Annexure
Product Cordial Labeling in the Context of Tensor Product of Graphs

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Received: March 10, 2011   Accepted: March 28, 2011   doi:10.5539/jmr.v3n3p83

Abstract
For the graph $G_1$ and $G_2$ the tensor product is denoted by $G_1(T_p)G_2$ which is the graph with vertex set $V(G_1(T_p)G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1(T_p)G_2) = \{(u_1,v_1),(u_2,v_2)/u_1u_2eE(G_1) and v_1v_2eE(G_2)\}.$ The graph $P_m(T_p)P_n$ is disconnected for $\forall m,n$ while the graphs $C_m(T_p)C_n$ and $C_m(T_p)P_n$ are disconnected for both $m$ and $n$ even. We prove that these graphs are product cordial graphs. In addition to this we show that the graphs obtained by joining the connected components of respective graphs by a path of arbitrary length also admit product cordial labeling.

Keywords: Cordial labeling, Product cordial labeling, Tensor product

AMS Subject classification (2010): 05C78.

1. Introduction
We begin with simple, finite and undirected graph $G = (V(G), E(G)).$ For standard terminology and notations we follow (West, D. B., 2001). The brief summary of definitions and relevant results are given below.

1.1 Definition: If the vertices of the graph are assigned values subject to certain condition(s) then it is known as graph labeling.

1.2 Definition: A mapping $f : V(G) \rightarrow \{0, 1\}$ is called binary vertex labeling of $G$ and $f(v)$ is called the label of vertex $v$ of $G$ under $f$. For an edge $e = uv$, the induced edge labeling $f^\ast : E(G) \rightarrow \{0, 1\}$ is given by $f^\ast(e = uv) = |f(u) - f(v)|$. Let $v_f(0), v_f(1)$ be the number of vertices of $G$ having labels 0 and 1 respectively under $f$ and let $e_f(0), e_f(1)$ be the number of edges of $G$ having labels 0 and 1 respectively under $f^\ast$.

1.3 Definition: A binary vertex labeling of graph $G$ is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is called cordial if admits cordial labeling.

The concept of cordial labeling was introduced by (Cahit,1987, p.201-207) and in the same paper he investigated several results on this newly introduced concept.

Motivated through cordial labeling the concept of product cordial labeling was introduced in (Sundaram, M., Ponraj, R. and Somsundaram, S., 2004, p.155-163 ) which has the flavour of cordial labeling but absolute difference of vertex labels is replaced by product of vertex labels.

1.4 Definition: A binary vertex labeling of graph $G$ with induced edge labeling $f^\ast : E(G) \rightarrow \{0, 1\}$ defined by $f^\ast(e = uv) = f(u)f(v)$ is called a product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is product cordial if it admits product cordial labeling.

In (Sundaram, M., Ponraj, R. and Somsundaram, S., 2004, p.155-163) it has been proved that trees, unicyclic graphs of odd order, triangular snakes, dragons, helms and union of two path graphs are product cordial. They also proved that a graph with $p$ vertices and $q$ edges with $p \geq 4$ is product cordial then $q < \frac{p^2-1}{4}$.

The graphs obtained by joining apex vertices of $k$ copies of stars, shells and wheels to a new vertex are proved to be product cordial in (Vaidya, S. K. and Dani, N. A., 2010, p.62-65). The product cordial labeling for some cycle related graphs is discussed in (Vaidya, S. K. and Kanani, K. K., 2010, p.109-116). In the same paper they have investigated product cordial labeling for the shadow graph of cycle $C_n$. 
We define binary vertex labeling for two components of $G$. We define binary vertex labeling for $G$ by a path $P_n$. Let $G$ be the graph obtained by joining two components of $P_m$. Denote the vertices of $G$ as $u_i$ and $v_j$ where $1 \leq i \leq m$, $1 \leq j \leq n$. We note that $|V(G)| = mn$ and $|E(G)| = 2(m-1)(n-1)$.

We define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ as follows.

$$f(u_i) = \begin{cases} 1 & \text{if } i + j \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

In view of the above defined labeling pattern the graph $G$ under consideration satisfies the conditions for product cordiality as shown in Table 1. Hence $P_m(T)P_n$ is product cordial.

**2.2 Example:** The product cordial labeling for $P_5(T)P_3$ is shown in Figure 1.

**2.3 Theorem:** $C_m(T)P_n$ is product cordial for both $m$ and $n$ even.

**Proof:** Let $G = C_m(T)P_n$ be the graph obtained by tensor product of $C_m$ and $P_n$. Denote the vertices of $G$ as $u_i$ where $1 \leq i \leq m$ and $1 \leq j \leq n$. We note that $|V(G)| = mn$ and $|E(G)| = 2m(n-1)$.

We define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ as follows.

$$f(u_i) = \begin{cases} 1 & \text{if } i + j \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

In view of the above defined labeling pattern the graph $G$ under consideration satisfies the conditions for product cordial labeling as shown in Table 2. Hence $C_m(T)P_n$ is product cordial for both $m$ and $n$ even.

**2.4 Example:** The product cordial labeling for $C_4(T)P_n$ is shown in Figure 2.

**2.5 Theorem:** $C_m(T)C_n$ is product cordial for $m$ and $n$ even.

**Proof:** Let $C_m$ and $C_n$ be the cycles with $m$ and $n$ vertices respectively. Let $G = C_m(T)C_n$ be the graph obtained by tensor product of $C_m$ and $C_n$ where $m$ and $n$ are even. Denote the vertices of $G$ as $u_i$ where $1 \leq i \leq m$ and $1 \leq j \leq n$. We note that $|V(G)| = mn$ and $|E(G)| = 2mn$.

We define binary vertex labeling $f : V(G) \rightarrow \{0, 1\}$ as follows.

$$f(u_i) = \begin{cases} 1 & \text{if } i + j \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

In view of the above defined labeling pattern the graph $G$ under consideration satisfies the conditions for product cordial labeling as shown in Table 3. Hence $C_m(T)P_n$ is product cordial for both $m$ and $n$ even.

**2.6 Example:** The product cordial labeling for $C_4(T)P_n$ is shown in Figure 3.

**2.7 Theorem:** The graph obtained by joining two components of $P_m(T)P_n$ with arbitrary path $P_k$ is product cordial.

**Proof:** Let $G = P_m(T)P_n$ be the graph obtained by tensor product of $P_m$ and $P_n$ and $G'$ be the graph obtained by joining two components of $G$ by a path $P_k$. Let $u_1, u_2, \ldots, u_j$ and $v_1, v_2, \ldots, v_j$ respectively be the vertices of first and second component of $G'$ where $j = \frac{mn}{k}$. Let $w_1, w_2, \ldots, w_j$ be the vertices of path $P_k$ such that $u_1 = w_1$ and $v_1 = w_k$. We note that $|V(G')| = mn + k - 2$ and $|E(G')| = 2mn + k - 1$.

We define binary vertex labeling $f : V(G') \rightarrow \{0, 1\}$ as follows.

Case: $1 \ k \equiv 0(mod 2)$

$$f(u_i) = 0; \quad 1 \leq i \leq j$$
$$f(v_i) = 1; \quad 1 \leq i \leq j$$
Let

2.11 Theorem: The graph obtained by joining two components of \( C_m \) shown in Figure 5.

2.10 Example: The product cordial labeling for the graph obtained by joining two components of

\( C_m \) for even \( m \)

Case: 1

\( P_k \)

The vertices of path \( u \)

\[ f(w_i) = \begin{cases} 0; & 1 \leq i \leq \frac{k}{2} \\ 1; & \frac{k}{2} < i \leq k \end{cases} \]

Case: 2 \( k \equiv 1 \pmod{2} \)

\[ f(u_i) = 0; \quad 1 \leq i \leq j \]
\[ f(v_i) = 1; \quad 1 \leq i \leq j \]

\[ f(w_i) = \begin{cases} 0; & 1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor \\ 1; & \frac{k}{2} \leq i \leq k \end{cases} \]

In view of the above defined labeling pattern \( f \) satisfies the conditions for product cordial labeling as shown in Table 4. Thus we prove that the graph obtained by joining two components of \( P_m(T_p)P_n \) with arbitrary path \( P_k \) is product cordial for even \( m \) and \( n \).

2.8 Example: The product cordial labeling for the graph obtained by joining two components of \( P_5(T_p)P_3 \) by path \( P_4 \) is shown in Figure 4.

2.9 Theorem: The graph obtained by joining two components of \( C_m(T_p)P_n \) with arbitrary path \( P_k \) is product cordial for \( m \) and \( n \) even.

Proof: Let \( C_m \) be the cycle with \( m \) vertices, \( P_n \) be the path of length \( n - 1 \) and \( G \) be the graph obtained by tensor product of \( C_m \) and \( P_n \) where \( m \) and \( n \) are even. Let \( G' \) be the graph obtained by joining two components of \( G \) by a path \( P_k \) and \( u_1, u_2, \ldots, u_j \) and \( v_1, v_2, \ldots, v_j \) be the vertices of first and second component of \( G' \) where \( j = \frac{mn}{2} \). Let \( w_1, w_2, \ldots, w_k \) be the vertices of path \( P_k \) with \( u_1 = v_1 = w_k \). We note that \( |V(G')| = mn + k - 2 \) and \( |E(G')| = 2m(n - 1) + k - 1 \).

We define binary vertex labeling \( f : V(G') \rightarrow \{0, 1\} \) as follows.

Case: 1 \( k \equiv 0 \pmod{2} \)

\[ f(u_i) = 0; \quad 1 \leq i \leq j \]
\[ f(v_i) = 1; \quad 1 \leq i \leq j \]

\[ f(w_i) = \begin{cases} 0; & 1 \leq i \leq \frac{k}{2} \\ 1; & \frac{k}{2} < i \leq k \end{cases} \]

Case: 2 \( k \equiv 1 \pmod{2} \)

\[ f(u_i) = 0; \quad 1 \leq i \leq j \]
\[ f(v_i) = 1; \quad 1 \leq i \leq j \]

\[ f(w_i) = \begin{cases} 0; & 1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor \\ 1; & \frac{k}{2} < i \leq k \end{cases} \]

In view of the above defined labeling pattern the graph \( G' \) satisfies the conditions for product cordial labeling as shown in Table 5. That is, the graph obtained by joining two components of \( C_m(T_p)P_n \) with arbitrary path \( P_k \) is product cordial for even \( m \) and \( n \).

2.10 Example: The product cordial labeling for the graph obtained by joining two components of \( C_4(T_p)P_3 \) by path \( P_3 \) is shown in Figure 5.

2.11 Theorem: The graph obtained by joining two components of \( C_m(T_p)C_n \) with arbitrary path \( P_k \) is product cordial for \( m \) and \( n \) even.

Proof: Let \( C_m \) and \( C_n \) be the cycle with \( m \) and \( n \) vertices respectively. Let \( G \) be the graph obtained by tensor product of \( C_m \) and \( C_n \) where \( m \) and \( n \) are even and \( G' \) be the graph obtained by joining two components of \( G \) by a path \( P_k \). Let \( u_1, u_2, \ldots, u_j \) and \( v_1, v_2, \ldots, v_j \) respectively be the vertices of first and second component of \( G' \) where \( j = \frac{mn}{2} \).
Let $w_1, w_2, \ldots, w_k$ be the vertices of path $P_k$ with $u_1 = w_1$ and $v_1 = w_k$. We note that $|V(G')| = mn + k - 2$ and $|E(G')| = 2mn + k - 1$.

We define binary vertex labeling $f : V(G') \rightarrow \{0, 1\}$ as follows.

Case:1 $k \equiv 0 (mod \ 2)$

\[
\begin{align*}
  f(u_i) &= 0; \quad 1 \leq i \leq j \\
  f(v_i) &= 1; \quad 1 \leq i \leq j \\
  f(w_i) &= \begin{cases} 
    0; & 1 \leq i \leq \frac{k}{2} \\
    1; & \frac{k}{2} < i \leq k 
  \end{cases}
\end{align*}
\]

Case:2 $k \equiv 1 (mod \ 2)$

\[
\begin{align*}
  f(u_i) &= 0; \quad 1 \leq i \leq j \\
  f(v_i) &= 1; \quad 1 \leq i \leq j \\
  f(w_i) &= \begin{cases} 
    0; & 1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor \\
    1; & \left\lfloor \frac{k}{2} \right\rfloor < i \leq k 
  \end{cases}
\end{align*}
\]

In view of the above defined labeling pattern the graph $G'$ under consideration satisfies the conditions for product cordial labeling as shown in Table 6. That is, the graph obtained by joining two components of $C_m(T_p)C_n$ with arbitrary path $P_k$ is product cordial for even $m$ and $n$.

2.12 Example: The product cordial labeling for the graph obtained by joining two components of $C_4(T_p)C_4$ by path $P_4$ is shown in Figure 6.

3. Concluding Remarks

As all the graphs are not product cordial graphs it is very interesting to investigate graphs or graph families which admit product cordial labeling. Here we investigate product cordial labeling for some graphs obtained by tensor product of two graphs. To derive similar results for other graph is an open area of research.

References


Table 1.

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<tr>
<th>Condition</th>
<th>Vertex condition</th>
<th>Edge condition</th>
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<td>$m$ and $n$ are odd</td>
<td>$v_f(0) + 1 = v_f(1) = \frac{m+1}{2}$</td>
<td>$e_f(0) = e_f(1) = (m-1)(n-1)$</td>
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<th>m and n are even</th>
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Table 4.

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<th>k even</th>
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<td>( v_f(0) = v_f(1) = \frac{mn+k-2}{2} )</td>
<td>( e_f(0) = e_f(1) + 1 = \frac{2(mn-1)(n-1)+2k}{2} )</td>
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<tr>
<td>k odd</td>
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<td>( e_f(0) = e_f(1) = \frac{2(mn-1)(n-1)+k}{2} )</td>
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Figure 1. The product cordial labeling for \( P_5(T_p)P_3 \)

Figure 2. The product cordial labeling for \( C_4(T_p)P_6 \)
Figure 3. The product cordial labeling for $C_4(T_p)C_4$

Figure 4. The product cordial labeling for the graph obtained by joining two components of $P_5(T_p)P_3$ by path $P_4$

Figure 5. The product cordial labeling for the graph obtained by joining two components of $C_4(T_p)P_6$ by path $P_5$

Figure 6. The product cordial labeling for the graph obtained by joining two components of $C_4(T_p)C_4$ by path $P_4$
E-Cordial Labeling for Cartesian Product of Some Graphs

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Received 10 October, 2011; accepted 27 November, 2011

Abstract
We investigate E-cordial labeling for some cartesian product of graphs. We prove that the graphs $K_n \times P_2$ and $P_n \times P_2$ are E-cordial for $n$ even while $W_n \times P_2$ and $K_{1,n} \times P_2$ are E-cordial for $n$ odd.

Key words
E-Cordial labeling; Edge graceful labeling; Cartesian product

1. INTRODUCTION

We begin with finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. For standard terminology and notations we refer to West (2001). The brief summary of definitions and relevant results are given below.

Definition 1.1 If the vertices of the graph are assigned values subject to certain condition(s) then it is known as graph labeling.

Most of the graph labeling techniques trace their origin to graceful labeling introduced independently by Rosa (1967) and Golomb (1972) which is defined as follows.

Definition 1.2 A function $f: V(G) \rightarrow \{0,1,2,\ldots,q\}$ is injective and the induced function $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits graceful labeling is called a graceful graph.

The famous Ringel-Kotzig graceful tree conjecture and illustrious work by Kotzig (1973) brought a tide of labeling problems having graceful theme.

Definition 1.3 A graph $G$ is said to be edge-graceful if there exists a bijection $f: E(G) \rightarrow \{1,2,\ldots,|E|\}$ such that the induced mapping $f^*: V(G) \rightarrow \{0,1,2,\ldots,|V| - 1\}$ given by $f^*(x) = \sum f(xy)(mod|V|)$, $xy \in E(G)$.

Definition 1.4 A mapping $f: V(G) \rightarrow \{0,1\}$ is called binary vertex labeling of $G$ and $f(v)$ is called the label of vertex $v$ of $G$ under $f$.

Notations 1.5 For an edge $e = uv$, the induced edge labeling $f^*: E(G) \rightarrow \{0,1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$. Then

$$
\begin{align*}
    v_f(i) &= \text{number of vertices of } G \text{ having label } i \text{ under } f \\
    e_f(i) &= \text{number of edges of } G \text{ having label } i \text{ under } f^* \\
\end{align*}
$$

where $i = 0 \text{ or } 1$
Definition 1.6 A binary vertex labeling of graph $G$ is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is called cordial if admits cordial labeling.

The concept of cordial labeling was introduced by Cahit (1987). He also investigated several results on this newly introduced concept.

Definition 1.7 A function $f : E(G) \rightarrow \{0, 1\}$ is called E-cordial labeling of graph $G$ if the induced function $f^* : V(G) \rightarrow \{0, 1\}$ defined by $f^*(v) = \sum_{uv \in E(G)} f(\text{mod} 2)$ is such that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph is called E-cordial if admits E-cordial labeling.

