Chapter 2

Construction of Minimal Balanced Crossover Designs for Carryover Models

2.1 Introduction

Guided by the constraint on the number of repeated measurements, a crossover experiment uses one of the three types of crossover designs, (i) $p < t$, (ii) $p = t$ and (iii) $p > t$. Literature review of their applications indicates that each of them have specific applications. The crossover designs having lesser periods are suitable in clinical trials and pharmaceutical studies, because each unit receives only a few of all the treatments. The crossover designs having periods equal to the number of treatments, so that each unit receives every treatment once, are employed in agriculture and for the sensory evaluation of food and products. The crossover designs having periods more than the number of treatments, so that a sequence of treatments including repetitions can be given are, useful in animal nutrition and educational experiments. Hedayat and Afsarinejad [18] emphasized on the construction of minimal balanced crossover designs, i.e. balanced crossover designs which require minimum possible number of experimental units for comparing a set of treatments.

Bailey [2] introduced the term *terrace* and used it for the construction of Quasi-complete Latin squares. Morgan [44] generalized the idea of *terrace* to *m-terrace* and used it for the construction of balanced polycross designs. The present chapter introduces modified forms of *terrace* called *complementary pair of terraces* and *complementary trio of terraces* and provides a simple method for the construction of the four series of minimal balanced crossover designs. Two series based on the modified forms of *terrace* are new series of crossover designs for even number of treatments in lesser periods. Some of the crossover designs of new series are strongly balanced crossover designs. The other two series of crossover designs are based on the directed *m-terraces* of Morgan [44]. Some of them are the same as those in Hedayat and Afsarinejad [18] and William [54], but our method of construction is quite simple and yields some better crossover designs. It is shown that the method of construction of the four series is common but each uses a specific *terrace*. Each series is illustrated by examples. Further, a comparison of crossover designs with those of Afsarinejad and Hedayat [1] in terms of efficiency of separability is given. At the end, a list of *directed 2-terraces*, *complementary pair of terraces* and *complementary trio of terraces* for the construction of minimal balanced crossover designs having three to nine treatments is provided for the benefit of the readers.

2.2 Preliminaries and new definitions

2.2.1 Model and ES

The crossover designs constructed in this chapter are suitable for the estimation of direct and the first order carryover effects under model 1.2.2 and 1.2.3.

Let we obtain simple formula to calculate the ES of direct and the first order carryover effects for balanced crossover designs. The contingency table relating direct treatment and carryover treatment incidence for balanced $COD(t, n, p)$ is of the form:

Direct	Carryover Treatment					
Treatment	No	1	2	...	t	Total
1	λ_1	λ_3	λ_2	...	λ_2	$\lambda_1 + \lambda_3 + (t-1)\lambda_2$
2	λ_1	λ_2	λ_3	...	λ_2	$\lambda_1 + \lambda_3 + (t-1)\lambda_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
t	λ_1	λ_2	λ_2	...	λ_3	$\lambda_1 + \lambda_3 + (t-1)\lambda_2$
Total	$t\lambda_1$	$\lambda_3 + (t-1)\lambda_2$	$\lambda_3 + (t-1)\lambda_2$	...	$\lambda_3 + (t-1)\lambda_2$	$t(\lambda_1 + \lambda_3 + (t-1)\lambda_2)$

Note that, No column in contingency table shows the observation which is free of carryover. The expected frequencies under the null hypothesis under the independence model are presented below.

Direct	Carryover Treatment				
Treatment	No	1	2	...	t
1	λ_1	$\frac{\lambda_3 + (t-1)\lambda_2}{t}$	$\frac{\lambda_3 + (t-1)\lambda_2}{t}$	...	$\frac{\lambda_3 + (t-1)\lambda_2}{t}$
2	λ_1	$\frac{\lambda_3 + (t-1)\lambda_2}{t}$	$\frac{\lambda_3 + (t-1)\lambda_2}{t}$	...	$\frac{\lambda_3 + (t-1)\lambda_2}{t}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
t	λ_1	$\frac{\lambda_3 + (t-1)\lambda_2}{t}$	$\frac{\lambda_3 + (t-1)\lambda_2}{t}$	...	$\frac{\lambda_3 + (t-1)\lambda_2}{t}$

Then Pearson chi-square value is,

$$\begin{aligned}\chi^2 &= \sum\sum[(O_{ij} - E_{ij})^2/E_{ij}] \\ &= t(0) + t\left(\frac{(t-1)^2(\lambda_3 - \lambda_2)^2}{t(\lambda_3 + (t-1)\lambda_2)}\right) + t(t-1)\left(\frac{(\lambda_3 - \lambda_2)^2}{t(\lambda_3 + (t-1)\lambda_2)}\right) \\ &= \frac{t(t-1)(\lambda_3 - \lambda_2)^2}{\lambda_3 + (t-1)\lambda_2}\end{aligned}$$

