Chapter 6

Antimagic Labeling of Graphs
6.1 Introduction

The magic labelings were introduced by Sedlacek [41] during 1963. The graphs labeled with numbers in which every vertex and its incident edges add up to the same number are called vertex-magic graphs and the corresponding number is called the magic number. In general for a magic-type labeling we require the sum of labels related to a vertex (a vertex magic labeling) or to an edge (an edge magic labeling) to be constant all over the graph.

Illustration 6.1.1. Vertex magic labeling and edge magic labeling of $C_5$ with magic number 14 is shown in Figure 6.1.

Motivated through the concept of magic labeling Hartsfield and Ringel [20] have introduced the concept of antimagic labeling which is defined as follows.

6.2 Antimagic Labeling

Definition 6.2.1. An antimagic labeling of a graph $G$ is a bijection from $E(G)$ to the set \{1, 2, \ldots, |E(G)|\} such that for any two distinct vertices $u$ and $v$, the sum of the labels on edges incident to $u$ is different from the sum of the labels on edges incident to $v$. 
A graph is called antimagic if it admits an antimagic labeling.

### 6.2.1 Some Existing Results

- Hartsfield and Ringel [20] have proved that
  - $P_n$ is antimagic graph for $n \geq 2$.
  - $C_n$ is antimagic graph.
  - $W_n$ is antimagic graph.
  - $K_n$ is antimagic graph for $n \geq 3$.
  - $K_{1,n}$ is antimagic graph for $n \geq 2$.
  - $K_{2,n}$ is antimagic graph.

- T. Wang [66] has shown that the toroidal grids $C_{n_1} \times C_{n_2} \times \ldots \times C_{n_k}$ are antimagic and, more generally, graphs of the form $G \times C_n$ are antimagic if $G$ is $r$-regular antimagic graph with $r > 1$.

- Cheng [7] has proved that all Cartesian products of two or more regular graphs of positive degree are antimagic and that if $G$ is $j$-regular and $H$ has maximum degree at most $k$, minimum degree at least one ($G$ and $H$ need not be connected), then $G \times H$ is antimagic provided that $j$ is odd and $j^2 - j \geq 2k$, or $j$ is even and $j^2 > 2k$.

- Wang and Hsiao [67] have proved the following graphs are antimagic
  - $G \times P_n (n > 1)$ where $G$ is regular.
  - $G \times K_{1,n}$ where $G$ is regular.
  - Compositions $G[H]$ where $H$ is $d$-regular with $d > 1$.
  - the Cartesian product of any double star (two stars with an edge joining their centers) and a regular graph.

- Cheng [6] also proved that $P_{n_1} \times P_{n_2} \times \ldots \times P_{n_t} (t \geq 2)$ is antimagic.
• Solairaju and Arockiasamy [45] have proved that various families of subgraphs of grids $P_m \times P_n$ are antimagic.

• Liang and Zhu [33] have proved that if $G$ is $k$-regular ($k \geq 2$), then for any graph $H$ with $|E(H)| \geq |V(H)| - 1 \geq 1$, the Cartesian product $H \times G$ is antimagic. They also showed that if $|E(H)| \geq |V(H)| - 1$ and each component of $H$ has a vertex of odd degree, or $H$ has at least $2|V(H)| - 2$ edges, then the prism of $H$ is antimagic.

• Lee, Lin and Tsai [28] have proved that $C_{2n}^2$ is antimagic and the vertex sums form a set of successive integers when $n$ is odd.

6.3 Antimagic Labeling of Some Path Related Graphs

Theorem 6.3.1. $M(P_n)$ is antimagic.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path $P_n$ and $G = M(P_n)$ be the middle graph of path $P_n$. According to the definition of middle graph $V(M(P_n)) = V(P_n) \cup E(P_n)$ and $E(M(P_n)) = \{v_ie_i; 1 \leq i \leq n-1, v_ie_{i-1}; 2 \leq i \leq n, e_ie_{i+1}; 1 \leq i \leq n-2\}$. Here $|V(G)| = 2n - 1$ and $|E(G)| = 3n - 4$. To define $f: E(G) \rightarrow \{1, 2, \ldots, 3n - 4\}$, we consider following two cases.