Yilmaz and Cahit (1997) introduced E-cordial labeling as a weaker version of edge-graceful labeling and having bland of cordial labeling. They proved that the trees with $n$ vertices, the complete graph $K_n$ and cycle $C_n$ are E-cordial if and only if $n \equiv 2 \pmod{4}$ while complete bipartite graph $K_{m,n}$ admits E-cordial labeling if and only if $m + n \equiv 2 \pmod{4}$.

Devaraj (2004) has shown that $M(m, n)$ (the mirror graph of $K_{m,n}$) is E-cordial when $m + n$ is even while the generalized Petersen graph $P(n, k)$ is E-cordial when $n$ is even. Vaidya and Vyas (2011) have proved that the mirror graphs of even cycle $C_n$, even path $P_n$ and hypercube $Q_k$ are E-cordial graphs.

Definition 1.8 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The cartesian product of $G_1$ and $G_2$ which is denoted by $G_1 \times G_2$ is the graph with vertex set $V = V_1 \times V_2$ consisting of vertices $V = \{ u = (u_1, u_2), v = (v_1, v_2) \} \cup u$ and $v$ are adjacent in $G_1 \times G_2$ whenever $u_1 = v_1$ and $u_2$ adjacent to $v_2$ or $u_1$ adjacent to $v_1$ and $u_2 = v_2$.

In this paper we have investigated some results on E-cordial labeling for cartesian product of some graphs.

2. MAIN RESULTS

Theorem-2.1: $K_n \times P_2$ is E-cordial for even $n$.

Proof: Let $G$ be the graph $K_n \times P_2$ where $n$ is even and $V(G) = \{ v_{ij} \mid i = 1, 2, \ldots, n \text{ and } j = 1, 2 \}$ be the vertices of graph $G$. We note that $|V(G)| = 2n$ and $|E(G)| = n^2$ as $|V(K_n)| = n$ and $|E(K_n)| = \frac{n(n-1)}{2}$.

Define $f : E(G) \rightarrow \{0, 1\}$ as follows:

For $1 \leq i, k \leq n$

$$f(v_{1i}, v_{1k}) = 0;$$

$$f(v_{2i}, v_{2k}) = 1;$$

$$f(v_{1i}, v_{2i}) = \begin{cases} 
1; & i \equiv 0 \pmod{2} \\
0; & \text{otherwise.}
\end{cases}$$

In view of the above defined labeling pattern $f$ satisfies the conditions for E-cordial labeling as shown in Table 1. That is, $K_n \times P_2$ is E-cordial for even $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$v_f(0) = v_f(1) = n$</th>
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<td></td>
<td>$e_f(0) = e_f(1) = \frac{n}{2}$</td>
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Illustration 2.2: The E-cordial labeling for $K_4 \times P_2$ is shown in Figure 1.
Theorem 2.3: $W_n \times P_2$ is E-cordial for odd $n$.

Proof: Let $G$ be the graph $W_n \times P_2$ where $n$ is odd and $V(G) = \{v_{ij}/i = 1, 2, \ldots, n + 1 \text{ and } j = 1, 2\}$ be the vertices of graph $G$. We note that $|V(G)| = 2(n + 1)$ and $|E(G)| = 5n + 1$ as $|V(W_n)| = n + 1$ and $|E(W_n)| = 2n$.

Define $f : E(G) \rightarrow \{0, 1\}$ as follows:

For $1 \leq i, k \leq n + 1$

- $f(v_{i1}, v_{k1}) = 0$;
- $f(v_{i2}, v_{k2}) = 1$;
- $f(v_{i1}, v_{i2}) = 1$; $i \equiv 0 (mod 2)$
- $= 0$; otherwise.

In view of the above defined labeling pattern $f$ satisfies conditions for E-cordial labeling as shown in Table 2. That is, $W_n \times P_2$ is E-cordial for odd $n$.

Table 2

<table>
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<th>vertex condition</th>
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<tr>
<td>$n$</td>
<td>$v_f(0) = v_f(1) = n + 1$</td>
</tr>
</tbody>
</table>

Illustration 2.4: The E-cordial labeling for $W_3 \times P_2$ is shown in Figure 2.

Figure 2

Theorem 2.5: $L_n = P_n \times P_2$ (also known as ladder graph) is E-cordial for even $n$.

Proof: Let $G$ be the graph $P_n \times P_2$ where $n$ is even and $V(G) = \{v_{ij}/i = 1, 2, \ldots, n \text{ and } j = 1, 2\}$ be the vertices of $G$. We note that $|V(G)| = 2n$ and $|E(G)| = 3n - 2$. Define $f : E(G) \rightarrow \{0, 1\}$ as follows:
For $1 \leq i, k \leq n$

\[ f(v_{i1}, v_{k1}) = 0; \]
\[ f(v_{i2}, v_{k2}) = 1; \]
\[ f(v_{i1}, v_{i2}) = \begin{cases} 1; & i \equiv 0 \pmod{2} \\ 0; & \text{otherwise.} \end{cases} \]

In view of the above defined labeling pattern $f$ satisfies conditions for E-cordial labeling as shown in Table 3. That is, $P_n \times P_2$ is E-cordial for even $n$.

**Table 3**

<table>
<thead>
<tr>
<th>$n$</th>
<th>vertex condition</th>
<th>edge condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_f(0) = v_f(1) = n$</td>
<td>$e_f(0) = e_f(1) = \frac{3n^2 - 2}{2}$</td>
<td></td>
</tr>
</tbody>
</table>

**Illustration 2.6:** The E-cordial labeling for $P_4 \times P_2$ is shown in Figure 3.

---

**Theorem 2.7:** $B_n = K_{1,n} \times P_2$ (also known as book graph) is E-cordial for odd $n$.

**Proof:** Let $G$ be the graph $K_{1,n} \times P_2$ where $n$ is odd and $V(G) = \{v_{ij} \mid i = 1, 2, \ldots, n + 1 \text{ and } j = 1, 2\}$ be the vertices of $G$. We note that $|V(G)| = 2(n + 1)$ and $|E(G)| = 3n + 1$. Define $f : E(G) \to \{0, 1\}$ as follows:

For $1 \leq i, k \leq n + 1$

\[ f(v_{i1}, v_{k1}) = 0; \]
\[ f(v_{i2}, v_{k2}) = 1; \]
\[ f(v_{i1}, v_{i2}) = \begin{cases} 1; & i \equiv 0 \pmod{2} \\ 0; & \text{otherwise.} \end{cases} \]

In view of the above defined labeling pattern $f$ satisfies the conditions for E-cordial labeling as shown in Table 4. That is, $K_{1,n} \times P_2$ is E-cordial for odd $n$.

**Illustration 2.8:** The E-cordial labeling for $K_{1,3} \times P_2$ is shown in Figure 4.
Table 4

<table>
<thead>
<tr>
<th>vertex condition</th>
<th>edge condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_f(0) = v_f(1) = n )</td>
<td>( e_f(0) = e_f(1) = \frac{3n+1}{2} )</td>
</tr>
</tbody>
</table>

Figure 4

3. CONCLUDING REMARKS

Here we investigate E-cordial labeling for cartesian product of some graphs. Similar results can be derived for other graph families and in the context of different graph labeling problems is an open area of research.

REFERENCES

International Journal of Contemporary Advanced Mathematics (IJCM)

ISSN : 2180-0030

Volume 2, Issue 1

Number of issues per year: 6
E-Cordial Labeling of Some Mirror Graphs

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Abstract

Let $G$ be a bipartite graph with a partite sets $V_1$ and $V_2$ and $G'$ be the copy of $G$ with corresponding partite sets $V'_1$ and $V'_2$. The mirror graph $M(G)$ of $G$ is obtained from $G$ and $G'$ by joining each vertex of $V_1$ to its corresponding vertex in $V_2$ by an edge. Here we investigate E-cordial labeling of some mirror graphs. We prove that the mirror graphs of even cycle $C_n$, even path $P_n$ and hypercube $Q_k$ are E-cordial graphs.

Keywords: E-Cordial labeling, Edge graceful labeling, Mirror graphs.

AMS Subject Classification Number(2010): 05C78

1. INTRODUCTION

We begin with finite, connected and undirected graph $G=(V(G), E(G))$ without loops and multiple edges. For standard terminology and notations we refer to West[1]. The brief summary of definitions and relevant results are given below.

Definition 1.1
If the vertices of the graph are assigned values subject to certain condition(s) then it is known as graph labeling.

Most of the graph labeling techniques trace their origin to graceful labeling introduced independently by Rosa[3] and Golomb[4] which is defined as follows.

Definition 1.2
A function $f$ is called graceful labeling of graph $G$ if $f: V(G) \rightarrow \{0,1,2,\ldots,q\}$ is injective and the induced function $f^*(e=uv)=|f(u)-f(v)|$ is bijective. A graph which admits graceful labeling is called a graceful graph.


Definition 1.3
A graph $G$ is said to be edge-graceful if there exists a bijection $f: E(G) \rightarrow \{1,2,\ldots,|E|\}$ such that the induced mapping $f^*: V(G) \rightarrow \{0,1,2,\ldots,|V|-1\}$ given by $f^*(x)=\sum f(xy)(mod |V|)$, $xy\in E(G)$.

Definition 1.4
A mapping $f: V(G) \rightarrow \{0,1\}$ is called binary vertex labeling of $G$ and $f(v)$ is called the label of vertex $v$ of $G$ under $f$. 
Notations 1.5
For an edge $e=uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0,1\}$ is given by $f^*(e=uv) = f(u) - f(v)$. Let $v_f(0), v_f(1)$ be the number of vertices of $G$ having labels 0 and 1 respectively under $f$ and let $e_f(0), e_f(1)$ be the number of edges of $G$ having labels 0 and 1 respectively under $f^*$.

Definition 1.6
A binary vertex labeling of graph $G$ is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is called cordial if admits cordial labeling.

The concept of cordial labeling was introduced by Cahit[6]. He also investigated several results on this newly introduced concept.

Definition 1.7
Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$ and let $f : E(G) \rightarrow \{0,1\}$. Define on $V(G)$ by $f(v) = \sum_{u \in E(G)} f(u,v) \pmod{2}$. The function $f$ is called an E-cordial labeling of $G$ if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph is called E-cordial if admits E-cordial labeling.

In 1997 Yilmaz and Cahit[7] introduced E-cordial labeling as a weaker version of edge-graceful labeling and having flavour of cordial labeling. They proved that the trees with $n$ vertices, $K_n$, $C_n$ are E-cordial if and only if $2(n-1) \equiv 0 \pmod{4}$ while $K_{m,n}$ admits E-cordial labeling if and only if $m+n \equiv 2 \pmod{4}$

Definition 1.8
For a bipartite graph $G$ with partite sets $V_1$ and $V_2$. Let $G'$ be the copy of $G$ and $V'_1$ and $V'_2$ be the copies of $V_1$ and $V_2$. The mirror graph $M(G)$ of $G$ is obtained from $G$ and $G'$ by joining each vertex of $V_2$ to its corresponding vertex in $V'_2$ by an edge.

Lee and Liu[8] have introduced mirror graph during the discussion of $k$-graceful labeling. Devaraj[9] has shown that $M(m,n)$, the mirror graph of $K_{m,n}$ is E-cordial when $m + n$ is even while the generalized Petersen graph $P(n,k)$ is E-cordial when $n$ is even.

In the following section we have investigated some new results on E-cordial labeling for some mirror graphs.

2. Main Results
Theorem 2.1 Mirror graph of even cycle $C_n$ is E-cordial.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of cycle $C_n$ where $n$ is even and $G = C_n$. Let $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_{n+1}\}$ be the partite sets of $C_n$. Let $G'$ be the copy of $G$ and $V'_1 = \{v'_1, v'_2, \ldots, v'_n\}$ and $V'_2 = \{v'_1, v'_2, \ldots, v'_{n+1}\}$ be the copies of $V_1$ and $V_2$ respectively. Let $e'_1, e'_2, \ldots, e'_n$ be the edges of $G'$. The mirror graph $M(G)$ of $G$ is obtained from $G$ and $G'$ by joining each vertex of $V_2$ to its corresponding vertex in $V'_2$ by additional edges $e'_1, e'_2, \ldots, e'_n$.

We note that $|V(M(G))| = 2n$ and $|E(M(G))| = 2n + \frac{n}{2}$. Let $f : E(M(G)) \rightarrow \{0,1\}$ as follows:

For $1 \leq i \leq n$: 

International Journal of Contemporary Advanced Mathematics (IJCM), Volume (2) : Issue (1) : 2011
\[ f(e_i) = 1; \quad i \equiv 0, 1 \pmod{4}. \]
\[ f(e_i) = 0; \quad \text{otherwise}. \]
\[ f(e'_i) = 1; \quad i \equiv 1, 2 \pmod{4}. \]
\[ f(e'_i) = 0; \quad \text{otherwise}. \]

For \( 1 \leq j < \frac{n}{2} \):
\[ f(e'_j) = 1; \quad j \equiv 1 \pmod{2}. \]
\[ f(e'_j) = 0; \quad \text{otherwise}. \]

For \( j = \frac{n}{2} \):
\[ f(e'_j) = 0. \]

In view of the above defined labeling pattern \( f \) satisfies conditions for E-cordial labeling as shown in Table 1. That is, the mirror graph of even cycle \( C_n \) is E-cordial.

<table>
<thead>
<tr>
<th>vertex condition</th>
<th>edge condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0 \pmod{4} )</td>
<td>( v_j(0) = v_j(1) = n )</td>
</tr>
<tr>
<td>( n \equiv 2 \pmod{4} )</td>
<td>( v_j(0) = v_j(1) = n )</td>
</tr>
</tbody>
</table>

Table 1

Illustration 2.2: The E-cordial labeling for the mirror graph of cycle \( C_6 \) is shown in Figure 1.

![Figure 1](image)

Theorem 2.3: Mirror graph of path \( P_n \) is E-cordial for even \( n \).

Proof: Let \( v_1, v_2, \ldots, v_n \) be the vertices and \( e_1, e_2, \ldots, e_{n-1} \) be the edges of path \( P_n \) where \( n \) is even and \( G = P_n \). \( P_n \) is a bipartite graph. Let \( V'_1 = \{v_2, v_4, \ldots, v_n\} \) and \( V'_2 = \{v_1, v_3, \ldots, v_{n-1}\} \) be the bipartition of \( P_n \). Let \( G' \) be a copy of \( G \) and \( V'_1' = \{v'_2, v'_4, \ldots, v'_n\} \) and \( V'_2' = \{v'_1, v'_3, \ldots, v'_{n-1}\} \) be the copies of \( V'_1 \) and \( V'_2 \). Let \( e'_1, e'_2, \ldots, e'_{n-1} \) be the edges of \( G' \). The mirror graph \( M(G) \) of \( G \) is obtained from \( G \) and \( G' \) by
joining each vertex of \( V_2 \) to its corresponding vertex in \( V'_2 \) by additional edges \( e_1^*, e_2^*, \ldots, e_n^* \).

We note that \(|V(M(G))| = 2n\) and \(|E(M(G))| = 2(n-1)+\frac{n}{2}\). Let \( f : E(M(G)) \rightarrow \{0,1\} \) as follows:

For \( 1 \leq i < n-1 \):

\[
\begin{align*}
f(e_i) &= 1; & i &\equiv 0,1 \pmod{4}. \\
&= 0; & \text{otherwise.}
\end{align*}
\]

For \( i = n-1 \):

\[
\begin{align*}
f(e_i) &= 1.
\end{align*}
\]

For \( 1 \leq i \leq n-1 \):

\[
\begin{align*}
f(e_i') &= 1; & i &\equiv 0,3 \pmod{4}. \\
&= 0; & \text{otherwise.}
\end{align*}
\]

For \( 1 \leq j \leq \frac{n}{2} \):

\[
\begin{align*}
f(e_j') &= 1; & j &\equiv 0 \pmod{2}. \\
&= 0; & \text{otherwise.}
\end{align*}
\]

In view of the above defined labeling pattern \( f \) satisfies the conditions for E-cordial labeling as shown in Table 2. That is, the mirror graph of path \( P_n \) is E-cordial for even \( n \).

<table>
<thead>
<tr>
<th>vertex condition</th>
<th>edge condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0 \pmod{4} )</td>
<td>( v_j(0) = v_j(1) = n )</td>
</tr>
<tr>
<td>( n \equiv 2 \pmod{4} )</td>
<td>( v_j(0) = v_j(1) = n )</td>
</tr>
</tbody>
</table>

Table 2

Illustration 2.4: The E-cordial labeling for mirror graph of path \( P_8 \) is shown in Figure 2.

Figure 2.
**Theorem 2.5:** Mirror graph of hypercube $Q_k$ is E-cordial.