The Cramers V is,

$$V_c = \left[\frac{(\lambda_3 - \lambda_2)^2}{(\lambda_3 + (t-1)\lambda_2)(\lambda_1 + \lambda_3 + (t-1)\lambda_2)}\right]^{1/2}$$

Hence, the simple formula to calculate the ES of direct and first order carryover effects for balanced crossover designs is given by

$$ES = \left[1 - \left\{\frac{(\lambda_3 - \lambda_2)^2}{(\lambda_3 + (t-1)\lambda_2)(\lambda_1 + \lambda_3 + (t-1)\lambda_2)}\right\}^{\frac{1}{2}}\right] \times 100\%. \qquad (2.2.1)$$

For example, the ES of the *cod*{AB, BA} calculated by substituting $\lambda_1 = 1$, $\lambda_2 = 1$ and $\lambda_3 = 0$ in the equation (2.2.1) is 29%, while the ES of the *cod*{AB, BA, AA, BB} obtained by substituting $\lambda_1 = 1$, $\lambda_2 = 1$ and $\lambda_3 = 1$ is 100%. The low ES indicates unsuitability while the high ES indicates suitability of crossover design for the estimation of direct and first order carryover effect of treatments under carryover models 1.2.2 and 1.2.3.

2.2.2 Definitions of terraces

Let $\mathbb{Z}_t$ be a group of order t with elements $0, 1, \ldots, t-1$. Let $x = (x_1, x_2, \ldots, x_p)$ be some arrangement of the elements of group $\mathbb{Z}_t$ with both repetitions and non-occurrences allowed. Corresponding to each such x, let x^* be the arrangement $(x_2 -$

$x_1, x_3 - x_2, \ldots, x_p - x_{p-1})$. Note that, the successive repetitions of elements in x contribute to the value of λ_3 (i.e., count of zero in x^*).

Definition 2.2.1. An arrangement $a = (a_1, a_2, \ldots, a_p)$ of p elements of $\mathbb{Z}_t$ where $p = 1 + \frac{m(t-1)}{2}$ for some positive even integer m is said to be a *directed m-terrace* if a^* modulo t consists of each non-zero element of $\mathbb{Z}_t$ exactly $m/2$ times.

For example, consider an arrangement $a = (0, 1, 3, 2, 3, 1, 0, 2, 3, 2)$ from the group of order 4, i.e., $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. Then $a^* = (1, 2, -1, 1, -2, -1, 2, 1, -1)$ and a^* modulo 4 is (1,2,3,1,2,3,2,1,3) because $1 \equiv -3$ (modulo 4), $2 \equiv -2$ (modulo 4) and $3 \equiv -1$ (modulo 4). Here, a^* modulo 4 consists of each non-zero element of $\mathbb{Z}_4$ exactly 3 times and hence a is a directed 6-terrace. Using this idea of directed m-terrace, three new forms of terraces, namely: uniform 2-terrace, complementary pair of terraces and complementary trio of terraces are defined.

Definition 2.2.2. A directed 2-terrace over $\mathbb{Z}_t$ for even t is said to be a *uniform 2-terrace* if it consists of all the elements of $\mathbb{Z}_t$ once.

For example, a directed 2-terrace $a = (0, 1, 3, 2)$ over $\mathbb{Z}_4$ is a uniform 2-terrace because it contains all the elements of $\mathbb{Z}_4$ once. A list of directed 2-terraces with uniform 2-terraces from groups of order three to eight is provided in Table 2.3.

Definition 2.2.3. A pair of arrangements $a = (a_1, a_2, \ldots, a_p)$ and $b = (b_1, b_2, \ldots, b_p)$ of the elements of $\mathbb{Z}_t$, where $p = int(\frac{t}{2}) + 1$ is said to be a *complementary pair of terraces* if (a^*, b^*) modulo t consists of each non-zero element of $\mathbb{Z}_t$ once, for odd t, while each element of $\mathbb{Z}_t$ once, for even t.

For example, a pair of arrangements $a = (0, 3, 1)$ and $b = (2, 3, 3)$ from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ is a complementary pair of terraces because (a^*, b^*) modulo 4 is (3,2,1,0).

A list of complementary pair of terraces from groups of order three to nine is provided in Table 2.4.