Case 1: $n = 3, 5$ and $n \equiv 0(mod\ 2)$

For $1 \leq i \leq n - 1$:

$$f(v_ie_i) = 2i - 1,$$

$$f(e_iv_{i+1}) = 2i,$$

For $1 \leq i \leq n - 2$:

$$f(e_ie_{i+1}) = 2(n - 1) + i,$$
Case 2: $n \equiv 1 \pmod{2}$ where $n > 5$

For $1 \leq i \leq n - 1$:

\[
  f(v_i e_i) = 2i - 1, \\
  f(e_i v_{i+1}) = 2i,
\]

For $1 \leq i \leq n - 4$:

\[
  f(e_i e_{i+1}) = 2(n - 1) + i, \\
  f(e_{n-3} e_{n-2}) = 3n - 4, \\
  f(e_{n-2} e_{n-1}) = 3n - 5.
\]

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling.

Hence $M(P_n)$ is antimagic.

Illustration 6.3.2. $M(P_5)$ and its antimagic labeling is shown in Figure 6.2.

![Figure 6.2: M(P_5) and its antimagic labeling.](image)
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\[ f(v_{i}v_{i+1}) = 4i, \]
\[ f(v_{n-1}v_{n}) = 4n - 5, \]
\[ f(e_{i}e_{i+1}) = 4i - 3, \]

For \(1 \leq i \leq n - 2:\)

\[ f(v_{i}e_{i}) = 4i - 1, \]

For \(2 \leq i \leq n - 1:\)

\[ f(v_{i}e_{i-1}) = 4i - 2, \]
\[ f(v_{n}e_{n-1}) = 4n - 7. \]

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling.

Hence \(T(P_{n})\) is antimagic. \(\blacksquare\)

**Illustration 6.3.4.** \(T(P_{6})\) and its antimagic labeling is shown in *Figure 6.3.*

**Theorem 6.3.5.** \(S'(P_{n})\) is antimagic.

**Proof.** Let \(v_{1}, v_{2}, \ldots, v_{n}\) be the vertices and \(e_{1}, e_{2}, \ldots, e_{n-1}\) be the edges of path \(P_{n}\). Let \(v'_{1}, v'_{2}, \ldots, v'_{n}\) be the newly added vertices to form the splitting graph of path \(P_{n}\). Let \(G = S'(P_{n})\) be the splitting graph of path \(P_{n}\). \(V(S'(P_{n})) = \{v_{i}\} \cup \{v'_{i}\}, 1 \leq i \leq n\) and \(E(S'(P_{n})) = \{v'_{i}v_{i+1}; 1 \leq i \leq n - 1, v'_{i}v_{i-1}; 2 \leq i \leq n, v_{i}v_{i+1}; 1 \leq i \leq n - 1\}. \) Here \(|V(G)| = 2n\) and \(|E(G)| = 3n - 3.\)
We define \( f : E(G) \to \{1, 2, \ldots, 3n - 3\} \) as follows.

For \( 1 \leq i \leq n - 1 \):

\[
\begin{align*}
f(v_iv_{i+1}) & = 3i, \\
f(v'_iv_{i+1}) & = 3i - 2, \\
f(v_iv'_{i+1}) & = 3i - 1.
\end{align*}
\]

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling.

Hence \( S'(P_n) \) is antimagic.

\[\blacksquare\]

**Illustration 6.3.6.** \( S'(P_6) \) and its antimagic labeling is shown in Figure 6.4.

\[\text{Figure 6.4: } S'(P_6) \text{ and its antimagic labeling.}\]

**Theorem 6.3.7.** \( D_2(P_n) \) is antimagic.

**Proof.** Let \( P'_n, P''_n \) be two copies of path \( P_n \). We denote the vertices of first copy of \( P_n \) by \( v'_1, v'_2, \ldots, v'_n \) and second copy by \( v''_1, v''_2, \ldots, v''_n \). Let \( G \) be \( D_2(P_n) \) with \( |V(G)| = 2n \) and \( |E(G)| = 4n - 4 \).

We define \( f : E(G) \to \{1, 2, \ldots, 4n - 4\} \) as follows.