**Proof:**

Let $G = Q_k$ be a hypercube with $n$ vertices where $n = 2^k$. Let $V_i$ and $V_j$ be the bipartition of $Q_k$ and $G'$ be a copy of $G$ with $V_i'$ and $V_j'$ be the copies of $V_i$ and $V_j$ respectively. Let $e_1, e_2, ..., e_m$ be the edges of graph $G$ and $e_1', e_2', ..., e_m'$ be the edges of graph $G'$ where $m = \frac{n}{2}$. The mirror graph $M(G)$ of $G$ is obtained from $G$ and $G'$ by joining each vertex of $V_2$ to its corresponding vertex in $V_2'$ by additional edges $e_1', e_2', ..., e_m'$ then $|V(M(G))| = 2n$ and $|E(M(G))| = \frac{n(2k+1)}{2}$.

Define $f : E(M(G)) \rightarrow \{0, 1\}$ as follows:

**Case 1** $k \equiv 0 \pmod{2}$

Let $V_i = \{v_{i1}, v_{i2}, ..., v_{i\frac{n}{2}}\}$ and $V_i' = \{v_{i1}', v_{i2}', ..., v_{i\frac{n}{2}}'\}$ where $i = 1, 2$. All the edges incident to the vertices $v_{ij}$ and $v_{ij}'$ where $j \equiv 1 \pmod{2}$ are assigned the label 0 while the edges incident to the vertices $v_{ij}$ and $v_{ij}'$ where $j \equiv 0 \pmod{2}$ are assigned label 1.

For $1 \leq j \leq \frac{n}{2}$:

- If $j \equiv 0 \pmod{2}$, then $f(e_j) = 1$.
- Otherwise, $f(e_j) = 0$.

**Case 2** $k \equiv 1 \pmod{2}$

For $1 \leq i \leq n$:

- If $i \equiv 1 \pmod{2}$, then $f(e_i) = 1$.

For $1 \leq i \leq n$:

- If $i \equiv 0 \pmod{2}$, then $f(e_i) = 0$.

For $1 \leq j \leq \frac{n}{2}$:

- If $j \equiv 0 \pmod{2}$, then $f(e_j) = 1$.

In view of the above defined labeling pattern $f$ satisfies the conditions for E-cordial labeling as shown in Table 3.

<table>
<thead>
<tr>
<th>vertex condition</th>
<th>edge condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$e_j(0) = e_j(1) = \frac{n(2k+1)}{4}$</td>
</tr>
</tbody>
</table>

**Table 3**

That is, the mirror graph of hypercube $Q_k$ is E-cordial.

**Illustration 2.6**

The E-cordial labeling for mirror graph of hypercube $Q_3$ is shown in Figure 3.
3. CONCLUDING REMARKS
Here we investigate E-cordial labeling for some mirror graphs. To investigate similar results for other graph families and in the context of different graph labeling problems is an open area of research.

REFERENCES
Antimagic Labeling in the Context of Switching of a Vertex

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Received 30 November 2012; accepted 10 December 2012

Abstract. A graph with \( q \) edges is called antimagic if its edges can be labeled with 1, 2, ..., \( q \) such that the sums of the labels of the edges incident to each vertex are distinct. Here we prove that the graphs obtained by switching of a pendant vertex in path \( P_n \), switching of a vertex in cycle \( C_n \), switching of a rim vertex in wheel \( W_n \), switching of an apex vertex in helm \( H_n \) and switching of a vertex of degree 2 in fan \( f_n \) admit antimagic labeling.

AMS Mathematics Subject Classification (2010): 05C78, 05C38

Keywords: Switching of a vertex, Antimagic labeling, Antimagic graphs.

1. Introduction

We begin with a finite, connected and undirected graph \( G = (V(G), E(G)) \) without loops and multiple edges. Throughout this paper \( |V(G)| \) and \( |E(G)| \) respectively denote the number of vertices and number of edges in \( G \). For any undefined notation and terminology we rely upon Gross and Yellen\textsuperscript{[3]}. A brief summary of definitions and existing results is provided in order to maintain the compactness.

Definition 1.1. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).

According to Beineke and Hegde\textsuperscript{[1]} labeling of discrete structure is a frontier between graph theory and theory of numbers. For an extensive survey of graph labeling as well as bibliographic references there in we refer to Gallian\textsuperscript{[2]}.
Definition 1.2. A graph with \( q \) edges is called antimagic if its edges can be labeled with \( 1, 2, \ldots, q \) such that the sums of the labels of the edges incident to each vertex are distinct.

The concept of antimagic graph was introduced by Hartsfield and Ringel[4]. They showed that paths \( P_n \) (\( n \geq 3 \)), cycles, wheels, and complete graphs \( K_n \) (\( n \geq 3 \)) are antimagic. They conjectured that

(i) all trees except \( K_2 \) are antimagic,
(ii) all connected graphs except \( K_2 \) are antimagic.

These conjectures are unsettled till today.

Definition 1.3. A vertex switching \( G_v \) of a graph \( G \) is the graph obtained by taking a vertex \( v \) of \( G \), removing all the edges to \( v \) and adding edges joining \( v \) to every other vertex which are not adjacent to \( v \) in \( G \).

Definition 1.4. The wheel graph \( W_n \) is defined to be the join \( K_1 + C_n \). The vertex corresponding to \( K_1 \) is known as apex vertex and vertices corresponding to cycle are known as rim vertices while the edges corresponding to \( C_n \) are known as rim edges. We continue to recognize apex of wheel as the apex of the respective graphs obtained from wheel.

Definition 1.5. The helm \( H_n \) is the graph obtained from a wheel \( W_n \) by attaching a pendant edge to each rim vertex.

Definition 1.6. A fan graph \( f_n \) is obtained by \( P_n + K_1 \).

In the following section we will investigate some new results on Antimagic labeling of graphs.

2. Main Results

Theorem 2.1. Switching of a pendant vertex in a path \( P_n \) is antimagic.

Proof: Let \( v_1, v_2, \ldots, v_n \) be the vertices of \( P_n \) and \( G_v \) denotes the graph obtained by switching of a pendant vertex \( v \) of \( G = P_n \). Without loss of generality let the switched vertex be \( v_1 \). We note that \( |V(G_{v_1})| = n \) and \( |E(G_{v_1})| = 2n - 4 \). We define

\[
f : E(G_{v_1}) \to \{1, 2, \ldots, 2n - 4\}
\]

as follows:

For \( 2 \leq i \leq n - 1 \):

\[
f(v_i v_{i+1}) = i - 1;
\]

For \( 3 \leq i \leq n \):

\[
f(v_i v_j) = n + i - 4 ,
\]

Above defined edge labeling function will generate all distinct vertex labels as per the definition of antimagic labeling. Hence the graph obtained by switching of a pendant vertex in a path \( P_n \) is antimagic.
Antimagic Labeling in the Context of Switching of a Vertex

**Illustration 2.2.** In Figure 1 the graph obtained by switching of a vertex \( v_1 \) in path \( P_3 \) and its antimagic labeling is shown.

**Theorem 2.3.** Switching of a vertex in cycle \( C_n \) is antimagic.

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be the successive vertices of \( C_n \) and \( G_v \) denotes graph obtained by switching of vertex \( v \) of \( G = C_n \). Without loss of generality let the switched vertex be \( v_1 \). We note that \( |V(G_{v_1})| = n \) and \( |E(G_{v_1})| = 2n - 5 \). We define

\[
f : E(G_{v_1}) \rightarrow \{1, 2, \ldots, 2n - 5\}
\]

as follows:

For \( 3 \leq i \leq n \):

\[
f(v_i v_{i-1}) = 2(i - 2);
\]

\[
f(v_i v_{i+1}) = 2i - 5,
\]

Above defined edge labeling function will generate all distinct vertex labels as per the definition of an antimagic labeling. Hence the graph obtained by switching of a vertex in a cycle \( C_n \) is antimagic.

**Illustration 2.4.** In Figure 2 the graph obtained by switching of a vertex \( v_1 \) in cycle \( C_7 \) and its antimagic labeling is shown.

**Theorem 2.5.** Switching of a rim vertex in a wheel \( W_n \) is antimagic.

**Proof.** Let \( v \) as the apex vertex and \( v_1, v_2, \ldots, v_n \) be the rim vertices of wheel \( W_n \). Let \( G_{v_1} \) denotes graph obtained by switching of a rim vertex \( v_1 \) of \( G = W_n \). We note that
\[ V(G_n) = n + 1 \text{ and } |E(G_n)| = 3n - 6. \]

We define \( f : E(G_n) \to \{1, 2, ..., 3n - 6\} \) as follows.

For \( 2 \leq i \leq n - 1 \):
\[ f(v_i v_i) = i - 1; \]

For \( 2 \leq i \leq n \):
\[ f(v_i v_1) = 2n - i - 1; \]

For \( 3 \leq i \leq n - 1 \):
\[ f(v_i v_2) = 2n + i - 5, \]

Above defined edge labeling function will generate all distinct vertex labels as per the definition of an antimagic labeling. Hence the graph obtained by switching of a rim vertex in a wheel \( W_n \) is antimagic.

**Illustration 2.6.** In Figure 3 the graph obtained by switching of a vertex \( v_1 \) in wheel \( W_8 \) and its antimagic labeling is shown.

**Theorem 2.7.** Switching of an apex vertex in helm \( H_n \) is antimagic.

**Proof.** Let \( H_n \) be a helm with \( v \) as the apex vertex, \( v_1, v_2, ..., v_n \) be the vertices of cycle and \( u_1, u_2, ..., u_n \) be the pendant vertices. Let \( G_v \) denotes graph obtained by switching of an apex vertex \( v \) of \( G = H_n \). We note that \( |V(G_v)| = 2n + 1 \) and \( |E(G_v)| = 3n \). We define \( f : E(G_v) \to \{1, 2, ..., 3n\} \) as follows.

**Case 1:** \( n \equiv 0 \pmod{3}; n \neq 3 \)
\[ f(v u_1) = 2; \]
\[ f(v u_i) = 1; \]
\[ f(v_1 v_2) = 3; \]
For \( 2 \leq i \leq n \)
Antimagic Labeling in the Context of Switching of a Vertex

\[ f(v_u_i) = 3i - 2; \]
\[ f(v_i u_t) = 3i - 1; \]
\[ f(v_i v_{i+1}) = 3i \] where \( v_{n+1} = v_1 \)

The case when \( n = 3 \) is to be dealt separately and the graph is labeled as shown in Figure 4.

Figure 4

Case 2: \( n \equiv 1, 2 \pmod{3} \)
For \( 1 \leq i \leq n \):
\[ f(v_u_i) = 3i - 2; \]
\[ f(v_i u_t) = 3i - 1; \]
\[ f(v_i v_{i+1}) = 3i \] where \( v_{n+1} = v_i \)

Above defined edge labeling function will generate all distinct vertex labels as per the definition of an antimagic labeling. Hence the graph obtained by switching of an apex vertex in a helm \( H_n \) is antimagic.

Figure 5
Illustration 2.8. In Figure 5 the graph obtained by switching of an apex vertex \( v \) in helm \( H_5 \) and antimagic labeling is shown.

Theorem 2.9. Switching of a vertex having degree 2 in fan \( f_n \) is antimagic.

Proof. Let \( v \) as the apex vertex and \( v_1, v_2, \ldots, v_n \) be the vertices of fan \( f_n \). Let \( G_v \) denotes graph obtained by switching of a vertex \( v_1 \) having degree 2 of \( G = f_n \). We note that \( |V(G_v)| = n + 1 \) and \( |E(G_v)| = 3n - 5 \).

We define \( f : E(G_v) \to \{1, 2, \ldots, 3n - 5\} \) as follows.

For \( 2 \leq i \leq n \):
\[ f(v_i) = i - 1; \]

For \( 2 \leq i \leq n - 1 \):
\[ f(v_i, v_{i+1}) = n + i - 2; \]

For \( 3 \leq i \leq n \):
\[ f(v_i, v) = 2n + i - 5; \]

Above defined edge labeling function will generate all distinct vertex labels as per the definition of an antimagic labeling. Hence the graph obtained by switching of a vertex having degree 2 in fan \( f_n \) is antimagic.

Illustration 2.10. In Figure 6 the graph obtained by switching of a vertex \( v_1 \) having degree 2 in fan \( f_5 \) and its antimagic labeling is shown.

3. Concluding Remarks

The investigations reported here is an effort to relate graph operations and antimagic labeling. To investigate some characterization(s) or sufficient condition(s) for any graph to be antimagic is an open area of research.
Antimagic Labeling in the Context of Switching of a Vertex

REFERENCES

FURTHER RESULTS ON E-CORDIAL LABELING

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Abstract

Here we investigate E-cordial labeling of some graphs. We prove that the Möbius ladder $M_n$, middle graph of $C_n$ and crown $C_n \odot K_1$ admit E-cordial labeling for even $n$ while bistar $B_{n,n}$ and its square graph $B_{n,n}^2$ admit E-cordial labeling for odd $n$.

1. Introduction

Throughout this work, by $G = (V(G), E(G))$ we mean a finite, connected and undirected graph without loops and multiple edges. We will give brief summary of definitions and existing results.

Definition 1.1. If the vertices of the graph are assigned values subject to certain condition(s), then it is known as graph labeling.

© 2012 Pushpa Publishing House
2010 Mathematics Subject Classification: 05C78.
Keywords and phrases: E-cordial labeling, Möbius ladder, middle graph, crown, bistar.
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Communicated by Kewen Zhao
Received March 15, 2012
Most of the graph labeling techniques trace their origin to graceful labeling introduced independently by Rosa [7] and Golomb [3] defined as follows.

**Definition 1.2.** A function \( f : V(G) \rightarrow \{0, 1, ..., |E(G)|\} \) is called **graceful labeling** of graph \( G \) if \( f \) is injective and the induced function \( f^* : E(G) \rightarrow \{1, 2, ..., |E(G)|\} \) defined by \( f^*(e = uv) = |f(u) - f(v)| \) is bijective. A graph which admits graceful labeling is called a **graceful graph**.

The famous Ringel-Kotzig conjecture [6] and many illustrious works on it brought a tide of labeling problems with graceful theme.

**Definition 1.3.** A graph \( G \) is said to be **edge-graceful** if there exists a bijection \( f : E(G) \rightarrow \{1, 2, ..., |E(G)|\} \) such that the induced function \( f^* : V(G) \rightarrow \{0, 1, 2, ..., |V(G)| - 1\} \) defined by \( f^*(x) = \sum f(xy)(\text{mod}|V(G)|) \), taken over all edges \( xy \) is a bijection.

The notion of edge gracefulness was introduced by Lo [5].

**Definition 1.4.** A mapping \( f : V(G) \rightarrow \{0, 1\} \) is called **binary vertex labeling** of \( G \) and \( f(v) \) is called the **label** of vertex \( v \) of \( G \) under \( f \).

**Notation.** For an edge \( e = uv \), the induced edge labeling \( f^* : E(G) \rightarrow \{0, 1\} \) is given by \( f^*(e = uv) = |f(u) - f(v)| \), then \( u_f(i) \) = the number of vertices of \( G \) having label \( i \) under \( f \) and let \( e_f(i) \) = the number of edges of \( G \) having label \( i \) under \( f^* \) for \( i = 0, 1 \).

**Definition 1.5.** A binary vertex labeling of graph \( G \) is called a **cordial labeling** if \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \). A graph \( G \) is called **cordial** if admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [1]. He also investigated several results on this newly defined concept.

**Definition 1.6.** Let \( G \) be a graph with vertex set \( V(G) \) and edge set
Further Results on $E$-cordial Labeling

$E(G)$ and let $f : E(G) \to \{0, 1\}$. Define $f^*$ on $V(G)$ by $f^*(v) = \sum \{f(uv) | uv \in E(G)\} \pmod{2}$. The function $f$ is called an $E$-cordial labeling of $G$ if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph is called $E$-cordial if admits $E$-cordial labeling.

In 1997, Yilmaz and Cahit [12] introduced $E$-cordial labeling as a weaker version of edge-graceful labeling having the blend of cordial labeling. They proved that the trees with $n$ vertices, $K_n, C_n$ are $E$-cordial if and only if $n \neq 2 \pmod{4}$ while $K_{m,n}$ admits $E$-cordial labeling if and only if $m + n \neq 2 \pmod{4}$.

Vaidya and Bijukumar [8] proved that the graphs obtained by duplication of an arbitrary vertex as well as an arbitrary edge in cycle $C_n$ admit $E$-cordial labeling. In addition to this, they also derived that the joint sum of two copies of cycle $C_n$, the split graph of even cycle $C_n$ and the shadow graph of path $P_n$ for even $n$ are $E$-cordial graphs. The same authors in [9] proved that the middle graph, total graph and split graph of $P_n$ and the composition of $P_n$ with $P_2$ admit $E$-cordial labeling.

Vaidya and Vyas [10] proved that the mirror graphs of even cycle $C_n$, even path $P_n$ and hypercube $Q_k$ are $E$-cordial graphs. The same authors in [11] proved that $K_n \times P_2$ and $P_n \times P_2$ are $E$-cordial graphs for even $n$ while $W_n \times P_2$ and $K_{1,n} \times P_2$ are $E$-cordial graphs for odd $n$.

**Definition 1.7.** The middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of $G$ or one is a vertex of $G$ and the other is an edge incident on it.