Definition 2.2.4. A trio of arrangements $a = (a_1, a_2, \ldots, a_p)$, $b = (b_1, b_2, \ldots, b_p)$ and $c = (c_1, c_2, \ldots, c_p)$ of the elements of $\mathbb{Z}_t$ for even $t(\geq 4)$, where $p = \frac{t}{2}$ is said to be a *complementary trio of terraces* if (a^*, b^*, c^*) modulo t consists of each non-zero element of $\mathbb{Z}_t$ once and zero element $\frac{3}{2}(t-2) - (t-1)$ times.

For example, a trio of arrangements $a = (0, 1)$, $b = (1, 0)$ and $c = (0, 2)$ from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ is a complementary trio of terraces because (a^*, b^*, c^*) modulo 4 is (1,3,2). A list of complementary trio of terraces from groups of order four to eight is provided in Table 2.5.

2.3 Method of construction

In this section, two series of minimal balanced $COD(t, n, p(< t))$, and one series each of minimal balanced $COD(t, t, t)$ and minimal balanced $COD(t, n, p(> t))$ are constructed using a simple method of construction. It is shown that, the simple method of construction applied on a specific form of terraces results in a specific series of crossover designs.

2.3.1 Minimal balanced $COD(t, n, p(< t))$

In several experimental situations, it is not convenient to measure each experimental unit for all treatments, especially when the number of treatments is large. Balanced crossover designs in which each experimental unit is measured only for fractions of all treatments is desirable.

Theorem 2.3.1. *A series of minimal balanced* $COD(t, 2t, int(\frac{t}{2}) + 1)$ *can be constructed by adding successively each element of* $\mathbb{Z}_t$ *to a complementary pair of terraces reduced modulo* t.

Proof. Consider a complementary pair of terraces $a = (a_1, a_2, \ldots, a_p)$ and $b = (b_1, b_2, \ldots, b_p)$ with $p = int(\frac{t}{2})+1$, as two adjacent columns $[a' : b']$. Adding successively each element of $\mathbb{Z}_t$ to $[a' : b']$ reduced modulo t gives a $p \times 2t$ array,

$$\left[\begin{array}{cc|cc|c|cc} a_1+0 & b_1+0 & a_1+1 & b_1+1 & \cdots & a_1+(t-1) & b_1+(t-1) \\ a_2+0 & b_2+0 & a_2+1 & b_2+1 & \cdots & a_2+(t-1) & b_2+(t-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_p+0 & b_p+0 & a_p+1 & b_p+1 & \cdots & a_p+(t-1) & b_p+(t-1) \end{array}\right] \text{modulo } t.$$

Now, considering the rows of the above array as periods and the columns as units constructs the said crossover design because, from the definition of complementary pair of terraces, $\lambda_3 = 1$ for even t and $\lambda_3 = 0$ for odd t, and hence from the equations (1.5.1)–(1.5.2), $\lambda_1 = 2$ and $\lambda_2 = 1$. Then from the equation (1.5.3), the crossover design is minimal balanced. □

Example 2.3.1. To construct $COD(4, 8, 3)$, consider the group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. Define a complementary pair of terraces such as $a = (0, 3, 1)$ and $b = (2, 3, 3)$. Consider them as two adjacent columns,

$$\begin{array}{cc} 0 & 2 \\ 3 & 3 \\ 1 & 3. \end{array}$$

Adding successively 0, 1, 2 and 3 to the above columns reduced modulo 4 constructs the minimal balanced $COD(4, 8, 3)$ given by

Periods	Subjects							
	1	2	3	4	5	6	7	8
1	0	2	1	3	2	0	3	1
2	3	3	0	0	1	1	2	2
3	1	3	2	0	3	1	0	2

Note that, $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 1$ and ES = 100%.

Example 2.3.2. To construct $COD(6, 12, 4)$, consider the group $\mathbb{Z}_6 = \{0, 1, \ldots, 5\}$. Define a complementary pair of terraces such as $a = (2, 0, 1, 4)$ and $b = (5, 1, 0, 0)$. Consider them as two adjacent columns $[a' : b']$. Adding successively 0, 1, 2, 3, 4 and 5 to $[a' : b']$ reduced modulo 6 constructs the minimal balanced $COD(6, 12, 4)$ given by

Periods	Subjects											
	1	2	3	4	5	6	7	8	9	10	11	12
1	2	5	3	0	4	1	5	2	0	3	1	4
2	0	1	1	2	2	3	3	4	4	5	5	0
3	1	0	2	1	3	2	4	3	5	4	0	5
4	4	0	5	1	0	2	1	3	2	4	3	5

Note that, $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 1$ and ES = 100%.

