^b \phi_4(r)e^{-2\alpha - 2\sigma - 1}dr \int_0^\infty E(\xi)(x - \xi)^{2\alpha - 2\nu - 1}(r - \xi)^{2\beta - 2\nu - 1}d\xi \\
= \frac{\Gamma(2\alpha - 2\nu)\Gamma(2\beta - 2\nu)}{2^{2\alpha - 2\nu}} e^xH(x), \quad \nu \leq x \leq f \]

Changing the order of integration and using the expression given by (6.5.15),

we obtain

\[ \int_0^\infty \frac{E(\xi)\phi_1(\xi)}{(x - \xi)^{2\alpha + 2\nu}}d\xi = \frac{\Gamma(2\alpha - 2\nu)\Gamma(2\beta - 2\nu)}{2^{2\alpha - 2\nu}} e^xH(x) \\
- \int_0^\infty \frac{E(\xi)}{(x - \xi)^{1 - 2\beta + 2\nu}}d\xi \int_0^\infty \phi_1(r)e^{-2\alpha - 2\sigma - 1} \\
- \int_0^\infty \frac{E(\xi)}{(x - \xi)^{1 - 2\beta + 2\nu}}d\xi \int_0^\infty \phi_2(r)e^{-2\alpha - 2\sigma - 1} \\
- \int_0^\infty \frac{E(\xi)}{(x - \xi)^{1 - 2\beta + 2\nu}}d\xi \int_0^\infty \phi_3(r)e^{-2\alpha - 2\sigma - 1} \\
- \int_0^\infty \frac{E(\xi)}{(x - \xi)^{1 - 2\beta + 2\nu}}d\xi \int_0^\infty \phi_4(r)e^{-2\alpha - 2\sigma - 1}, \quad (6.5.45) \]
Solution of this Abel–type integral equation is given as

\[
E(\xi)\phi_3(\xi) = \frac{\sin(1 - 2\alpha + 2\nu)\pi}{\pi} \int \frac{\Gamma(2\alpha - 2\nu)\Gamma(2\beta - 2\nu)}{2^{2\alpha - 2\nu}} \cdot \frac{d}{d\xi} \int_0^\infty \frac{e^x H(x)}{(\xi - x)^{2\alpha - 2\nu}} dx \\
- \int_0^\infty E(y) dy \frac{d}{d\xi} \int_0^\infty \frac{\phi_1(r)}{(r - \xi)^{2\alpha - 2\nu} (r - x)^{1 - 2\alpha + 2\nu}} dr \\
- \int_0^\infty E(y) dy \frac{d}{d\xi} \int_0^\infty \frac{\phi_2(r)}{(r - \xi)^{2\alpha - 2\nu} (r - x)^{1 - 2\alpha + 2\nu}} dr \\
- \int_0^\infty E(y) dy \frac{d}{d\xi} \int_0^\infty \frac{\phi_3(r)}{(r - \xi)^{2\alpha - 2\nu} (r - x)^{1 - 2\alpha + 2\nu}} dr
\]

(6.5.46)

Let

\[
H_1(\xi) = \frac{\sin(1 - 2\alpha + 2\nu)\pi}{\pi} \frac{\Gamma(2\alpha - 2\nu)\Gamma(2\beta - 2\nu)}{2^{2\alpha - 2\nu}} \frac{d}{d\xi} \int_0^\infty \frac{e^x H(x)}{(\xi - x)^{2\alpha - 2\nu}} dx \quad (6.5.47)
\]

we know that

\[
\frac{d}{d\xi} \int_0^\infty \frac{dx}{(\xi - x)^{2\alpha - 2\nu} (x - y)^{1 - 2\alpha + 2\nu}} = \frac{(e - y)^{2\alpha - 2\nu}}{(\xi - y)(\xi - e)^{2\alpha - 2\nu}}
\]

(6.5.48)

Using the expressions from (6.5.19), (6.5.47) and (6.5.48) in equation (6.5.46), we get