**Definition 1.8.** For a simple graph $G$, the square of graph $G$ is denoted by $G^2$ and defined as the graph obtained with the same vertex set as of $G$ and two vertices are adjacent in $G^2$ if they are at a distance 1 or 2 apart in $G$. 
In the following section, we investigate some new results on $E$-cordial labeling of graphs. The definition of special graph families can be found in Gallian [2] while for standard graph theoretic notations and terminology we refer to Harary [4].

2. Main Results

Theorem 2.1. Möbius ladder $M_n$ is $E$-cordial for even $n$.

Proof. Let $G = M_n$ be a Möbius ladder graph for even $n$. We note that $|V(G)| = 2n$ and $|E(G)| = 3n$. We define $f : E(G) \to \{0, 1\}$ as follows:

$f(v_1v'_n) = 1$,

$f(v_nv'_1) = 1$.

For $1 \leq i < n$:

$f(v_iv'_{i+1}) = \begin{cases} 1, & i \equiv 0 \pmod{2}, \\ 0, & \text{otherwise,} \end{cases}$

$f(v'_{i+1}v'_i) = \begin{cases} 1, & i \equiv 0 \pmod{2}, \\ 0, & \text{otherwise.} \end{cases}$

For $1 \leq i \leq n$:

$f(v_iv'_i) = \begin{cases} 1, & i \equiv 0 \pmod{2}, \\ 0, & \text{otherwise.} \end{cases}$

In view of the above defined labeling pattern $f$ satisfies the conditions for $E$-cordial labeling as shown in Table 1. Hence $M_n$ is $E$-cordial for even $n$.

<table>
<thead>
<tr>
<th>even $n$</th>
<th>$v_f(0) = v_f(1) = n$</th>
<th>$e_f(0) = e_f(1) = \frac{3n}{2}$</th>
</tr>
</thead>
</table>

Table 1
Illustration 2.2. $E$-cordial labeling of $M_6$ is shown in Figure 1.

Figure 1

Theorem 2.3. $M(C_n)$ admits $E$-cordial labeling for even $n$.

Proof. Let $G = M(C_n)$ be a middle graph of cycle $C_n$ for even $n$. We note that $|V(G)| = 2n$ and $|E(G)| = 3n$. We define $f : E(G) \to \{0, 1\}$ as follows:

For $1 \leq i < n$:

$$f(v_i v_{i+1}) = \begin{cases} 0, & i \equiv 0 \pmod{2}, \\ 1, & \text{otherwise}, \end{cases}$$

$$f(v_n v_1) = 0,$$

$$f(v'_n v_1) = 1,$$

$$f(v'_i v_1) = 0; \quad 1 \leq i \leq n,$$

$$f(v'_i v_{i+1}) = 1; \quad 1 \leq i < n.$$

In view of the above defined labeling pattern $f$ satisfies the conditions for $E$-cordial labeling as shown in Table 2. Hence $M(C_n)$ is $E$-cordial for even $n$. 

Table 2

<table>
<thead>
<tr>
<th>vertex condition</th>
<th>edge condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>even $n$</td>
<td>$v_f(0) = v_f(1) = n$</td>
</tr>
</tbody>
</table>

Illustration 2.4. $E$-cordial labeling of $M(C_6)$ is shown in Figure 2.

Theorem 2.5. The crown $C_n \odot K_1$ is $E$-cordial for even $n$.

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of $C_n$, where $n$ is even, while $u_1, u_2, ..., u_n$ be the newly added pendant vertices in order to construct $G$ and denote $C_n \odot K_1 = G$. We note that $|V(G)| = 2n$ and $|E(G)| = 2n$. We define $f : E(G) \to \{0, 1\}$ as follows:

For $1 \leq i \leq n - 1$:

$$f(v_iu_i) = f(v_iw_{i+1}) = \begin{cases} 1, & i \equiv 1 \text{ (mod 2)}, \\ 0, & \text{otherwise}, \end{cases}$$

$$f(v_nu_n) = f(v_nv_1) = 0.$$
Further Results on $E$-cordial Labeling

In view of the above defined labeling pattern $f$ satisfies the conditions for $E$-cordial labeling as shown in Table 3. Hence crown $C_n \odot K_1$ is $E$-cordial for even $n$.

| Table 3 |
|-----------------|-----------------|
| even $n$        | vertex condition | edge condition |
|                 | $v_f(0) = v_f(1) = n$ | $e_f(0) = e_f(1) = n$ |

Illustration 2.6. $E$-cordial labeling of crown $C_6 \odot K_1$ is shown in Figure 3.

Figure 3

**Theorem 2.7.** The bistar $B_{n, n}$ is $E$-cordial for odd $n$.

**Proof.** Let $G = B_{n, n}$ for odd $n$ with vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$, where $u_i, v_i$ are pendant vertices. Then $|V(G)| = 2n + 2$ and $|E(G)| = 2n + 1$. We define $f : E(G) \to \{0, 1\}$ as follows:

For $1 \leq i \leq n$:

\[
\begin{align*}
    f( uv ) &= 0, \\
    f( uu_i ) &= 0, \\
    f( vv_i ) &= 1.
\end{align*}
\]
In view of the above defined labeling pattern $f$ satisfies the conditions for $E$-cordial labeling as shown in Table 4. Hence bistar $B_{n,n}$ is $E$-cordial for odd $n$.

<table>
<thead>
<tr>
<th>Table 4</th>
<th>vertex condition</th>
<th>edge condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>odd $n$</td>
<td>$v_f(0) = v_f(1) = n + 1$</td>
<td>$e_f(0) = e_f(1) + 1 = n + 1$</td>
</tr>
</tbody>
</table>

**Illustration 2.8.** $E$-cordial labeling of $B_{5,5}$ is shown in Figure 4.

**Figure 4**

**Theorem 2.9.** $B_{n,n}^2$ is $E$-cordial for odd $n$.

**Proof.** Consider $B_{n,n}$ for odd $n$ with vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$, where $u_i$, $v_i$ are pendant vertices. Let $G$ be the graph $B_{n,n}^2$. Then $|V(G)| = 2n + 2$ and $|E(G)| = 4n + 1$. We define $f : E(G) \rightarrow \{0, 1\}$ as follows:

For $1 \leq i \leq n$:

$$f(vu) = 1,$$

$$f(vv_i) = 1,$$

$$f(uu_i) = 0,$$

$$f(uu_i) = f(vu_i) = \begin{cases} 1, & 1 \leq i \leq \frac{n-1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$
In view of the above defined labeling pattern $f$ satisfies the conditions for $E$-cordial labeling as shown in Table 5. Hence bistar $B_{n,n}^2$ is $E$-cordial for odd $n$.

| Table 5 |
|-----------------|-----------------------|
| vertex condition | edge condition         |
| odd $n$          | $v_f(0) = v_f(1) = n + 1$ | $e_f(0) = e_f(1) + 1 = 2n + 1$ |

Illustration 2.10. $E$-cordial labeling of $B_{5,5}^2$ is shown in Figure 5.

Figure 5

3. Concluding Remarks

Some new families of $E$-cordial graphs are investigated. To investigate similar results for other graph families and in the context of different graph labeling problems is an open area of research.

References


E-CORDIAL LABELING IN THE CONTEXT OF SWITCHING OF A VERTEX

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[Received-07/07/2012, Accepted-03/09/2012]

ABSTRACT:

Let G=(V(G),E(G)) be a graph and f : E(G) → {0,1} be a binary edge labeling. Define f' : E(G) → \{0,1\} by 
\[
f'(uv) = \sum_{uv \in E(G)} f(uv) \mod 2
\]
The function f is called E-cordial labeling of G if \(|f_v(0) - f_v(1)| \leq 1\) and \(|f_e(0) - f_e(1)| \leq 1\). In the present work we discuss E-cordial labeling in the context of switching of a vertex in cycle, wheel, helm and closed helm.

Keywords: E-cordial labeling, Cycle, Wheel, Vertex switching

[I] INTRODUCTION

We begin with finite, connected and undirected graph \(G=(V(G),E(G))\) without loops and multiple edges. For standard terminology and notations we follow Harary[4]. For extensive survey of graph labeling as well as bibliographic references we refer Gallian[2].

1.1. Definition If the vertices of the graph are assigned values subject to certain condition(s) then it is known as graph labeling.

Most of the graph labeling techniques trace their origin to graceful labeling introduced independently by Rosa[7] and Golomb[3] which is defined as follows.

1.2. Definition

A function \(f : V(G) \rightarrow \{0,1,\ldots,|E(G)|\}\) is called graceful labeling of graph G if f is injective and the induced function \(f^* : E(G) \rightarrow \{1,2,\ldots,|E(G)|\}\) defined by
E-CORDIAL LABELING IN THE CONTEXT OF SWITCHING OF A VERTEX

S. K. Vyaidya and N. B. Vyas

1.3. Definition
A graph $G$ is said to be edge-graceful if there exists a bijection $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$ such that the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, |V(G)| - 1\}$ defined by $f^*(x) = \sum (f(xy)) \mod |V(G)|$, taken over all edges $xy$, is a bijection.

1.4. Definition
A mapping $f : V(G) \to \{0, 1\}$ is called binary vertex labeling of $G$ and $f(v)$ is called the label of vertex $v$ of $G$ under $f$.

1.5. Notation
For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$ then $v_f(i) = \text{the number of vertices of } G \text{ having label } i \text{ under } f$ and let $e_f(i) = \text{the number of edges of } G \text{ having label } i \text{ under } f^*$ for $i = 0, 1$.

1.6. Definition
A binary vertex labeling of graph $G$ is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is called cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit[1]. He also investigated several results on this newly defined concept.

1.7. Definition
Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$ and let $f : E(G) \to \{0, 1\}$. Define $f^*$ on $V(G)$ by $f^*(v) = \sum \{f(uv) \mod |E(G)|\}$, taken over all edges $uv$. The function $f$ is called an E-cordial labeling of $G$ if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph is called E-cordial if it admits E-cordial labeling.

In 1997 Yilmaz and Cahit[12] have introduced E-cordial labeling as a weaker version of edge-graceful labeling having the blend of cordial labeling. They proved that the trees with $n$ vertices, $K_n$, $C_n$ are E-cordial if and only if $2(2n) \mod 4 = /$ while $K_{m,n}$ admits E-cordial labeling if and only if $m + n \not\equiv 2(\mod 4)$. Vaidya and Lekha[8] have proved that the graphs obtained by duplication of an arbitrary vertex as well as an arbitrary edge in cycle $C_n$ admit E-cordial labeling. In addition to this they also derived that the joint sum of two copies of cycle $C_n$, the split graph of even cycle $C_n$ and the shadow graph of path $P_n$ for even $n$ are E-cordial graphs. The same authors in [9] proved that the middle graph, total graph and split graph of $P_n$ and the composition of $P_n$ with $P_2$ admit E-cordial labeling.

Vaidya and Vyas[10] proved that the mirror graphs of even cycle $C_n$, even path $P_n$ and hypercube $Q_k$ are E-cordial graphs. The same authors in [11] proved that $K_n \times P_2$ and $P_n \times P_2$ are E-cordial graphs for even $n$ while $W_n \times P_2$ and $K_{l,n} \times P_2$ are E-cordial graphs for odd $n$.

1.8. Definition
A vertex switching $G_v$ of a graph $G$ is the graph obtained by taking a vertex $v$ of $G$,
removing all the edges incident to \( v \) and adding edges joining \( v \) to every other vertex which are not adjacent to \( v \) in \( G \).

1.9. Definition
The wheel graph \( W_n \) is defined to be the join \( K_1 + C_n \). The vertex corresponding to \( K_1 \) is known as apex vertex and vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges. We continue to recognize apex of wheel as the apex of respective graphs obtained from wheel.

1.10. Definition
The helm \( H_n \) is the graph obtained from a wheel \( W_n \) by attaching a pendant edge to each rim vertex.

1.11. Definition
The closed helm \( CH_n \) is the graph obtained from a helm \( H_n \) by joining each pendant vertex to form a cycle.

We continue to recognize terminology used in Definition 1.9 for Definitions 1.10 and 1.11 also.

In the following section we will investigate some new results on E-cordial labeling of graphs.

[II] MAIN RESULTS

2.1. Theorem
The graph obtained by switching of an arbitrary vertex in cycle \( C_n \) admits E-cordial labeling except for \( n \equiv 2(\text{mod} \ 4) \).

Proof: Let \( v_1, v_2, ..., v_n \) be the successive vertices of \( C_n \) and \( G_v \) denotes graph obtained by switching of vertex \( v \) of \( G \). Without loss of generality let the switched vertex be \( v_1 \) and we initiate the labeling from \( v_1 \). To define \( f : E(G_v) \rightarrow \{0, 1\} \) we consider following cases.

Case 1: When \( n \) is odd

For \( 3 \leq i \leq n-1 \):

\[
f(v_i, v_i) = \begin{cases} 
1, & i \equiv 1(\text{mod} \ 2) \\
0, & \text{otherwise}; 
\end{cases}
\]

Subcase 1: \( n \equiv 3(\text{mod} \ 4) \)

For \( 2 \leq i \leq n-1 \):

\[
f(v_i, v_i) = \begin{cases} 
1, & i \equiv 0(\text{mod} \ 2) \\
0, & \text{otherwise}; 
\end{cases}
\]

Subcase 2: \( n \equiv 1(\text{mod} \ 4) \)

For \( 2 \leq i \leq n-1 \):

\[
f(v_i, v_i) = \begin{cases} 
0, & i \equiv 0(\text{mod} \ 2) \\
1, & \text{otherwise}; 
\end{cases}
\]

Case 2: When \( n \) is even

Subcase 1: \( n \equiv 0(\text{mod} \ 4) \)

\[
f(v_1, v_1) = 1;
\]

For \( 3 \leq i \leq n-1 \):

\[
f(v_i, v_i) = \begin{cases} 
0, & i \equiv 1(\text{mod} \ 2) \\
1, & \text{otherwise}; 
\end{cases}
\]

\[
f(v_i, v_i) = \begin{cases} 
1, & i \equiv 1(\text{mod} \ 2) \\
0, & \text{otherwise}; 
\end{cases}
\]

Subcase 2: \( n \equiv 2(\text{mod} \ 4) \)

A graph with \( n \) vertices is not E-cordial when \( n \equiv 2(\text{mod} \ 4) \) as observed by Yilmaz and Cahit [12].

In view of the labeling pattern defined above \( f \) satisfies the condition \( |v_j(0) - v_j(1)| \leq 1 \) and \( |e_j(0) - e_j(1)| \leq 1 \) as shown in Table 1.

Hence \( G_v \) admits E-cordial labeling.

2.2. Illustration
Consider the graph obtained by switching of a vertex \( v_1 \) in cycle \( C_7 \). The E-cordial labeling is as shown in Figure 1.
2.3. Theorem
The graph obtained by switching of a rim vertex in wheel $W_n$ admits E-cordial labeling except for $n \equiv 1(\text{mod}4)$.

Proof: Let $v$ as the apex vertex and $v_1, v_2, \ldots, v_n$ be the rim vertices of wheel $W_n$. Let $G_v$ denote the graph obtained by switching of a rim vertex $v_i$ of $G=W_n$. We define $f : E(G_v) \rightarrow \{0, 1\}$ as follows.

Case 1: When $n$ is even
For $2 \leq i \leq n$:
$$f(v_i v_{i+1}) = \begin{cases} 1, & i \equiv 0(\text{mod } 2) \\ 0, & \text{otherwise} \end{cases}$$
For $2 \leq i \leq n-1$:
$$f(v_i v_{i+1}) = \begin{cases} 1, & i \equiv 0(\text{mod } 2) \\ 0, & \text{otherwise} \end{cases}$$
For $3 \leq i \leq n-1$:
$$f(v_i v_{i+1}) = \begin{cases} 0, & 3, i, \frac{n+1}{2} \\ 1, & \text{otherwise} \end{cases}$$

Case 2: When $n$ is odd
Subcase 1: $n \equiv 3(\text{mod } 4)$
For $2 \leq i \leq n$:
$$f(v_i v_{i+1}) = 0;$$
For $3 \leq i \leq n-1$:
$$f(v_i v_{i+1}) = \begin{cases} 0, 3, i, \frac{n+1}{2} \\ 1, & \text{otherwise} \end{cases}$$
Subcase 2: $n \equiv 1(\text{mod } 4)$
In this case $|V(W_n)| = n + 1 \equiv 2(\text{mod } 4)$ equivalently $n \equiv 1(\text{mod } 4)$. This graph is not E-cordial because the graph $G$ with number of vertices congruent to $2(\text{mod } 4)$ does not admits E-cordial labeling as observed by Yilmaz and Cahit [12].

In view of the labeling pattern defined above $f$ satisfies the condition $|v_i(0) - v_i(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in Table 2.

Hence $G_v$ admits E-cordial labeling.

2.4. Illustration
Consider the graph obtained by switching of a rim vertex in wheel $W_8$. The E-cordial labeling is as shown in Figure 2.