Example 2.3.3. To construct $COD(7, 14, 4)$, consider the group $\mathbb{Z}_7 = \{0, 1, \ldots, 6\}$. Define a complementary pair of terraces such as $a = (0, 1, 3, 6)$ and $b = (0, 6, 4, 1)$. Consider them as two adjacent columns $[a' : b']$. Adding successively 0, 1, 2, 3, 4, 5 and 6 to $[a' : b']$ reduced modulo 7 constructs the minimal balanced $COD(7, 14, 4)$

given by

	Subjects													
Periods	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	0	1	1	2	2	3	3	4	4	5	5	6	6
2	1	6	2	0	3	1	4	2	5	3	6	4	0	5
3	3	4	4	5	5	6	6	0	0	1	1	2	2	3
4	6	1	0	2	1	3	2	4	3	5	4	6	5	0

Note that, $\lambda_1 = 2, \lambda_2 = 1$, $\lambda_3 = 0$ and ES $= 86\%$.

Remark 2.3.1. Some minimal balanced crossover designs with $p < t$, t odd, provides balanced incomplete block designs for nearest neighbor model (EBIBD) (for details see Kiefer and Wynn [31]). Example 2.3.3 is EBIBD(7,14,8,4,4).

Theorem 2.3.2. *A series of minimal balanced* $COD(t, 3t, \frac{t}{2})$ *for even* $t(\geq 4)$ *can be constructed by adding successively each element of* $\mathbb{Z}_t$ *to a complementary trio of terraces reduced modulo* t*.*

Proof. Consider a complementary trio of terraces $a = (a_1, a_2, \ldots, a_p)$, $b = (b_1, b_2, \ldots, b_p)$ and $c = (c_1, c_2, \ldots, c_p)$ with $p = \frac{t}{2}$, as three adjacent columns $[a' : b' : c']$. Adding successively each element of $\mathbb{Z}_t$ to $[a' : b' : c']$ reduced modulo t form a $p \times 3t$ array. Now, considering the rows of the array as periods and the columns as units, constructs the said crossover design because, from the definition of complementary trio of terraces, $\lambda_3 = \frac{3}{2}(t-2) - (t-1)$, and hence from the equations (1.5.1)–(1.5.2), $\lambda_1 = 3$ and $\lambda_2 = 1$. Then from the equation (1.5.3), the crossover design is minimal balanced. □

Table 2.1: ES of crossover design with and without repeating last period

		Without Repeating last period		With Repeating last period	
t	n	p	ES	p	ES
3	3	3	59	4	100
	3	5	55	6	82
	6	2	65	3	80
4	4	4	71	5	100
	8	3	100*	-	-
	12	2	76*	-	-
	4	7	69	8	87
	4	10	68	11	81
5	5	5	78	6	100
	10	3	80	4	86
	5	9	76	10	86
6	6	6	82	7	100
	12	4	100*	-	-
	18	3	100*	-	-
7	7	7	85	8	100
	14	4	86	5	89
8	8	8	87	9	100
	16	5	100*	-	-
	24	4	90*	-	-
9	9	9	88	10	100
	18	5	89	6	91
10	10	10	89	11	100
	20	6	100*	-	-
	30	5	85*	-	-

* indicate the new crossover design.

Table 2.1 lists ES calculated using equation (2.2.1) of crossover designs having three to ten treatments and three to thirty units for with and without repetition of the last period. As per our literature search so far, no strongly balanced crossover designs are available for even number of treatments in $\frac{t}{2}+1$ periods. Theorem 2.3.1 can be used to construct such crossover designs, because any complementary pair of terraces for even t, necessarily hold $\lambda_3 = \lambda_2 = 1$. One more strongly balanced crossover design, $COD(6, 18, 3)$ can be constructed using Theorem 2.3.2 owing to incidental equality of λ_2 and λ_3. Table 2.1 shows five new strongly balanced crossover designs, $COD(4, 8, 3)$, $COD(6, 12, 4)$, $COD(6, 18, 3)$, $COD(8, 16, 5)$ and $COD(10, 20, 6)$.

Table 2.2: ES of Afsarinejad & Hedayat [1] crossover design comparable to our crossover design with $p < t$

Afsarinejad & Hedayat [1]		Our paper	
Design	ES	Design	ES
COD(3,6,2)	65	COD(3,6,2)	65
COD(4,8,2)	59	COD(4,8,3)	100
		COD(4,12,2)	76
COD(5,15,2) D1	71	COD(5,10,3)	80
D2	68		
D3	71		
D4	63		
COD(6,18,2)	68	COD(6,12,4)	100
		COD(6,18,3)	100
COD(7,21,2)	68	COD(7,14,4)	86
COD(7,28,2)	75		
COD(8,56,2)	67	COD(8,16,5)	100
		COD(8,24,4)	90
COD(10,90,2)	70	COD(10,20,6)	100
		COD(10,30,5)	85