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\[ E(\xi) \phi_2(\xi) = G_1(\xi) - \frac{\sin(1 - 2\alpha + 2\nu)\pi}{\pi} \left\{ \int_0^{\xi-e} \frac{E(y)(c-y)^{2\alpha-2\nu}}{(\xi-y)(\xi-c)^{2\alpha-2\nu}} \, dy + \int_{\xi-e}^{\xi} \frac{E(y)(c-y)^{2\alpha-2\nu}}{(\xi-y)(\xi-c)^{2\alpha-2\nu}} \, dy \right. \\
\left. \times \int_0^{\xi-e} \frac{\phi_1(r)r^{-2\alpha-2\sigma-1}}{(r-\xi)^{1-2\beta+2\nu}} \, dr + \int_{\xi-e}^{\xi} \frac{E(y)(c-y)^{2\alpha-2\nu}}{(\xi-y)(\xi-c)^{2\alpha-2\nu}} \, dy \right. \\
\left. \times \int_0^{\xi-e} \frac{\phi_2(r)r^{-2\alpha-2\sigma-1}}{(r-\xi)^{1-2\beta+2\nu}} \, dr + \int_{\xi-e}^{\xi} \frac{E(y)(c-y)^{2\alpha-2\nu}}{(\xi-y)(\xi-c)^{2\alpha-2\nu}} \, dy \right. \\
\left. \times \int_0^{\xi-e} \frac{\phi_3(r)r^{-2\alpha-2\sigma-1}}{(r-\xi)^{1-2\beta+2\nu}} \, dr + \int_{\xi-e}^{\xi} \frac{\phi_3(r)r^{-2\alpha-2\sigma-1}}{(r-\xi)^{1-2\beta+2\nu}} \, dr \right\} d\xi \]
\[ e < \xi < f \quad (6.5.49) \]

Now substituting the expressions from (6.5.24) to (6.5.26) in equation (6.5.49), we get

\[ E(\xi) \phi_3(\xi) = H_1(\xi) - \frac{\sin(1 - 2\alpha + 2\nu)\pi \sin(1 - 2\beta + 2\nu)\pi}{\pi^2(\xi-e)^{2\alpha-2\nu}} \]
\[ \times \left\{ \int_0^{\xi-e} \frac{E(y)(e-y)^{2\alpha-2\nu}}{(\xi-y)(a-y)^{2\beta-2\nu}} \, dy \int_a^b \frac{\phi_1(t)}{(t-y)(t-a)^{2\beta-2\nu}} \, dt \right. \\
\left. \int_0^{\xi-e} \frac{E(y)(e-y)^{2\alpha-2\nu}}{(\xi-y)(c-y)^{2\beta-2\nu}} \, dy \int_c^e \frac{\phi_2(t)}{(t-y)(t-c)^{2\beta-2\nu}} \, dt \right. \\
\left. + \int_{\xi-e}^{\xi} \frac{E(y)(e-y)^{2\alpha-2\nu}}{(\xi-y)(c-y)^{2\beta-2\nu}} \, dy \int_c^e \frac{\phi_2(t)}{(t-y)(t-c)^{2\beta-2\nu}} \, dt \right. \\
\left. \int_0^e \frac{E(y)(e-y)^{2\alpha-2\nu}}{(\xi-y)} \, dy \int_e^\xi \frac{\phi_3(t)}{(t-y)(t-e)^{2\beta-2\nu}} \, dt \right\} \\
\left. - \frac{\sin(1 - 2\beta + 2\nu)\pi}{\pi} \left\{ \int_0^{\xi-e} \frac{d}{d\xi} E(y)(a-y)^{2\beta-2\nu} \, dy \right. \\
\left. \times \int_a^b \frac{\phi_1(t)}{(t-y)(t-a)^{2\beta-2\nu}} \, dt \right. \\
\left. + \int_{\xi-e}^{\xi} \frac{d}{d\xi} E(y)(c-y)^{2\beta-2\nu} \, dy \right. \\
\left. \times \int_c^e \frac{\phi_2(t)}{(t-y)(t-c)^{2\beta-2\nu}} \, dt \right\}, \quad e < \xi < f \quad (6.5.50) \]
Now equation (6.5.50) can be written as

\[
E(\xi)\phi_3(\xi) + \int_c^t W(t, \xi)\phi_3(t) dt = H(t, \xi) \\
- \int_c^t U(t, \xi)\phi_1(t) dt - \int_c^t V(t, \xi)\phi_2(t) dt, \quad e < t < f
\]  