2.5. Theorem
The graph obtained by switching of an apex vertex in helm $H_n$ admits E-cordial labeling.

Proof: Let $H_n$ be a helm with $v$ as the apex vertex, $v_1, v_2, \ldots, v_n$ be the vertices of cycle and $u_1, u_2, \ldots, u_n$ be the pendant vertices. Let $G_v$ denote the graph obtained by switching of
an apex vertex $v$ of $G=H_n$. We define $f : E(G_v) \rightarrow \{0,1\}$ as follows.
For $1 \leq i \leq n$:

\[
\begin{align*}
  f(v_iu_i) & = 1; \\
  f(v_iu_{i+1}) & = 0; \quad (v_{n+1} = v_1) \\
  f(v_iu_{i+1}) & = \begin{cases} 0, & i, \lfloor \frac{n}{2} \rfloor \\
  1, & \text{otherwise}. 
\end{cases}
\end{align*}
\]

In view of the labeling pattern defined above $f$ satisfies the condition $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in Table 3.
Hence $G_v$ admits E-cordial labeling.

2.6. Illustration
Consider the graph obtained by switching of an apex vertex in Helm $H_6$. The E-cordial labeling is as shown in Figure 3.

![Figure 3](image)

2.7. Theorem
The graph obtained by switching of an apex vertex in closed helm $CH_n$ admits E-cordial labeling.

Proof: Let $v$ as the apex vertex, $v_1, v_2, \ldots, v_n$ be the vertices of inner cycle and $u_1, u_2, \ldots, u_n$ be the vertices of outer cycle $CH_n$. Let $G_v$ denotes graph obtained by switching

of an apex vertex $v$ of $G=CH_n$. We define $f : E(G_v) \rightarrow \{0,1\}$ as follows.
For $1 \leq i \leq n$:

\[
\begin{align*}
  f(v_iu_i) & = 1; \\
  f(v_iu_{i+1}) & = 0; \quad (v_{n+1} = v_1) \\
  f(u_iu_{i+1}) & = 0; \quad (u_{n+1} = u_i) \\
  f(v_iu_i) & = 1.
\end{align*}
\]

In view of the labeling pattern defined above $f$ satisfies the condition $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ as shown in Table 4.
Hence $G_v$ admits E-cordial labeling.

2.8. Illustration
Consider the graph obtained by switching of an apex vertex in closed helm $CH_5$. The E-cordial labeling is as shown in Figure 4.

![Figure 4](image)

[III] CONCLUDING REMARKS
Here we investigate E-cordial labeling in the context of switching of a vertex of some graphs. To investigate similar results for other graph families and in the context of different graph labeling problems is an open area of research.
E-CORDIAL LABELING IN THE CONTEXT OF SWITCHING OF A VERTEX

<table>
<thead>
<tr>
<th>vertex condition</th>
<th>edge condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 (mod 4)$</td>
<td>$v_j(0) = v_j(1) = \frac{n}{2}$ $e_j(0)+1 = e_j(1) = \frac{2n-5}{2}$</td>
</tr>
<tr>
<td>$n \equiv 1 (mod 4)$</td>
<td>$v_j(0) = v_j(1)+1 = \frac{n}{2}$ $e_j(0) = e_j(1)+1 = \frac{2n-5}{2}$</td>
</tr>
<tr>
<td>$n \equiv 2 (mod 4)$</td>
<td>$v_j(0)+1 = v_j(1)+1 = \frac{n}{2}$ $e_j(0)+1 = e_j(1) = \frac{2n-5}{2}$</td>
</tr>
</tbody>
</table>

Table 1: vertex and edge conditions corresponding to Theorem 2.1

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 (mod 4)$</td>
<td>$v_j(0)-1 = v_j(1) = \frac{n}{2}$ $e_j(0) = e_j(1) = \frac{3(n-2)}{2}$</td>
</tr>
<tr>
<td>$n \equiv 2 (mod 4)$</td>
<td>$v_j(0) = v_j(1)-1 = \frac{n}{2}$ $e_j(0) = e_j(1) = \frac{3(n-2)}{2}$</td>
</tr>
<tr>
<td>$n \equiv 3 (mod 4)$</td>
<td>$v_j(0) = v_j(1) = \frac{n+1}{2}$ $e_j(0)+1 = e_j(1) = \frac{3(n-2)+1}{2}$</td>
</tr>
</tbody>
</table>

Table 2: vertex and edge conditions corresponding to Theorem 2.3

<table>
<thead>
<tr>
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<th>edge condition</th>
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<tbody>
<tr>
<td>$n \equiv 0 (mod 2)$</td>
<td>$v_j(0)-1 = v_j(1) = n$ $e_j(0) = e_j(1) = n + \frac{n}{2}$</td>
</tr>
<tr>
<td>$n \equiv 1 (mod 2)$</td>
<td>$v_j(0) = v_j(1)-1 = n$ $e_j(0) = e_j(1)+1 = n + \frac{n}{2}$</td>
</tr>
</tbody>
</table>

Table 3: vertex and edge conditions corresponding to Theorem 2.5

<table>
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<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 (mod 2)$</td>
<td>$v_j(0)-1 = v_j(1) = n$ $e_j(0) = e_j(1) = 2n$</td>
</tr>
<tr>
<td>$n \equiv 1 (mod 2)$</td>
<td>$v_j(0) = v_j(1)-1 = n$ $e_j(0) = e_j(1) = 2n$</td>
</tr>
</tbody>
</table>

Table 4: vertex and edge conditions corresponding to Theorem 2.7

REFERENCES


E-CORDIAL LABELING IN THE CONTEXT OF SWITCHING OF A VERTEX

Journal of Mathematics Research, 3: 105-111. doi:10.5539/jmr.v3n4p105


Some Results on E-cordial Labeling

S. K. Vaidya and N. B. Vyas

Abstract—A binary vertex labeling \( f : V(G) \rightarrow \{0, 1\} \) with induced labeling \( f^* : E(G) \rightarrow \{0, 1\} \) defined by \( f^*(v) = \sum \{ f(uv) | uv \in E(G) \} (\text{mod} 2) \) is called E-cordial labeling of a graph \( G \) if the number of vertices labeled 0 and number of vertices labeled 1 differ by at most 1 and the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph which admits E-cordial labeling is called E-cordial graph. Here we prove that flower graph \( Fl_n \), closed helm \( CH_n \), double triangular snake \( DT_n \) and gear graph \( G_n \) are E-cordial graphs.

Index Terms—Binary vertex labeling, Cordial labeling, E-cordial labeling, E-cordial graphs.

MSC 2010 Codes - 05C78, 05C38.

I. INTRODUCTION

We begin with finite, connected and undirected graph \( G = (V(G), E(G)) \) without loops and multiple edges. Throughout this paper \( |V(G)| \) and \( |E(G)| \) respectively denote the number of vertices and number of edges in \( G \). For any undefined notation and terminology we rely upon Gross and Yellen [5]. In order to maintain compactness we will provide a brief summary of definitions and existing results.

Definition 1.1: A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).

Beineke and Hegde[1] describe labeling of discrete structure as a frontier between graph theory and theory of numbers. For extensive survey of graph labeling as well as bibliographic references we refer Gallian[3].

Most of the graph labeling techniques trace their origin to graceful labeling introduced independently by Rosa[8] and Golomb[4] defined as follows.

Definition 1.2: A function \( f : V(G) \rightarrow \{0, 1, \ldots, |E(G)|\} \) is called graceful labeling of graph \( G \) if \( f \) is injective and the induced function \( f^*(e = uv) = |f(u) - f(v)| \) is bijective. A graph which admits graceful labeling is called a graceful graph.

The famous Ringel-Kotzig conjecture [7] and many illustrious work on it brought a tide of labeling problems with graceful theme.

Definition 1.3: A graph \( G \) is said to be edge-graceful if there exists a bijection \( f : E(G) \rightarrow \{1, 2, \ldots, |E(G)|\} \) such that the induced mapping \( f^* : V(G) \rightarrow \{0, 1, 2, \ldots, |V(G)| - 1\} \) given by \( f^*(x) = \sum f(xy) (\text{mod} |V|) \), taken over all edges \( xy \) is a bijective.

The notion of edge gracefulness was introduced by Lo[6].

Definition 1.4: A mapping \( f : V(G) \rightarrow \{0, 1\} \) is called binary vertex labeling of \( G \) and \( f(v) \) is called the label of vertex \( v \) of \( G \) under \( f \).

Notation: For an edge \( e = uv \), the induced edge labeling \( f^* : E(G) \rightarrow \{0, 1\} \) is given by \( f^*(e = uv) = |f(u) - f(v)| \) then \( v_f(i) = \) the number of vertices of \( G \) having label \( i \) under \( f \) and let \( e_f(i) = \) the number of edges of \( G \) having label \( i \) under \( f^* \) for \( i = 0, 1 \).

Definition 1.5: A binary vertex labeling of graph \( G \) is called cordial labeling if \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \). A graph \( G \) is called cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit[2]. He also investigated several results on this newly defined concept.

Definition 1.6: Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \) and let \( f : E(G) \rightarrow \{0, 1\} \). Define \( f^* \) on \( V(G) \) by \( f^*(v) = \sum \{ f(uv) | uv \in E(G) \} (\text{mod} 2) \). The function \( f \) is called an \( E - \text{cordial} \) labeling of \( G \) if \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \). A graph is called \( E - \text{cordial} \) if it admits \( E - \text{cordial} \) labeling.

In 1997 Yilmaz and Cahit[13] introduced E-cordial labeling as a weaker version of edge-graceful labeling and with the blend of cordial labeling. They proved that the trees with \( n \) vertices, \( K_n, C_n \) are E-cordial if and only if \( n \neq 2(\text{mod} 4) \) while \( K_{m,n} \) admits E-cordial labeling if and only if \( m + n \neq 2(\text{mod} 4) \).

Vaidya and Lekha[9] have proved that the graphs obtained by duplication of an arbitrary vertex as well as an arbitrary edge in cycle \( C_n \) admit E-cordial labeling. In addition to this they show that the joint sum of two copies of cycle \( C_n \), the split graph of even cycle \( C_n \) and the shadow graph of path \( P_n \) for even \( n \) are E-cordial graphs. The same authors in [10] proved that the middle graph, total graph and split graph of \( P_n \) and the composition of \( P_n \) with \( P_2 \) admit E-cordial labeling.
Vaidya and Vyas[11] have proved that the mirror graphs of even cycle \( C_n \), even path \( P_n \), and hypercube \( Q_k \) are E-cordial graphs. The same authors in [12] proved that \( K_n \times P_2 \) and \( P_n \times P_2 \) are E-cordial graphs for even \( n \) while \( W_n \times P_2 \) and \( K_{1,n} \times P_2 \) are E-cordial graphs for odd \( n \).

**Definition 1.7:** The wheel graph \( W_n \) is defined to be the join \( K_1 + C_n \). The vertex corresponding to \( K_1 \) is known as apex vertex and vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges. We continue to recognize apex of wheel as the apex of respective graphs obtained from wheel.

**Definition 1.8:** The helm \( H_n \) is the graph obtained from a wheel \( W_n \) by attaching a pendant edge to each rim vertex.

**Definition 1.9:** The closed helm \( CH_n \) is the graph obtained from a helm \( H_n \) by joining each pendant vertex to form a cycle.

**Definition 1.10:** The flower graph \( Fl_n \) is the graph obtained from a helm \( H_n \) by joining each pendant vertex to the apex of the helm.

**Definition 1.11:** The double triangular snake \( DT_n \) is obtained from a path \( P_n \) with vertices \( v_1, v_2, \ldots, v_n \) by joining \( v_i \) and \( v_{i+1} \) to a new vertex \( w_i \) for \( i = 1, 2, \ldots, n-1 \) and to a new vertex \( u_i \) for \( i = 1, 2, \ldots, n-1 \).

**Definition 1.12:** Let \( e = uv \) be an edge of graph \( G \) and \( w \) is not a vertex of \( G \). The edge \( e \) is subdivided when it is replaced by edges \( e' = uw \) and \( e'' = vw \).

**Definition 1.13:** The gear graph \( G_n \) is obtained from the wheel by subdividing each of its rim edges.

## II. Main Results

**Theorem 2.1:** \( Fl_n \) is E - cordial.

**Proof:** Let \( H_n \) be a helm with \( v \) as the apex vertex, \( v_1, v_2, \ldots, v_n \) be the vertices of cycle and \( u_1, u_2, \ldots, u_n \) be the pendant vertices for \( n > 3 \). Let \( Fl_n \) be the flower graph obtained from helm \( H_n \) then \( |V(Fl_n)| = 2n + 1 \) and \( |E(Fl_n)| = 4n \). We define \( f : E(Fl_n) \rightarrow \{0, 1\} \) as follows:

For \( 1 \leq i \leq n \):

\[
\begin{align*}
    f(vv_i) &= f(vu_i) = 1 \\
    f(vu_i) &= 0 \\
    f(vi_1v_{i+1}) &= 0 \quad (v_{n+1} = v_1)
\end{align*}
\]

In view of above defined labeling pattern \( f \) satisfies the vertex and edge conditions for E-cordial labeling as shown in Table I. Hence \( Fl_n \) is E-cordial graph.

**Illustration 2.2:** \( Fl_6 \) and its E-cordial labeling is shown in Figure 1.

**Theorem 2.3:** \( CH_n \) is E-cordial.

**Proof:** Let \( v \) be the apex vertex, \( v_1, v_2, \ldots, v_n \) be the vertices of inner cycle and \( u_1, u_2, \ldots, u_n \) be the vertices of outer cycle of \( CH_n \). We note that \( |V(CH_n)| = 2n + 1 \) and \( |E(CH_n)| = 4n \). We define \( f : E(CH_n) \rightarrow \{0, 1\} \) as follows:

For \( 1 \leq i \leq n \):

\[
\begin{align*}
    f(vv_i) &= 1 \\
    f(vu_i) &= 0 \\
    f(vi_1v_{i+1}) &= 1 \quad (v_{n+1} = v_1) \\
    f(u_iu_{i+1}) &= 0 \quad (u_{n+1} = u_1)
\end{align*}
\]

In view of above defined labeling pattern \( f \) satisfies the vertex and edge conditions for E-cordial labeling as shown in Table II. Hence \( CH_n \) is E-cordial graph.

**Illustration 2.4:** \( CH_6 \) and its E-cordial labeling is shown in Figure 2.

**Theorem 2.5:** \( DT_n \) is E-cordial.

**Proof:** Let \( v_1, v_2, \ldots, v_n \) be the vertices of path \( P_n \) and \( u_1, u_2, \ldots, u_n \) and \( u'_1, u'_2, \ldots, u'_n \) be the newly added vertices in order to obtain \( DT_n \). We note that \( |V(DT_n)| = 3n - 2 \) and \( |E(DT_n)| = 5n - 5 \). We define \( f : E(DT_n) \rightarrow \{0, 1\} \) as follows:
Case 1: \( n \equiv 1(\text{mod} 2) \)

For \( 1 \leq i \leq n - 1 \)

\[
\begin{align*}
f(v_i v_{i+1}) &= \begin{cases} 
0 & i \equiv 1, 2(\text{mod} 4) \\
1 & \text{otherwise}
\end{cases} \\
f(v_i u_i) &= \begin{cases} 
0 & i \equiv 1(\text{mod} 2) \\
1 & \text{otherwise}
\end{cases} \\
f(u_i v_{i+1}) &= \begin{cases} 
0 & i \equiv 1(\text{mod} 2) \\
1 & \text{otherwise}
\end{cases} \\
f(v_i u'_i) &= 1 \\
f(u'_i v_{i+1}) &= 0
\end{align*}
\]

Case 2: \( n \equiv 0(\text{mod} 2) \)

\[
\begin{align*}
f(v_1 v_2) &= 1 \\
\text{For } 2 \leq i \leq n - 1 \quad f(v_i v_{i+1}) &= \begin{cases} 
1 & i \equiv 0(\text{mod} 2) \\
0 & \text{otherwise}
\end{cases}
\]

For \( 1 \leq i \leq n - 1 \)

\[
\begin{align*}
f(v_i u_i) &= \begin{cases} 
0 & i \equiv 1(\text{mod} 2) \\
1 & \text{otherwise}
\end{cases} \\
f(u_i v_{i+1}) &= \begin{cases} 
0 & i \equiv 1(\text{mod} 2) \\
1 & \text{otherwise}
\end{cases} \\
f(v_i u'_i) &= 1 \\
f(u'_i v_{i+1}) &= 0
\end{align*}
\]

In view of above defined labeling pattern \( f \) satisfies the vertex and edge conditions for E-cordial labeling as shown in Table III. Hence \( DT_n \) is E-cordial graph.

**Illustration 2.6**: \( DT_5 \) and its E-cordial labeling is shown in Figure 3.

![Figure 3](image-url)

**Theorem 2.7**: \( G_n \) is E-cordial.

**Proof**: Let \( W_n \) be a wheel with apex vertex \( v \) and rim vertices \( v_1, v_2, \ldots, v_n \). To obtain the gear graph \( G_n \) subdivide each rim edges of wheel by the vertices \( u_1, u_2, \ldots, u_n \). Where each \( u_i \) is added between \( v_i \) and \( v_{i+1} \) for \( i = 1, 2, \ldots, n - 1 \) and \( u_n \) is added between \( v_1 \) and \( v_n \). Then \( |V(G_n)| = 2n + 1 \) and \( |E(G_n)| = 3n \). We define \( f : E(G_n) \to \{0, 1\} \) as follows.