Afsarinejad and Hedayat [1] have especially considered two period crossover designs for comparing t treatments. Their crossover designs has several units which do not receive crossover treatments, i.e. same treatment is given twice. Table 2.2 provides ES of Afsarinejad and Hedayat [1] two period crossover designs and alternative useful crossover designs of this paper. Table 2.2 shows that, the $COD(t, n, p(< t))$ are better alternative to their two period designs. In particular, the $COD(5, 15, 2)$ of Afsarinejad and Hedayat [1] and this paper $COD(5, 10, 3)$, both uses equal 30 number of observations and the latter crossover design possesses higher ES. Further, Table 2.1 shows that, the ES for the $COD(t, 2t, int(\frac{t}{2}) + 1)$ for odd t constructed using Theorem 2.3.1, improves when the last period is repeated. Note that, the ES improves considerably for the crossover design with three treatments, $COD(3, 6, 2)$ has 65% ES while, $COD(3, 6, 3)$ has 80% ES.

Example 2.3.4. To construct $COD(4, 12, 2)$, consider the group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. Define a complementary trio of terraces such as $a = (0, 1)$, $b = (1, 0)$ and $c = (0, 2)$. Consider them as three adjacent columns $[a' : b' : c']$. Adding successively 0, 1, 2 and 3 to $[a' : b' : c']$ reduced modulo 4 constructs the minimal balanced $COD(4, 12, 2)$ given by

	Subjects											
Periods	1	2	3	4	5	6	7	8	9	10	11	12
1	0	1	0	1	2	1	2	3	2	3	0	3
2	1	0	2	2	1	3	3	2	0	0	3	1

Note that, $\lambda_1 = 3, \lambda_2 = 1$, $\lambda_3 = 0$ and ES = 76%. A comparative $COD(4, 8, 2)$ in Table 2.2 has lower ES (59%) as λ_2 is not constant.

Example 2.3.5. To construct $COD(6, 18, 3)$, consider the group $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. Define a complementary trio of terraces such as $a = (2, 0, 1)$, $b = (3, 0, 5)$ and $c = (4, 0, 0)$. Consider them as three adjacent columns $[a' : b' : c']$. Adding successively 0, 1, 2, 3, 4 and 5 to $[a' : b' : c']$ reduced modulo 6 constructs the minimal balanced $COD(6, 18, 3)$ given by

	Subjects																	
Periods	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	2	3	4	3	4	5	4	5	0	5	0	1	0	1	2	1	2	3
2	0	0	0	1	1	1	2	2	2	3	3	3	4	4	4	5	5	5
3	1	5	0	2	0	1	3	1	2	4	2	3	5	3	4	0	4	5

Note that, $\lambda_1 = 3, \lambda_2 = 1$, $\lambda_3 = 1$ and ES = 100%.

2.3.2 Minimal balanced $COD(t, t, t)$

When it is possible to measure each experimental unit repeatedly for t times, a minimal balanced $COD(t, t, t)$ is suitable. This design is constructed using a directed 2-terrace.

Theorem 2.3.3. *A series of minimal balanced $COD(t, t, t)$ can be constructed by adding successively each element of $\mathbb{Z}_t$ to a directed 2-terrace reduced modulo t.*

Proof. Consider a directed 2-terrace $a = (a_1, a_2, \ldots, a_p)$ with $p = t$, as a column a'. Adding successively each element of $\mathbb{Z}_t$ to a' reduced modulo t form a $p \times t$ array. Now, considering the rows of the array as periods and the columns as units, constructs the said crossover design because, from the definition of directed 2-terrace, $\lambda_3 = 0$, and hence from the equations (1.5.1)–(1.5.2), $\lambda_1 = 1$ and $\lambda_2 = 1$. Then from the equation (1.5.3), the crossover design is minimal balanced. □

From equation (2.2.1) and Table 2.1, it is clear that, the ES of $COD(t, t, t)$ increases with t and it is reasonably high (more than 75%) for $t \geq 5$. $COD(t, t, t)$ for even t are the same as those given in William [54]. Cheng and Wu [7] have shown that $COD(t, t, t)$ becomes $COD(t, t, t+1)$ when last period is repeated, and is optimal for the estimation of direct and first order carryover effects, which in terms of ES means, ES is necessarily 100%.

Example 2.3.6. To construct $COD(6,6,6)$, consider the group $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$. Define a directed 2-terrace such as $a = (0,4,5,2,1,3)$. Consider column a' as a sequence for the first unit. Adding successively 0, 1, 2, 3, 4 and 5 to a' reduced modulo 6 constructs the minimal balanced $COD(6,6,6)$ given by

	Subjects					
Periods	1	2	3	4	5	6
1	0	1	2	3	4	5
2	4	5	0	1	2	3
3	5	0	1	2	3	4
4	2	3	4	5	0	1
5	1	2	3	4	5	0
6	3	4	5	0	1	2

Note that, $\lambda_1 = 1, \lambda_2 = 1$ and $\lambda_3 = 0$. ES of this design is 82% and that of minimal balanced $COD(6,6,7)$ is 100%.