(6.5.51)

where

\[
U(t, \xi) = \frac{\sin(1 - 2\alpha + 2\nu)\pi\sin(1 - 2\beta + 2\nu)\pi}{\pi^2(\xi - e)^{2\alpha - 2\nu}(t - a)^{2\beta - 2\nu}} \int_0^\infty \frac{E(y)(e - y)^{2\alpha - 2\nu}(a - y)^{2\beta - 2\nu}}{(\xi - y)(t - y)} dy \\
+ \frac{\sin(1 - 2\beta + 2\nu)\pi}{\pi} \frac{1}{(t - a)^{2\beta - 2\nu}} \int \frac{d}{d\xi} \frac{E(y)(a - y)^{2\beta - 2\nu}}{(t - y)} dy
\]

(6.5.52)

\[
V(t, \xi) = \frac{\sin(1 - 2\alpha + 2\nu)\pi\sin(1 - 2\beta + 2\nu)\pi}{\pi^2(\xi - e)^{2\alpha - 2\nu}(t - c)^{2\beta - 2\nu}} \int_0^\infty \frac{E(y)(e - y)^{2\alpha - 2\nu}(c - y)^{2\beta - 2\nu}}{(\xi - y)(t - y)} dy \\
+ \frac{\sin(1 - 2\beta + 2\nu)\pi}{\pi} \frac{1}{(t - c)^{2\beta - 2\nu}} \int \frac{d}{d\xi} \frac{E(y)(c - y)^{2\beta - 2\nu}}{(t - y)} dy
\]

(6.5.53)

\[
W(t, \xi) = \frac{\sin(1 - 2\alpha + 2\nu)\pi\sin(1 - 2\beta + 2\nu)\pi}{\pi(\xi - e)^{2\alpha - 2\nu}(t - e)^{2\beta - 2\nu}} \int \frac{E(y)(e - y)^{2\alpha + 2\beta - 4\nu}}{(\xi - y)(t - y)} dy
\]

(6.5.54)

Equations (6.5.28), (6.5.40) and (6.5.51) are simultaneous Fredholm integral equations of the Second kind which determine \( \phi_1(\xi), \phi_2(\xi) \) and \( \phi_3(\xi) \). Equations (6.5.21), (6.5.22) and (6.5.23) yield the values of \( \phi_1(t), \phi_2(t) \) and \( \phi_3(t) \) respectively. After that, we can compute the unknown coefficient \( A_n \) with the help of equation (6.5.4).
PARTICULAR CASES

If we let $f \rightarrow 1$ in equations (6.3.1) and (6.3.2), that reduce to the corresponding Six Series equations and solution follows for such equations. Similarly, we can reduce above equations to dual, triple, quadruple and five series equations and their solutions can be written down with the help of solutions obtained here.

6.6 SEVEN SERIES EQUATIONS INVOLVING GENERALISED LAGUERRE POLYNOMIALS

Solutions of seven series equations involving generalised Laguerre functions has been obtained in this problem by reducing them to simultaneous Fredholm integral equations of second kind.

6.7 THE EQUATIONS

We shall solve the following sets of seven series equations involving generalised Laguerre polynomials

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha + n + 1)} L_n^\alpha(x) = \begin{cases} f_1(x), & 0 < x < a \\ f_3(x), & b < x < c \\ f_5(x), & d < x < e \\ f_7(x), & f < x < \infty \end{cases} \quad (6.7.1)
\]

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta + n + 1)} L_n^\beta(x) = \begin{cases} f_2(x), & a < x < b \\ f_4(x), & c < x < d \\ f_6(x), & e < x < f \end{cases} \quad (6.7.2)
\]
Where $L_n^\nu(x)$ is the generalised Laguerre polynomial, $A_n$ is unknown coefficient, $f_i(x)$, $i = 1,2,...,7$ are known functions and the parameters $\alpha, \beta, \nu, \sigma, > -1$. Here we shall solve the given set of series equations by reducing them to simultaneous Fredholm integral equations of the second kind.