Case 1: \( n \equiv 1(\text{mod} 2) \)

For \( 1 \leq i \leq n - 1 \):

\[
\begin{align*}
f(v_1 u_1) &= 1 \\
f(u_1 v_2) &= 0 \\
f(v_i) &= \begin{cases} 
1 & i \equiv 1(\text{mod} 2) \\
0 & \text{otherwise}
\end{cases}
\]

Sub Case 1: \( n \equiv 1(\text{mod} 4) \)

\[
f(v_n) = 1
\]

For \( 2 \leq i \leq n \):

\[
\begin{align*}
f(v_3 u_i) &= \begin{cases} 
1 & i \equiv 2, 3(\text{mod} 4) \\
0 & \text{otherwise}
\end{cases} \\
f(u_i v_{i+1}) &= \begin{cases} 
1 & i \equiv 2, 3(\text{mod} 4) \\
0 & \text{otherwise}
\end{cases} \quad (\text{consider } v_{n+1} = v_1)
\end{align*}
\]

Sub Case 2: \( n \equiv 3(\text{mod} 4) \)

\[
\begin{align*}
f(v_n) &= 0 \\
f(v_{n-1} u_{n-1}) &= 1 \\
f(u_{n-1} v_n) &= 1 \\
f(v_n u_n) &= 0 \\
f(u_n v_1) &= 0
\end{align*}
\]

For \( 2 \leq i \leq n - 2 \):

\[
\begin{align*}
f(v_i u_i) &= \begin{cases} 
1 & i \equiv 2, 3(\text{mod} 4) \\
0 & \text{otherwise}
\end{cases} \\
f(u_i v_{i+1}) &= \begin{cases} 
1 & i \equiv 2, 3(\text{mod} 4) \\
0 & \text{otherwise}
\end{cases}
\]

Case 2: \( n \equiv 0(\text{mod} 2) \)

Sub Case 1: \( n \equiv 0(\text{mod} 4) \):

\[
\begin{align*}
f(v_1) &= \begin{cases} 
0 & i \equiv 1(\text{mod} 2) \\
1 & \text{otherwise}
\end{cases} \\
f(v_i) &= \begin{cases} 
1 & i \equiv 1, 2(\text{mod} 4) \\
0 & \text{otherwise}
\end{cases} \\
f(u_i v_{i+1}) &= \begin{cases} 
1 & i \equiv 1, 2(\text{mod} 4) \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

Sub Case 2: \( n \equiv 2(\text{mod} 4) \):

\[
\begin{align*}
f(v_1) &= 1 \\
f(v_n) &= 0 \\
f(v_n u_n) &= 1 \\
f(u_n v_1) &= 0
\end{align*}
\]

For \( 2 \leq i \leq n - 1 \):

\[
f(v_i) = \begin{cases} 
1 & i \equiv 0(\text{mod} 2) \\
0 & \text{otherwise}
\end{cases}
\]

For \( 1 \leq i \leq n - 1 \):

\[
\begin{align*}
f(v_i u_i) &= \begin{cases} 
1 & i \equiv 1, 2(\text{mod} 4) \\
0 & \text{otherwise}
\end{cases} \\
f(u_i v_{i+1}) &= \begin{cases} 
1 & i \equiv 1, 2(\text{mod} 4) \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

In view of above defined labeling pattern \( f \) satisfies the vertex and edge conditions for E-cordial labeling as shown in Table IV. Hence \( G_n \) is E-cordial graph.

**Illustration 2.8**: \( G_7 \) and its E-cordial labeling is shown in Figure 4.
CONCLUDING REMARKS

Some new E-cordial graphs are investigated. To investigate some characterization(s) or sufficient condition(s) for the graph to be E-cordial is an open area of research.

REFERENCES

### TABLE I

<table>
<thead>
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</tr>
<tr>
<td>( n \equiv 1 \pmod{2} )</td>
<td>( v_f(0) = v_f(1) - 1 = n ) ( e_f(0) = e_f(1) = 2n )</td>
</tr>
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### TABLE II

<table>
<thead>
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<tbody>
<tr>
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</tr>
<tr>
<td>( n \equiv 1 \pmod{2} )</td>
<td>( v_f(0) = v_f(1) - 1 = n ) ( e_f(0) = e_f(1) = 2n )</td>
</tr>
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### TABLE III

<table>
<thead>
<tr>
<th>Vertex Condition</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0 \pmod{2} )</td>
<td>( v_f(0) = v_f(1) = \frac{3n-2}{2} ) ( e_f(0) - 1 = e_f(1) = \left\lfloor \frac{5n-5}{2} \right\rfloor )</td>
</tr>
<tr>
<td>( n \equiv 1 \pmod{4} )</td>
<td>( v_f(0) - 1 = v_f(1) = \left\lfloor \frac{3n-2}{2} \right\rfloor ) ( e_f(0) = e_f(1) = \frac{5n-5}{2} )</td>
</tr>
<tr>
<td>( n \equiv 3 \pmod{4} )</td>
<td>( v_f(0) = v_f(1) - 1 = \left\lfloor \frac{3n-2}{2} \right\rfloor ) ( e_f(0) = e_f(1) = \frac{5n-5}{2} )</td>
</tr>
</tbody>
</table>

### TABLE IV

<table>
<thead>
<tr>
<th>Vertex Condition</th>
<th>Edge Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 1 \pmod{2} )</td>
<td>( v_f(0) = v_f(1) - 1 = n ) ( e_f(0) + 1 = e_f(1) = \frac{3n+1}{2} )</td>
</tr>
<tr>
<td>( n \equiv 0 \pmod{2} )</td>
<td>( v_f(0) - 1 = v_f(1) = n ) ( e_f(0) = e_f(1) = \frac{3n}{2} )</td>
</tr>
</tbody>
</table>
Antimagic Labeling of Some Path and Cycle Related Graphs

S. K. Vaidya ¹ and N. B. Vyas ²

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Received 29 June 2013; accepted 23 July 2013

Abstract. An edge labeling of a graph is a bijection from $E(G)$ to the set $\{1,2,\ldots , |E(G)|\}$. If for any two distinct vertices $u$ and $v$, the sum of labels on the edges incident to $u$ is different from the sum of labels on the edges incident to $v$ then an edge labeling is called antimagic labeling. We investigate antimagic labeling for some path and cycle related graphs.

Keywords: Antimagic labeling, Antimagic graph, Middle graph, Total graph.

AMS Mathematics Subject Classification (2010): 05C78

1. Introduction

We begin with a finite, connected and undirected graph $G=(V(G),E(G))$ without loops and multiple edges. Throughout this paper $|V(G)|$ and $|E(G)|$ denote the number of vertices and number of edges respectively. For any graph theoretic notation and terminology we rely upon Balakrishnan and Ranganathan [1].

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges) then the labeling is called a vertex labeling (an edge labeling).

According to Beineke and Hegde[2] labeling of discrete structure is a frontier between graph theory and theory of numbers. For an extensive survey of graph labeling as well as bibliographic references we refer to Gallian[3].

The concept of magic labeling was introduced during 1963 by Seldacek[5]. A graph is said to be magic if it has a real-valued edge labeling such that;

(i) distinct edges have distinct non-negative labels;

(ii) the sum of the labels of the edges incident to a particular vertex is same for all the vertices.
S. K. Vaidya and N. B. Vyas

An antimagic labeling of a graph $G$ is a bijection from $E(G)$ to the set $\{1, 2, \ldots, |E(G)|\}$ such that for any two distinct vertices $u$ and $v$, the sum of the labels on edges incident to $u$ is different from the sum of the labels on edges incident to $v$.

Hartsfield and Ringel[4] have introduced the concept of an antimagic graph in 1990. They proved that paths $P_n (n \leq 3)$, cycles, wheels, and complete graphs $K_n (n \leq 3)$ admit antimagic labeling. They have also conjectured that;

(i) all trees except $K_2$ are antimagic.
(ii) all connected graphs except $K_2$ are antimagic.

These two conjectures are still not settled.

The graphs obtained by switching of vertex in path $P_n$, cycle $C_n$, wheel $W_n$, helm $H_n$ and fan $f_n$ are proved to be antimagic by Vaidya and Vyas[6].

The middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of $G$ or one is a vertex of $G$ and the other is an edge incident on it. The total graph $T(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in $G$. For a graph $G$ the splitting graph $S'(G)$ is obtained by adding a new vertex $v'$ corresponding to each vertex $v$ of $G$ such that $N(v)=N(v')$ where $N(v)$ and $N(v')$ are the neighborhood sets of $v$ and $v'$ respectively. The shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$, say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbours of the corresponding vertex $u''$ in $G''$.

2. Main Results

**Theorem 2.1.** Middle graph of path $P_n$ is antimagic.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of path $P_n$ and $G = M(P_n)$ be the middle graph of path $P_n$. According to the definition of middle graph $V(M(P_n)) = V(P_n) \cup E(P_n)$ and $E(M(P_n)) = \{v_ie_i; 1 \leq i \leq n-1, v_ie_{i+1}; 2 \leq i \leq n, e_ie_{i+1}; 1 \leq i \leq n-2\}$. Here $|V(G)| = 2n-1$ and $|E(G)| = 3n-4$. To define $f: E(G) \rightarrow \{1, 2, \ldots, 3n-4\}$, we consider following two cases.

**Case 1:** $n = 3, 5$ and $n \equiv 0 \pmod{2}$

For $1 \leq i \leq n-1$:

$f(v_ie_i) = 2i-1$; \hspace{1cm} $f(e_{i+1}) = 2i$;

For $1 \leq i \leq n-2$:

$f(e_{i+1}) = 2(n-1) + i$;

**Case 2:** $n \equiv 1 \pmod{2}$ where $n > 5$

For $1 \leq i \leq n-1$:

$f(v_ie_i) = 2i-1$; \hspace{1cm} $f(e_{i+1}) = 2i$;

For $1 \leq i \leq n-4$:

$f(e_{i+1}) = 2(n-1) + i$;

$f(e_{n-3}e_{n-2}) = 3n-4$; \hspace{1cm} $f(e_{n-2}e_{n-1}) = 3n-5$;
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Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence $M(P_n)$ is antimagic.

**Illustration 2.2.** Middle graph of path $P_5$ and its antimagic labeling is shown in Figure 1.

![Figure 1](image)

**Theorem 2.3.** Middle graph of cycle $C_n$ is antimagic.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of cycle $C_n$ and $G = M(C_n)$ be the middle graph of cycle $C_n$. According to the definition of middle graph $V(M(C_n)) = V(C_n) \cup E(C_n)$ and $E(M(C_n)) = \{v_ie_i, 1 \leq i \leq n, v_ie_{i+1}, 2 \leq i \leq n, e_ie_{i+1}, 1 \leq i \leq n-1, e_ne_1\}$. Here $|V(G)| = 2n$ and $|E(G)| = 3n$. To define $f: E(G) \rightarrow \{1, 2, \ldots, 3n\}$, we consider following two cases.

**Case 1:** $n \equiv 1 \pmod{2}$
- For $1 \leq i \leq n$:
  - $f(v_ie_i) = 2i$;
- For $1 \leq i \leq n-1$:
  - $f(e_{i+1}v_i) = 2i + 1$;
  - $f(e_{i+1}e_i) = 2n + 1 + i$;

**Case 2:** $n \equiv 0 \pmod{2}$
- $f(v_1e_1) = 2$;
- For $2 \leq i \leq n$:
  - $f(v_ie_i) = 2i$;
  - For $1 \leq i \leq n-1$:
    - $f(e_{i+1}v_i) = 2i + 1$;
    - $f(e_{i+1}e_i) = 2n + 1 + i$;

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence $M(C_n)$ is antimagic.

**Illustration 2.4.** Middle graph of cycle $C_5$ and its antimagic labeling is shown in Figure 2.
Figure 2

**Theorem 2.5.** Total graph of path $P_n$ is antimagic.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path $P_n$ and $G = T(P_n)$ be the total graph of path $P_n$ with $V(T(P_n)) = V(P_n) \cup E(P_n)$ and $E(T(P_n)) = \{v_i v_{i+1} ; 1 \leq i \leq n - 1, v_1 e_1 ; 1 \leq i \leq n, e_i e_{i+1} ; 1 \leq i \leq n - 2, v_1 e_{n-1} ; 2 \leq i \leq n \}$. Here $|V(G)| = 2n - 1$ and $|E(G)| = 4n - 5$. Define $f : E(G) \to \{1, 2, \ldots, 4n - 7\}$ as follows.

For $1 \leq i \leq n - 2$:  
$f(v_i v_{i+1}) = 4i$;  
$f(e_i e_{i+1}) = 4i - 3$;  
For $1 \leq i \leq n - 2$:  
$f(v_i e_i) = 4i - 1$;  
For $2 \leq i \leq n - 1$:  
$f(v_i e_{i-1}) = 4i - 2$;  
$f(v_n e_{n-1}) = 4n - 7$.

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence $T(P_n)$ is antimagic.

**Illustration 2.6.** Total graph of path $T(P_6)$ and its antimagic labeling is shown in Figure 3.

Figure 3

**Theorem 2.7.** Total graph of cycle $C_n$ is antimagic.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of cycle $C_n$ and $G = T(C_n)$ be the total graph of cycle $C_n$ with $V(T(C_n)) = V(C_n) \cup E(C_n)$ and $E(T(C_n)) = \{v_i v_{i+1} ; 1 \leq i \leq n - 1, v_n v_1, v_i e_i ; 1 \leq i \leq n, e_i e_{i+1} ; 1 \leq i \leq n - 1, e_n e_1, v_i e_{n-1} ; 2 \leq i \leq n, v_1 e_n \}$. Here $|V(G)| = 2n$ and $|E(G)| = 4n$. To define $f : E(G) \to \{1, 2, \ldots, 4n\}$ we consider following two cases.
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Case 1: \( n \equiv 1 \pmod{2} \)
For 1 \( \leq i \leq n - 1 \):
\[ f(v_iv_{i+1}) = 3n + i; \quad f(v_nv_1) = 4n; \]
For 2 \( \leq i \leq n - 1 \):
\[ f(e_{i-1}e_i) = i + 1; \quad f(e_1e_2) = 1; \]
For 1 \( \leq i \leq n \):
\[ f(v_ie_i) = n + 2i; \quad f(v_ne_1) = n + 1; \]
For 2 \( \leq i \leq n \):
\[ f(v_{i-1}e_i) = n - 1 + 2i; \]

Case 2: \( n \equiv 0 \pmod{2} \)
Sub case 1: \( n \equiv 0 \pmod{4} \)
For 1 \( \leq i \leq n - 1 \):
\[ f(v_iv_{i+1}) = 3n + i; \quad f(v_nv_1) = 4n; \]
For 2 \( \leq i \leq n - 1 \):
\[ f(e_{i-1}e_i) = i + 1; \quad f(e_1e_2) = 1; \]
For 1 \( \leq i \leq n \):
\[ f(v_ie_i) = n + 2i; \quad f(v_ne_1) = n + 1; \]
For 2 \( \leq i \leq n \):
\[ f(v_{i-1}e_i) = n - 1 + 2i; \]

Sub case 2: \( n \equiv 2 \pmod{4} \)
\[ f(v_1v_2) = 3n + 2; \quad f(v_2v_3) = 3n + 1; \]
For 3 \( \leq i \leq n - 1 \):
\[ f(v_iv_{i+1}) = 3n + i; \]
For 2 \( \leq i \leq n - 1 \):
\[ f(e_{i-1}e_i) = i + 1; \]
\[ f(e_1e_2) = 2; \quad f(e_1e_3) = 1; \]
For 1 \( \leq i \leq n \):
\[ f(v_1e_i) = n + 2i; \quad f(v_ne_1) = n + 1; \]
For 2 \( \leq i \leq n \):
\[ f(v_{i-1}e_i) = n - 1 + 2i; \]

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence \( T(C_n) \) is antimagic.

Illustration 2.8. Total graph of cycle \( C_6 \) and its antimagic labeling is shown in Figure 4.
Theorem 2.9. Splitting graph of path $P_n$ is antimagic.

Proof. Let $v_1, v_2, ..., v_n$ be the vertices and $e_1, e_2, ..., e_{n-1}$ be the edges of path $P_n$. Let $v'_1, v'_2, ..., v'_n$ be the newly added vertices to form the splitting graph of path $P_n$. Let $G=S(P_n)$ be the splitting graph of path $P_n$. $V(S'(P_n)) = \{v_i, v'_i / 1 \leq i \leq n\}$ and $E(S'(P_n)) = \{v'_iv_{i+1}; 1 \leq i \leq n-1, v'_iv_{i+1}; 2 \leq i \leq n, v_i,v_{i+1}; 1 \leq i \leq n-1\}$. Here $|V(G)| = 2n$ and $|E(G)| = 3n - 3$.