Remark 2.3.2. Minimal balanced crossover designs, obtained using uniform 2-terrace will be uniform and hence, following Theorem 1.5.1 they are universally optimal for the estimation of direct effects (e.g. Example 2.3.6).

Example 2.3.7. To construct $COD(7,7,7)$, consider the group $\mathbb{Z}_7 = \{0,1,2,3,4,5,6\}$. Define a directed 2-terrace such as $a = (0,1,3,6,3,1,0)$. Consider column a' as a sequence for the first unit. Adding successively 0, 1, 2, 3, 4, 5 and 6 to a' reduced modulo 7 constructs the minimal balanced $COD(7,7,7)$ given by

	Subjects						
Periods	1	2	3	4	5	6	7
1	0	1	2	3	4	5	6
2	1	2	3	4	5	6	0
3	3	4	5	6	0	1	2
4	6	0	1	2	3	4	5
5	3	4	5	6	0	1	2
6	1	2	3	4	5	6	0
7	0	1	2	3	4	5	6

Note that, $\lambda_1 = 1, \lambda_2 = 1$ and $\lambda_3 = 0$. ES of this design is 85% and that of minimal balanced $COD(7,7,8)$ is 100%.

2.3.3 Minimal balanced $COD(t, t, p(> t))$

If an experimental situation demands minimal balanced crossover design with $p > t$, such design can be constructed by directed m-terrace.

Theorem 2.3.4. *A series of minimal balanced $COD(t, t, 1 + \frac{m(t-1)}{2})$ for even $m(\geq 4)$ can be constructed by adding successively each element of $\mathbb{Z}_t$ to a directed m-terrace reduced modulo t.*

Proof. Consider a directed m-terrace $a = (a_1, a_2, \ldots, a_p)$ with $p = 1 + \frac{m(t-1)}{2}$, as a column a'. Adding successively each element of $\mathbb{Z}_t$ to a' reduced modulo t form a $p \times t$ array. Now, considering the rows of the array as periods and the columns as units, constructs the said crossover design because, from the definition of directed m-terrace, $\lambda_3 = 0$, and hence from the equations (1.5.1)–(1.5.2), $\lambda_1 = 1$ and $\lambda_2 = \frac{m}{2}$. Then from the equation (1.5.3), the crossover design is minimal balanced. □

From equation (2.2.1) and Table 2.1, it is clear that, in spite of a larger number of periods the ES is not much affected. Similar to other crossover designs, the ES improves with repetition of the last period. It is interesting to note that repeating the last period λ_2 times improves the ES to 100%.

Example 2.3.8. To construct $COD(4, 4, 10)$, consider the group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. Define a directed 6-terrace such as $a = (0, 1, 3, 2, 3, 1, 0, 2, 3, 2)$. Consider column a' as a sequence for the first unit. Adding successively 0, 1, 2 and 3 to a' reduced modulo 4 constructs the minimal balanced $COD(4, 4, 10)$ given by

	Subjects			
Periods	1	2	3	4
1	0	1	2	3
2	1	2	3	0
3	3	0	1	2
4	2	3	0	1
5	3	0	1	2
6	1	2	3	0
7	0	1	2	3
8	2	3	0	1
9	3	0	1	2
10	2	3	0	1

Note that, $\lambda_1 = 1, \lambda_2 = 3$ and $\lambda_3 = 0$. ES of this design is 68% and that of minimal balanced $COD(4, 4, 11)$ is 81%.

Example 2.3.9. To construct $COD(5,5,9)$, consider the group $\mathbb{Z}_5 = \{0,1,2,3,4\}$. Define a directed 4-terrace such as $a = (0,4,2,3,0,1,3,2,0)$. Consider column a' as a sequence for the first unit. Adding successively 0, 1, 2, 3 and 4 to a' reduced modulo 5 constructs the minimal balanced $COD(5,5,9)$ given by

	Subjects				
Periods	1	2	3	4	5
1	0	1	2	3	4
2	4	0	1	2	3
3	2	3	4	0	1
4	3	4	0	1	2
5	0	1	2	3	4
6	1	2	3	4	0
7	3	4	0	1	2
8	2	3	4	0	1
9	0	1	2	3	4

Note that, $\lambda_1 = 1, \lambda_2 = 2$ and $\lambda_3 = 0$. ES of this design is 76% and that of minimal balanced $COD(5,5,11)$ as $COD(5,5,9)$ with repetition of last period twice is 100%.