Define $f:E(G) \to \{1, 2, ..., 3n - 3\}$ as follows.

$$f(v_iv_{i+1}) = 3i;$$
$$f(v'_iv_{i+1}) = 3i - 2;$$
$$f(v_iv'_i) = 3i - 1;$$

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence $S'(P_n)$ is antimagic.

Illustration 2.10. Splitting graph of path $P_6$ and its antimagic labeling is shown in Figure 5.

Theorem 2.11. Splitting graph of cycle $C_n$ is antimagic.

Proof. Let $v_1, v_2, ..., v_n$ be the vertices and $e_1, e_2, ..., e_n$ be the edges of cycle $C_n$. Let $v'_1, v'_2, ..., v'_n$ be the newly added vertices to form the splitting graph of cycle $C_n$. Let $G=S'(C_n)$ be the splitting graph of cycle $C_n$. $V(S'(C_n)) = \{v_i, v'_i / 1 \leq i \leq n\}$ and $E(S'(C_n))$
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\[ V(G) = \{v_i^1, v_i^2, v_i^3; 1 \leq i \leq n-1, v_n, v_{n-1}, v_{n-2}; 2 \leq i \leq n, v_i^1, v_i^2; 1 \leq i \leq n-1, v_n \} \]. Here \(|V(G)| = 2n\) and \(|E(G)| = 3n\). To define \(f: E(G) \rightarrow \{1, 2, \ldots, 3n\}\) we consider following two cases.

**Case 1:** \(n \equiv 1 (\text{mod } 2)\)

For \(1 \leq i \leq n-1:\)

\[ f(v_i v_{i+1}) = 2n + 1 + i; \quad f(v_nv_1) = 2n; \]

\[ f(v'_i v_{i+1}) = \begin{cases} 2i + 1, & i \equiv 1 \text{ (mod } 2) ; \\ 2i, & \text{otherwise.} \end{cases} \]

\[ f(v'_{n-1} v_1) = 2n + 1; \]

For \(2 \leq i \leq n:\)

\[ f(v'_i v_{i-1}) = \begin{cases} 2i - 1, & i \equiv 1 \text{ (mod } 2) ; \\ 2i - 2, & \text{otherwise.} \end{cases} \]

**Case 2:** \(n \equiv 0 (\text{mod } 2)\)

For \(1 \leq i \leq n-1:\)

\[ f(v_i v_{i+1}) = 2n + 1 + i; \quad f(v_nv_1) = 3n; \]

**Sub Case 1:** \(n \equiv 0 (\text{mod } 4), n \neq 4\)

\[ f(v_1^2 v_1) = 4; \quad f(v_2^2 v_3) = 2; \]

For \(1 \leq i \leq n, (i \neq 2):\)

\[ f(v_i^1 v_{i+1}) = \begin{cases} 2i + 1, & i \equiv 1 \text{ (mod } 2) ; \\ 2i, & \text{otherwise.} \end{cases} \]

\[ f(v'_i v_n) = 1; \]

For \(3 \leq i \leq n:\)

\[ f(v'_i v_{i-1}) = \begin{cases} 2i - 1, & i \equiv 1 \text{ (mod } 2) ; \\ 2i - 2, & \text{otherwise.} \end{cases} \]

**Sub Case 2:** \(n = 4\) and \(n \equiv 1 (\text{mod } 4)\)

\[ f(v_1^1 v_n) = 3; \quad f(v_1^1 v_2) = 1; \]

For \(2 \leq i \leq n:\)

\[ f(v_i^1 v_{i+1}) = \begin{cases} 2i + 1, & i \equiv 1 \text{ (mod } 2) ; \\ 2i, & \text{otherwise.} \end{cases} \]

\[ f(v'_{i-1} v_{i+1}) = \begin{cases} 2i - 1, & i \equiv 1 \text{ (mod } 2) ; \\ 2i - 2, & \text{otherwise.} \end{cases} \]

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence \(S'(C_n)\) is antimagic.

**Illustration 2.12.** Splitting graph of cycle \(C_4\) and its antimagic labeling is shown in Figure 6.
Theorem 2.13. Shadow graph of path $P_n$ is antimagic.

Proof. Let $P_n'$, $P_n''$ be two copies of path $P_n$. We denote the vertices of first copy of $P_n$ by $v'_1, v'_2, \ldots, v'_n$ and second copy by $v''_1, v''_2, \ldots, v''_n$. Let $G$ be $D_2(P_n)$ with $|V(G)| = 2n$ and $|E(G)| = 4n - 4$. Define $f : E(G) \rightarrow \{1, 2, \ldots, 4n-4\}$ as follows.

For $1 \leq i \leq n - 1$:
- $f(v'_i v'_{i+1}) = 4i$;
- $f(v'_i v''_{i+1}) = 4i - 3$;
- $f(v'_i v''_{i+1}) = 4i - 1$;

For $2 \leq i \leq n$:
- $f(v'_i v'_{i-1}) = 4i - 2$;

Above defined edge labeling function will generate all distinct vertex labels satisfying the condition for antimagic labeling. Hence $D_2(P_n)$ is antimagic.

Illustration 2.14. Shadow graph of path $P_6$ and its antimagic labeling is shown in Figure 7.

Theorem 2.15. Shadow graph of cycle $C_n$ is antimagic.

Proof. Let $C_n'$, $C_n''$ be two copies of cycle $C_n$. We denote the vertices of first copy of $C_n$ by $v'_1, v'_2, \ldots, v'_n$ and second copy by $v''_1, v''_2, \ldots, v''_n$. Let $G$ be $D_2(C_n)$ with $|V(G)| = 2n$ and $|E(G)| = 4n$. To define $f : E(G) \rightarrow \{1, 2, \ldots, 4n\}$ we consider following three cases.

Case 1: $n \equiv 1(\text{mod } 2)$

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For $1 \leq i \leq n - 1$:

\[
\begin{align*}
    f(v_i^s v_{i+1}^s) &= i; & f(v_i^s v'_i) &= n; \\
    f(v_i^s v_{i+1}') &= 3n + i; & f(v_i^s v'_i) &= 4n; \\
    f(v_i^s v_{i+1}') &= n + 2i - 1; & f(v_i^s v'_i) &= 3n - 1; \\
    f(v_i^s v''_{i-1}) &= n + 2i; & f(v_i^s v''_i) &= 3n;
\end{align*}
\]

**Case 2:** $n \equiv 0(\text{mod } 2)$, $n \neq 6$

\[
\begin{align*}
    f(v_i^s v'_i) &= 1; & f(v_i^s v''_i) &= 4n; \\
    f(v_i^s v'_i) &= n + 1; & f(v_i^s v''_i) &= 3n; \\
    \text{For } 1 \leq i \leq n - 1: & & \\
    f(v_i^s v''_{i+1}) &= 3n + i; & f(v_i^s v'_{i+1}) &= i + 1; \\
    f(v_i^s v''_{i+1}) &= n + 1 + 2i; &
\end{align*}
\]

**Case 3:** For $n = 6$, antimagic labeling of $D_2(C_6)$ is shown in below Figure 8.

![Figure 8](image)

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence $D_2(C_6)$ is antimagic.

**Illustration 2.16.** Shadow graph of cycle $C_5$ and its antimagic labeling is shown in Figure 9.
3. Concluding Remarks
We have investigated antimagic labeling for shadow graph, middle graph and total graph of $P_n$ and $C_n$. More exploration is possible for other graph families and in the context of different graph labeling problems.

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SOME NEW RESULTS ON MEAN LABELING

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Abstract

We investigate some new results on mean labeling of graphs. We prove that the shadow graph of path \( P_n \), square graph of bistar \( B_{n,n} \), splitting graph of \( B_{n,n} \), middle graph of \( C_n \) are mean graphs. We also show that \( P_n + 2K_1 \) admits mean labeling for even \( n \).

1. Introduction

Throughout this work, by a graph we mean finite, connected, undirected, simple graph \( G = (V(G), E(G)) \) of order \( |V(G)| \) and size \( |E(G)| \). For any undefined term in graph theory we rely upon West [9].

A graph labeling is an assignment of integers to the vertices or edges or
both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges), then the labeling is called a vertex labeling (an edge labeling).

According to Beineke and Hegde [1] labeling of discrete structure is a frontier between graph theory and theory of numbers. A latest survey on various graph labeling problems can be found in Gallian [2].

The function \( f \) is called mean labeling of graph \( G \) if \( f : V(G) \to \{0, 1, 2, ..., |E(G)|\} \) is injective and the induced function \( f^* : E(G) \to \{1, 2, ..., |E(G)|\} \) defined as

\[
f^*(e = uv) = \begin{cases} 
\frac{f(u) + f(v)}{2}, & \text{if } f(u) + f(v) \text{ is even;} \\
\frac{f(u) + f(v) + 1}{2}, & \text{if } f(u) + f(v) \text{ is odd}
\end{cases}
\]

is bijective. A graph which admits mean labeling is called a mean graph.

The mean labeling was introduced by Somasundaram [3] and they proved that the graphs \( P_n, C_n, P_n \times P_m, P_m \times C_n \) are mean graphs. The same authors in [4] have discussed the mean labeling for subdivision of \( K_{1,n} \) for \( n < 4 \) while in [5] they proved that \( W_n \) does not admit mean labeling for \( n > 3 \). Vaidya and Lekha [6] proved that the graphs \( P_m[P_2], P_n^2, M(P_n) \) and some cycle related graphs admit mean labeling. The same authors in [7] have proved that the graphs obtained by duplicating an arbitrary vertex as well as an arbitrary edge in cycle \( C_n \), the joint sum of two copies of \( C_n \) and identifying two vertices of cycle \( C_n \) are mean graphs.

For a connected graph \( G \), let \( G' \) be the copy of \( G \). Then shadow graph \( D_2(G) \) is obtained by joining each vertex \( u \) in \( G \) to the neighbours of the corresponding vertex \( u' \) in \( G' \). Square of a graph \( G \) denoted by \( G^2 \) has the same vertex set as of \( G \) and two vertices are adjacent in \( G^2 \) if they are at a distance 1 or 2 apart in \( G \). For a graph \( G \) the splitting graph \( S'(G) \) is obtained
by adding new vertex \( v' \) corresponding to each vertex \( v \) of \( G \) such that \( N(v) = N(v') \), where \( N(v) \) and \( N(v') \) are the neighborhood sets of \( v \) and \( v' \) respectively. The middle graph \( M(G) \) of a graph \( G \) is the graph whose vertex set is \( V(G) \cup E(G) \) and in which two vertices are adjacent if and only if either they are adjacent edges of \( G \) or one is a vertex of \( G \) and the other is an edge incident on it. The double fan \( DF_n \) is obtained by \( P_n + 2K_1 \).

2. Main Results

**Theorem 2.1.** The graph \( D_2(P_n) \) is a mean graph.

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be the vertices of path \( P_n \) and \( v'_1, v'_2, \ldots, v'_n \) be the newly added vertices corresponding to the vertices \( v_1, v_2, \ldots, v_n \) in order to obtain \( D_2(P_n) \). Denoting \( G = D_2(P_n) \), then \( |V(G)| = 2n \) and \( |E(G)| = 4(n - 1) \). We define \( f : V(G) \to \{0, 1, 2, \ldots, |E(G)|\} \) as follows:

**Case 1.** \( n \equiv 0 \pmod{2} \)

\[
f(v_1) = 0; \quad f(v'_1) = 3;
\]
\[
f(v_2) = 4; \quad f(v'_2) = 2.
\]

For \( 3 \leq i \leq n \):

\[
f(v_i) = \begin{cases} 4i - 4, & i \equiv 1 \pmod{2}; \\ 4i - 1, & \text{otherwise}; \end{cases}
\]
\[
f(v'_i) = \begin{cases} 4i - 4, & i \equiv 1 \pmod{2}; \\ 4i - 6, & \text{otherwise}. \end{cases}
\]

**Case 2.** \( n \equiv 1 \pmod{2} \)

\[
f(v_1) = 2; \quad f(v'_1) = 4;
\]
\[
f(v_2) = 0; \quad f(v'_2) = 7;
\]
\[
f(v_3) = 8; \quad f(v'_3) = 6.
\]
For $4 \leq i \leq n$:

$$f(v_i) = \begin{cases} 
4i - 1, & i \equiv 0 \pmod{2}; \\
4i - 4, & \text{otherwise}; 
\end{cases}$$

$$f(v'_i) = \begin{cases} 
4i - 4, & i \equiv 0 \pmod{2}; \\
4i - 6, & \text{otherwise}. 
\end{cases}$$

The above defined function $f$ provides a mean labeling for $D_2(P_n)$.

That is, $D_2(P_n)$ is a mean graph.

**Illustration 2.2.** Shadow graph of path $P_6$ and its mean labeling is shown in Figure 1.

![Figure 1](image)

**Theorem 2.3.** $B_{n,n}^2$ is a mean graph.

**Proof.** Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$, where $u_i, v_i$ are pendant vertices. Let $G$ be the graph $B_{n,n}^2$. Then $|V(G)| = 2n + 2$ and $|E(G)| = 4n + 1$. We define $f : V(G) \rightarrow \{0, 1, \ldots, |E(G)|\}$ as follows:

$f(u) = 0$;

$f(v) = 4n + 1$.

For $1 \leq i \leq n$:

$f(v_i) = 2i$;

$f(u_i) = 2n + 2i$. 
The above defined function $f$ provides mean labeling for $B_{n,n}^2$.

That is, $B_{n,n}^2$ is a mean graph.

**Illustration 2.4.** $B_{5,5}^2$ and its mean labeling is shown in Figure 2.

![Figure 2](image)

**Theorem 2.5.** $S'(B_{n,n})$ is a mean graph.

**Proof.** Consider $B_{n,n}$ with the vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$, where $u_i, v_i$ are pendant vertices. In order to obtain $S'(B_{n,n})$, add $u', v', u'_i, v'_i$ vertices corresponding to $u, v, u_i, v_i$, where, $1 \leq i \leq n$. If $G = S'(B_{n,n})$, then $|V(G)| = 4(n + 1)$ and $|E(G)| = 6n + 3$. We define $f : V(G) \to \{0, 1, ..., |E(G)|\}$ as follows:

\[
\begin{align*}
    f(u) &= 2n + 2; \\
    f(v) &= 6n + 2; \\
    f(u') &= 1; \\
    f(v') &= 4n - 1.
\end{align*}
\]
For $1 \leq i \leq n$:

\[ f(u_i) = 2i - 2; \]

\[ f(u'_i) = 2n. \]

For $2 \leq i \leq n$:

\[ f(u'_i) = 2n - 3 + 2i. \]

Now for the remaining vertex labels we consider following two cases.

**Case 1.** $n \equiv 1 \pmod{2}$

For $1 \leq i \leq n + 1, i \neq 3$:

\[ f(v_i) = 4n + 2(i - 1); \]

\[ f(v'_i) = 6n + 3; \]

\[ f(v'_2) = 6n + 1; \]

\[ f(v'_3) = 4n + 4. \]

For $4 \leq i \leq n$:

\[ f(v'_i) = 2i - 5. \]

**Case 2.** $n \equiv 0 \pmod{2}$

For $1 \leq i \leq n + 1, i \neq 4$:

\[ f(v_i) = 4(n - 1) + 2i; \]

\[ f(v'_1) = 6n + 3; \]

\[ f(v'_2) = 6n + 1; \]

\[ f(v'_3) = 6n; \]

\[ f(v'_4) = 4n + 4. \]
For $5 \leq i \leq n$:

$$f(v'_i) = 2i - 7.$$ 

The above defined function $f$ provides mean labeling for $S'(B_{n,n})$.

That is, $S'(B_{n,n})$ is a mean graph.

**Illustration 2.6.** $S'(B_{6,6})$ and its mean labeling is shown in Figure 3.

![Figure 3](image)

**Theorem 2.7.** *Middle graph of cycle $C_n$ is a mean graph.*

**Proof.** Let $G = M(C_n)$ be a middle graph of cycle $C_n$. We note that $|V(G)| = 2n$ and $|E(G)| = 3n$. We define $f : V(G) \rightarrow \{0, 1, ..., |E(G)|\}$ as follows.

Case 1: $n \equiv 1 \pmod{2}$

For $1 \leq i \leq \frac{n-1}{2}$:

$$f(v_i) = 3(i - 1);$$

$$f(v'_i) = 3i - 1;$$
\[ f(v_i) = \frac{3n - 1}{2}; \text{ for } i = \frac{n - 1}{2} \]
\[ f(v'_i) = \frac{3n + 5}{2}; \text{ for } i = \frac{n - 1}{2}. \]

For \( \frac{n + 1}{2} < i \leq n \):

\[ f(v_i) = 3i; \]
\[ f(v'_i) = 3i - 1. \]

**Case 2: \( n \equiv 0 \pmod{2} \)**

For \( 1 \leq i \leq \frac{n}{2} \):

\[ f(v_i) = 3(i - 1); \]
\[ f(v'_i) = 3i - 1. \]

For \( \frac{n}{2} < i < n \):

\[ f(v_i) = 3i + 2; \]
\[ f(v_n) = 3; \]
\[ f(v'_i) = \frac{3n}{2}; \text{ for } i = \frac{n}{2} + 1; \]
\[ f(v'_i) = \frac{3n + 2}{2}; \text{ for } i = \frac{n}{2} + 2. \]

For \( \frac{n}{2} + 2 < i < n - 1 \):

\[ f(v'_i) = 3i - 5; \]
\[ f(v'_{n-1}) = 3n - 7; \]
\[ f(v'_n) = 3n - 2. \]

The above defined function \( f \) provides mean labeling for \( M(C_n) \).