Table 2.3: List of directed 2-terraces with uniform 2-terraces from groups of order 3 to 8

Group	Directed 2-terraces
$\mathbb{Z}_3$	$(0,1,0)$
$\mathbb{Z}_4$	$(0,1,0,2)$, $\mathbf{(0,1,3,2)}$, $(0,2,1,2)$, $(0,2,3,2)$
$\mathbb{Z}_5$	$(0,1,3,1,0)$, $(0,1,3,2,0)$, $(0,1,4,1,0)$, $(0,1,4,3,0)$
$\mathbb{Z}_6$	$\mathbf{(0,4,5,2,1,3)}$, $(0,1,3,0,4,3)$, $(0,1,3,0,5,3)$, $(0,1,3,1,4,3)$, $(0,1,3,1,0,3)$, $(0,1,4,0,4,3)$, $(0,1,4,0,5,3)$, $(0,1,4,2,1,3)$, $(0,1,4,2,4,3)$, $(0,1,4,3,5,3)$, $(0,1,5,1,0,3)$, $\mathbf{(0,1,5,2,4,3)}$, $(0,1,5,2,1,3)$, $(0,1,5,4,1,3)$, $(0,1,5,4,0,3)$, $(0,1,0,2,5,3)$, $(0,1,0,3,1,3)$, $(0,1,0,3,5,3)$, $(0,1,0,4,1,3)$, $(0,2,3,0,4,3)$, $(0,2,3,1,4,3)$, $(0,2,3,1,0,3)$, $(0,2,3,2,5,3)$, $(0,2,3,2,0,3)$, $(0,2,5,0,4,3)$, $(0,2,5,3,4,3)$, $(0,2,5,3,2,3)$, $(0,2,5,4,5,3)$, $(0,2,5,4,2,3)$, $(0,2,0,3,4,3)$, $(0,2,0,1,4,3)$, $(0,2,0,5,2,3)$, $(0,2,1,2,5,3)$, $(0,2,1,2,0,3)$, $\mathbf{(0,2,1,4,5,3)}$, $(0,2,1,4,2,3)$, $(0,2,1,5,0,3)$, $(0,2,1,5,2,3)$, $(0,3,4,0,4,3)$, $(0,3,4,0,5,3)$, $(0,3,4,2,1,3)$, $(0,3,5,0,4,3)$, $(0,3,5,0,5,3)$, $(0,3,5,4,5,3)$, $(0,3,5,4,2,3)$, $(0,1,3,2,5,3)$, $(0,1,3,2,0,3)$, $(0,1,4,3,1,3)$, $(0,2,3,0,5,3)$, $(0,2,5,0,5,3)$, $(0,2,0,3,2,3)$, $(0,3,4,2,4,3)$, $(0,1,5,1,4,3)$
$\mathbb{Z}_7$	$(0,1,3,6,3,1,0)$, $(0,1,3,6,3,2,0)$, $(0,1,3,6,4,1,0)$, $(0,1,3,6,4,3,0)$, $(0,1,3,6,5,3,0)$, $(0,1,3,1,5,4,0)$, $(0,1,3,1,4,3,0)$, $(0,1,3,2,5,2,0)$, $(0,1,3,2,6,2,0)$, $(0,1,3,2,6,4,0)$, $(0,1,4,6,3,1,0)$, $(0,1,4,6,3,2,0)$, $(0,1,4,6,4,1,0)$, $(0,1,4,2,4,1,0)$, $(0,1,4,2,6,5,0)$, $(0,1,4,2,6,1,0)$, $(0,1,4,2,1,5,0)$, $(0,1,4,1,3,2,0)$, $(0,1,4,1,6,5,0)$, $(0,1,4,3,5,2,0)$, $(0,1,4,3,1,3,0)$, $(0,1,4,3,1,5,0)$, $(0,1,5,1,3,2,0)$, $(0,1,5,1,6,5,0)$, $(0,1,5,3,5,4,0)$, $(0,1,5,3,6,1,0)$, $(0,1,5,3,6,5,0)$, $(0,1,5,3,2,4,0)$, $(0,1,5,4,6,2,0)$, $(0,1,5,4,6,4,0)$, $(0,1,5,4,2,4,0)$, $(0,1,5,4,2,5,0)$, $(0,1,6,2,6,1,0)$, $(0,1,6,2,6,5,0)$, $(0,1,6,2,1,5,0)$, $(0,1,6,2,1,3,0)$, $(0,1,6,3,5,4,0)$, $(0,1,6,3,6,1,0)$, $(0,1,6,3,6,5,0)$, $(0,1,6,3,2,4,0)$, $(0,1,6,5,1,3,0)$, $(0,1,6,5,1,5,0)$, $(0,1,6,5,2,4,0)$, $(0,1,6,5,2,5,0)$, $(0,1,4,6,5,3,0)$, $(0,1,4,2,4,3,0)$, $(0,1,6,1,5,4,0)$, $(0,1,6,2,4,1,0)$, $(0,1,3,6,5,2,0)$, $(0,1,3,2,5,3,0)$, $(0,1,4,6,4,3,0)$, $(0,1,4,2,1,3,0)$, $(0,1,4,3,5,3,0)$, $(0,1,5,3,5,1,0)$, $(0,1,5,3,2,5,0)$, $(0,1,6,1,4,3,0)$, $(0,1,6,3,5,1,0)$, $(0,1,6,3,2,5,0)$, $(0,1,4,6,5,2,0)$, $(0,1,6,2,4,3,0)$
@$\mathbb{Z}_8$	$\mathbf{(0,1,3,6,2,7,5,4)}$, $\mathbf{(0,1,6,5,3,7,2,4)}$, $\mathbf{(0,1,7,2,6,3,5,4)}$, $\mathbf{(0,2,1,5,3,6,7,4)}$, $\mathbf{(0,2,3,6,5,1,7,4)}$, $\mathbf{(0,2,5,1,7,6,3,4)}$, $\mathbf{(0,1,7,3,6,5,2,4)}$, $\mathbf{(0,2,7,6,1,5,3,4)}$