That is, \( M(C_n) \) is a mean graph.
**Illustration 2.8.** $M(C_6)$ and its mean labeling is shown in Figure 4.

![Figure 4](image)

**Theorem 2.9.** $DF_n$ is a mean graph for $n$ even.

**Proof.** Let $v_1, v_2, ..., v_n$ be the vertices of $P_n$ for $n$ even. Vertices $u$ and $v$ are added to obtain $DF_n = P_n + 2K_1$. We note that $|V(DF_n)| = n + 2$ and $|E(DF_n)| = 3n - 1$. We define $f : V(DF_n) \to \{0, 1, ..., |E(G)|\}$ as follows:

$$f(u) = 0;$$
$$f(v) = 3n - 1.$$

For $1 \leq i \leq n$:

$$f(v_i) = \begin{cases} 3i - 1, & i \equiv 1 \pmod{2} \\ 3i - 2, & \text{otherwise.} \end{cases}$$

The above defined function $f$ provides mean labeling for $DF_n$.

That is, $DF_n$ is a mean graph.
Illustration 2.10. $DF_6$ and its mean labeling is shown in Figure 5.

![Graph Image]

Figure 5

3. Concluding Remarks

As every graph does not possess mean labeling it is very interesting to find graph or graph families which admits mean labeling. Here we investigate some new mean graphs arising from graph operations. More exploration is possible for other graph families and in the context of different graph labeling problems as well as graph operations.

References


EVEN MEAN LABELING FOR PATH AND BISTAR RELATED GRAPHS

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[Received 28/06/2013 | Received in revised form 31/07/2013 | Accepted 31/07/2013]

ABSTRACT

An even mean labeling is a variant of mean labeling. Here we investigate even mean labeling for path and bistar related graphs.

Keywords: Shadow graph; Splitting graph; Middle graph; Even Mean labeling.

2010 Mathematics Subject Classification: 05C78.

1. INTRODUCTION

Throughout this work, by a graph we mean finite, connected, undirected, simple graph \( G = (V(G),E(G)) \) of order \( |V(G)| \) and size \( |E(G)| \).

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges) then the labeling is called a vertex labeling (an edge labeling).

According to Beineke and Hegde[1] labeling of discrete structure is a frontier between graph theory and theory of numbers. A latest survey on various graph labeling problems can be found in Gallian[2].

The function \( f \) is called mean labeling of graph \( G \) if \( f : V(G) \rightarrow \{0, 1, 2, \ldots , |E(G)|\} \) is injective and the induced function \( f^* : E(G) \rightarrow \{1, 2, \ldots , |E(G)|\} \) defined as

\[
f^*(e = uv) = \begin{cases} \frac{f(u) + f(v)}{2}, & \text{if } f(u) + f(v) \text{ is even;} \\ \frac{f(u) + f(v) + 1}{2}, & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}
\]

is bijective. A graph which admits mean labeling is called a mean graph.
The mean labeling was introduced by Somasundaram and Ponraj [3] and they proved that the graphs $P_n, C_n, P_n \times P_m, P_m \times C_n$ are mean graphs. Vaidya and Lekha [4] proved that the graphs $P_m [P_2], P_n^2, M(P_n)$ and some cycle related graphs admit mean labeling.

According to Pricilla [5], function $f$ is called even mean labeling of graph $G$ if $f : V(G) \rightarrow \{2, 4, \ldots, 2|E(G)|\}$ is injective and each edge $uv$ assigned the label $\frac{f(u) + f(v)}{2}$ such that the resulting edge labels are distinct.

For a connected graph $G$, let $G'$ be the copy of $G$ then shadow graph $D_2(G)$ is obtained by joining each vertex $u$ in $G$ to the neighbours of the corresponding vertex $u'$ in $G'$. The middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of $G$ or one is a vertex of $G$ and the other is an edge incident on it. The total graph $T(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in $G$. For a graph $G$ the splitting graph $S'(G)$ is obtained by adding new vertex $v'$ corresponding to each vertex $v$ of $G$ such that $N(v) = N(v')$ where $N(v)$ and $N(v')$ are the neighbourhood sets of $v$ and $v'$ respectively. Duplication of a vertex $v$ by a new edge $e = v'v''$ in a graph $G$ produces a new graph $G'$ such that $N(v') \cap N(v'') = v$. A vertex switching $G_v$ of a graph $G$ is the graph obtained by taking a vertex $v$ of $G$, removing all the edges to $v$ and adding edges joining $v$ to every other vertex which are not adjacent to $v$ in $G$. Square of a graph $G$ denoted by $G^2$ has the same vertex set as of $G$ and two vertices are adjacent in $G^2$ if they are at a distance at most 2 apart in $G$. Cube of a graph $G$ denoted by $G^3$ has the same vertex set as of $G$ and two vertices are adjacent in $G^3$ if they are at a distance at most 3 apart in $G$. The double fan $DF_n$ is obtained by $P_n + 2K_1$. For any undefined term in graph theory we rely upon West [6].

2. RESULTS

**Theorem 2.1.** The graph $D_2(P_n)$ is an even mean graph.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices of path $P_n$ and $v'_1, v'_2, \ldots, v'_n$ be the newly added vertices corresponding to the vertices $v_1, v_2, \ldots, v_n$ in order to obtain $D_2(P_n)$. Denoting $G = D_2(P_n)$ then $|V(G)| = 2n$ and $|E(G)| = 4(n - 1)$.

We define $f : V(G) \rightarrow \{2, 4, \ldots, 2|E(G)|\}$ as follows.

For $1 \leq i \leq n$:

$$f(v_i) = \begin{cases} 
2i - 6, & i \equiv 1 \text{ (mod 2)}; \\
2i - 12, & \text{otherwise}.
\end{cases}$$

$$f(v'_i) = \begin{cases} 
2i, & i \equiv 1 \text{ (mod 2)}; \\
2i - 10, & \text{otherwise}.
\end{cases}$$

The above defined function $f$ provides an even mean labelling for $D_2(P_n)$. That is, $D_2(P_n)$ is an even mean graph.

**Illustration 2.2.** Shadow graph of path $P_5$ and its even mean labeling is shown in Fig.1.

![Shadow graph of path P5](image)

**Theorem 2.3.** Middle graph of path $P_n$ is an even mean graph.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path $P_n$ and $G = M(P_n)$ be the middle graph of path $P_n$. According to the definition of middle graph $V(M(P_n)) = V(P_n) \cup E(P_n)$ and $E(M(P_n)) = \{v_i e_i; 1 \leq i \leq n-1, v_i e_{i-1}; 2 \leq i \leq n, e_i e_{i+1}; 1 \leq i \leq n-2\}$. Here $|V(G)| = 2n-1$ and $|E(G)| = 3n - 4$.

We define $f : V(G) \rightarrow \{2, 4, \ldots, 2|E(G)|\}$ as follows.

For $1 \leq i \leq n$:

$f(v_i) = 4i - 2$;

For $1 \leq i \leq n-1$:

$f(e_i) = 4i$;

The above defined function $f$ provides an even mean labeling for $M(P_n)$. Hence, $M(P_n)$ is an even mean graph.

**Illustration 2.4.** $M(P_5)$ and its even mean labeling is shown in Fig.2.

![Middle graph of path P5](image)

**Theorem 2.5.** Total graph of path $P_n$ is an even mean graph.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path $P_n$ and $G = T(P_n)$ be the total graph of path $P_n$ with $V(T(P_n)) = V(P_n) \cup E(P_n)$ and $E(T(P_n)) = \{v_i v_{i+1}; 1 \leq i \leq n-1, v_i e_i; 1 \leq i \leq n-1, e_i e_{i+1}; 1 \leq i \leq n-2, v_i e_{i-1}; 2 \leq i \leq n\}$. Here $|V(G)| = 2n-1$ and $|E(G)| = 4n - 5$.

We define $f : V(G) \rightarrow \{2, 4, \ldots, 2|E(G)|\}$ as follows.

For $1 \leq i \leq n$:

$f(v_i) = 4i - 2$;

For $1 \leq i \leq n-1$:

$f(e_i) = 4i$;
The above defined function $f$ provides an even mean labeling for $T(P_n)$. Hence, $T(P_n)$ is an even mean graph.

Illustration 2.6. $T(P_n)$ and its even mean labeling is shown in Fig.3.

![Fig.3](image)

### Theorem 2.7. Splitting graph of path $P_n$ is an even mean graph.

**Proof.** Let $v_1,v_2,...,v_n$ be the vertices of path $P_n$. Let $v'_1,v'_2,...,v'_n$ be the newly added vertices to form the splitting graph of path $P_n$. Let $G = S'(P_n)$ be the splitting graph of path $P_n$. $V(S'(P_n)) = \{v_i \cup \{v'_i\}, 1 \leq i \leq n \}$ and $E(S'(P_n)) = \{v'_iv_{i+1}, 1 \leq i \leq n-1, v'_iv_{i+1}, 2 \leq i \leq n, v_1v_{i+1}: 1 \leq i \leq n-1 \}$. Here $|V(G)| = 2n$ and $|E(G)| = 3n-3$.

We define $f : V(G) \rightarrow \{2,4,...,|E(G)|\}$ as follows.

For $1 \leq i \leq n$:

\[
\begin{align*}
  f(v_i) &= \begin{cases} 
    4i, & i \equiv 1 \pmod{2}; \\
    4i - 2, & \text{otherwise}.
  \end{cases} \\
  f(v'_i) &= \begin{cases} 
    4i - 2, & i \equiv 1 \pmod{2}; \\
    4i, & \text{otherwise}.
  \end{cases}
\end{align*}
\]

The above defined function $f$ provides an even mean labeling for $S'(P_n)$. Hence, $S'(P_n)$ is an even mean graph.

Illustration 2.8. $S'(P_n)$ and its even mean labeling is shown in Fig.4.

![Fig.4](image)

### Theorem 2.9. Duplicating each vertex by an edge in path $P_n$ is an even mean graph.

**Proof.** Let $v_1,v_2,...,v_n$ be the vertices of path $P_n$. Let $G$ be the graph obtained by duplicating each vertex $v_i$ of $P_n$ by an edge $v'_i v''_i$ at a time, where $1 \leq i \leq n$. Note that $|V(G)| = 3n$ and $|E(G)| = 4n-1$.

We define $f : V(G) \rightarrow \{2,4,...,|E(G)|\}$ as follows.
For $1 \leq i \leq n$:
\[
\begin{align*}
  f(v_i) &= 8i - 6; \\
  f(v'_i) &= 8i - 4; \\
  f(v''_i) &= 8i - 2;
\end{align*}
\]

The above defined function $f$ provides an even mean labeling for graph $G$. Hence, duplicating each vertex by edge in path $P_n$ is an even mean graph.

Illustration 2.10. Duplicating each vertex by edge in path $P_5$ and its even mean labeling is shown in Fig.5.

Theorem 2.11. Switching of a pendant vertex in path $P_n$ is an even mean graph.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of $P_n$ and $G_v$ denotes the graph obtained by switching of a pendant vertex $v$ of $G = P_n$. Without loss of generality let the switched vertex be $v_1$. We note that $|V(G_v)| = n$ and $|E(G_v)| = 2n - 4$.

We define $f : V(G_v) \rightarrow \{2, 4, \ldots, 4n - 8\}$ as follows:
\[
\begin{align*}
  f(v_1) &= 2; \\
  f(v'_i) &= 4n - 4 - 2i;
\end{align*}
\]

The above defined function $f$ provides an even mean labeling for $G_v$. Hence, the graph obtained by switching of a pendant vertex in a path $P_n$ is an even mean graph.

Illustration 2.12. Switching of a pendant vertex in path $P_5$ and its even mean labeling is shown in Fig.6.

Theorem 2.13. $P_n^2$ is an even mean graph.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of path $P_n$. Let $G = P_n^2$ then note that $|V(G)| = n$ and $|E(G)| = 2n - 3$.

We define $f : V(G) \rightarrow \{2, 4, \ldots, 4n - 6\}$ as follows.
For $1 \leq i \leq n$:

$$f(v_i) = 2i;$$

The above defined function $f$ provides an even mean labeling for $P_n^2$. Hence, $P_n^2$ is an even mean graph.

**Illustration 2.14.** $P_6^2$ and its even mean labeling is shown in Fig.7.

![Graph](image1)

**Theorem 2.15.** $P_n^3$ is an even mean graph.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices of path $P_n$. Let $G = P_n^3$, then note that $|V(G)| = n$ and $|E(G)| = 3n - 6$.

We define $f: V(G) \rightarrow \{2, 4, \ldots, 6n - 12\}$ as follows.

For $1 \leq i \leq n$:

$$f(v_i) = \begin{cases} 
12 \left\lfloor \frac{i}{4} \right\rfloor - 10, & i \equiv 1 \text{ (mod 4)}; \\
12 \left\lfloor \frac{i}{4} \right\rfloor - 8, & i \equiv 2 \text{ (mod 4)}; \\
12 \left\lfloor \frac{i}{4} \right\rfloor - 6, & i \equiv 3 \text{ (mod 4)}; \\
12 \left\lfloor \frac{i}{4} \right\rfloor - 2, & i \equiv 0 \text{ (mod 4)}. 
\end{cases}$$

The above defined function $f$ provides an even mean labeling for $P_n^3$. Hence, $P_n^3$ is an even mean graph.

**Illustration 2.16.** $P_5^3$ and it’s even mean labeling is shown in Fig.8.

![Graph](image2)

**Theorem 2.17.** $B_{n,n}^2$ is a mean graph.

**Proof.** Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where $u_i, v_i$ are pendant vertices. Let $G$ be the graph $B_{n,n}^2$. Then $|V(G)| = 2n + 2$ and $|E(G)| = 4n + 1$.

We define $f: V(G) \rightarrow \{2, 4, \ldots, 8n + 2\}$ as follows.

$$f(u) = 8n + 2;$$
The above defined function $f$ provides an even mean labeling for $B_{n,n}^2$. Hence, $B_{n,n}^2$ is an even mean graph.

Illustration 2.18. $B_{5,5}^2$ and its even mean labeling is shown in Fig.9.

Fig.9

Theorem 2.19. $S'(B_{n,n})$ is an even mean graph.

Proof. Consider $B_{n,n}$ with the vertex set $\{u,v,u',v',1 \leq i \leq n\}$ where $u_i,v_i$ are pendant vertices. In order to obtain $S'(B_{n,n})$, add $u'_i,v'_i,u'_i,v'_i$ vertices corresponding to $u,v,u,v_i$ where, $1 \leq i \leq n$. If $G = S'(B_{n,n})$ then $|V(G)| = 4(n+1)$ and $|E(G)| = 6n + 3$.

We define $f : V(G) \rightarrow \{2,4,..,12n+6\}$ as follows.

- $f(u) = 6n + 2$; $f(v) = 12n + 6$
- $f(u'_i) = 2$; $f(v'_i) = 6n + 4$

For $1 \leq i \leq n$:

- $f(u_i) = 2i + 2$
- $f(u'_i) = 2n + 2 + 2i$
- $f(v_i) = 6n + 4 + 2i$
- $f(v'_i) = 8n + 4 + 2i$

The above defined function $f$ provides an even mean labeling for $S'(B_{n,n})$. Hence, $S'(B_{n,n})$ is an even mean graph.
Illustration 2.20. $S'(B_{4,4})$ and it’s even mean labeling is shown in Fig.10.

![Illustration 2.20](image)

Fig.10

Theorem 2.21. $DF_n$ is an even mean graph.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices of $P_n$ for $n$ even. Vertices $u$ and $v$ are added to obtain $DF_n = P_n + 2K_1$. We note that $|V(DF_n)| = n + 2$ and $|E(DF_n)| = 3n - 1$.

We define $f : V(DF_n) \rightarrow \{2, 4, \ldots, 6n - 2\}$ as follows.

$f(u) = 2$;

$f(v) = 6n - 2$;

For $1 \leq i \leq n$:

$f(v_i) = 2n + 2i - 2$;

The above defined function $f$ provides an even mean labeling for $DF_n$. Hence, $DF_n$ is an even mean graph. $\square$

Illustration 2.22. $DF_6$ and it’s even mean labeling is shown in Fig.11.

![Illustration 2.22](image)

Fig.11

**CONCLUSIONS**

It is always interesting to find out graph or graph families which admit a particular labeling. Here we investigate some new graph families which admit even mean labeling. To investigate similar results for other graph families is an open area of research.
REFERENCES