[@]only uniform 2-terraces are mentioned to save space.

Terraces in bold denote uniform 2-terrace.

Table 2.4: List of complementary pair of terraces from groups of order 3 to 9

Group	a	b (any one)
$\mathbb{Z}_3$	$(0,1)$	$(1,0)$
$\mathbb{Z}_4$	$(0,3,1)$	$(2,3,3)$, $(0,0,1)$, $(0,1,1)$, $(1,1,2)$, $(1,2,2)$, $(2,2,3)$
$\mathbb{Z}_5$	$(0,1,3)$	$(0,3,2)$, $(1,4,3)$, $(2,0,4)$, $(3,1,0)$, $(4,2,1)$, $(0,4,2)$
$\mathbb{Z}_6$	$(0,2,3,1)$	$(0,0,3,2)$, $(0,0,5,2)$, $(1,0,0,3)$, $(1,1,0,3)$, $(2,1,1,4)$, $(2,2,1,4)$, $(3,2,2,5)$, $(4,1,0,0)$, $(4,3,3,0)$, $(5,2,1,1)$, $(3,2,5,5)$, $(5,2,2,1)$
$\mathbb{Z}_7$	$(0,1,6,3)$	$(1,3,6,5)$, $(2,4,3,6)$, $(3,6,1,0)$, $(4,0,6,1)$, $(5,4,0,2)$, $(6,5,0,3)$
$\mathbb{Z}_8$	$(0,1,7,4,6)$	$(5,1,4,4,3)$, $(0,3,2,2,6)$, $(5,1,1,4,3)$, $(3,6,5,5,1)$, $(0,0,7,3,6)$, $(3,6,6,5,1)$, $(3,3,6,2,1)$, $(3,2,6,1,1)$, $(0,3,3,7,6)$, $(5,1,0,3,3)$
$\mathbb{Z}_9$	$(0,1,6,4,7)$	$(1,0,2,8,3)$, $(2,1,3,7,4)$, $(3,2,6,8,5)$, $(4,3,0,2,6)$, $(5,0,2,8,7)$, $(6,1,3,2,8)$, $(7,2,8,1,0)$, $(8,3,2,4,1)$, $(7,4,6,1,0)$, $(6,3,5,4,8)$, $(3,0,4,6,5)$, $(2,8,7,0,4)$

Table 2.5: List of complementary trio of terraces from groups of order 4 to 8

Group	a	b	c (any one)
$\mathbb{Z}_4$	$(0,1)$	$(1,0)$	$(0,2)$, $(0,3)$, $(2,1)$, $(3,2)$
$\mathbb{Z}_6$	$(0,1,3)$	$(0,3,1)$	$(0,5,5)$, $(1,0,0)$, $(2,1,1)$, $(3,2,2)$, $(4,3,3)$, $(5,4,4)$, $(1,1,0)$, $(2,2,1)$, $(3,3,2)$, $(4,4,3)$, $(5,5,4)$, $(0,0,5)$
$\mathbb{Z}_8$	$(0,1,3,6)$	$(0,4,1,1)$	$(0,0,7,5)$, $(0,6,5,5)$, $(0,6,6,5)$, $(0,7,5,5)$, $(0,7,7,5)$, $(1,1,7,6)$