For \( 1 \leq i \leq n - 1 \):

\[
\begin{align*}
f(v'_iv'_{i+1}) & = 4i, \\
f(v''_iv''_{i+1}) & = 4i - 3, \\
f(v'_iv''_{i+1}) & = 4i - 1,
\end{align*}
\]
For $2 \leq i \leq n$:

$$f(v'_i v''_{i-1}) = 4i - 2.$$ 

Above defined edge labeling function will generate all distinct vertex labels satisfying the condition for antimagic labeling.

Hence $D_2(P_n)$ is antimagic. ■

Illustration 6.3.8. $D_2(P_6)$ and its antimagic labeling is shown in Figure 6.5.

\[
\begin{array}{c}
\text{Figure 6.5: } D_2(P_6) \text{ and its antimagic labeling.}
\end{array}
\]

6.4 Antimagic Labeling of Some Cycle Related Graphs

Theorem 6.4.1. $M(C_n)$ is antimagic.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of cycle $C_n$ and $G = M(C_n)$ be the middle graph of cycle $C_n$. According to the definition of middle graph $V(M(C_n)) = V(C_n) \cup E(C_n)$ and $E(M(C_n)) = \{v_i e_i; 1 \leq i \leq n, v_i e_{i-1}; 2 \leq i \leq n, v_1 e_n, e_i e_{i+1}; 1 \leq i \leq n - 1, e_n e_1\}$. Here $|V(G)| = 2n$ and $|E(G)| = 3n$.

To define $f: E(G) \to \{1, 2, \ldots, 3n\}$, we consider following two cases.

Case 1: $n \equiv 1 \pmod{2}$

For $1 \leq i \leq n$:
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\[ f(v_i) = 2i, \]

For \(1 \leq i \leq n-1\):

\[ f(e_i v_{i+1}) = 2i + 1, \]
\[ f(e_n v_1) = 1, \]
\[ f(e_i e_{i+1}) = 2n + 1 + i, \]
\[ f(e_n e_1) = 2n + 1, \]

**Case 2:** \( n \equiv 0 \pmod{2} \)

\[ f(v_1 e_1) = 2, \]

For \(2 \leq i \leq n\):

\[ f(v_i e_i) = 2i, \]

For \(1 \leq i \leq n-1\):

\[ f(e_i v_{i+1}) = 2i + 1, \]
\[ f(e_n v_1) = 1, \]

For \(1 \leq i \leq n-1\):

\[ f(e_i e_{i+1}) = 2n + 1 + i, \]
\[ f(e_n e_1) = 2n + 1. \]

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling.

Hence \( M(C_n) \) is antimagic.

**Illustration 6.4.2.** \( M(C_5) \) and its antimagic labeling is shown in Figure 6.6.

**Theorem 6.4.3.** \( T(C_n) \) is antimagic.
Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of cycle $C_n$ and $G = T(C_n)$ be the total graph of cycle $C_n$ with $V(T(C_n)) = V(C_n) \cup E(C_n)$ and $E(T(C_n)) = \{v_iv_{i+1}; 1 \leq i \leq n-1, v_nv_1, v_ie_i; 1 \leq i \leq n, e_ie_{i+1}; 1 \leq i \leq n-1, e_ne_1, v_ie_{i+1}; 2 \leq i \leq n, v_1e_n\}$. Here $|V(G)| = 2n$ and $|E(G)| = 4n$.

To define $f : E(G) \rightarrow \{1, 2, \ldots, 4n\}$ we consider following two cases.

Case 1: $n \equiv 1 \pmod{2}$

For $1 \leq i \leq n - 1$:

\[
f(v_iv_{i+1}) = 3n + i, \quad f(v_nv_1) = 4n,
\]

For $2 \leq i \leq n - 1$:

\[
f(e_ie_{i+1}) = i + 1, \quad f(e_ne_1) = 1,
\]

For $1 \leq i \leq n$:

\[
f(v_ie_i) = n + 2i,
\]

For $2 \leq i \leq n$: 

...
\[ f(v_ie_{i-1}) = n - 1 + 2i, \]
\[ f(v_1e_n) = n + 1, \]

**Case 2**: \( n \equiv 0 \pmod{2} \)

**Sub case 1**: \( n \equiv 0 \pmod{4} \)

For \( 1 \leq i \leq n - 1 \):

\[ f(v_iv_{i+1}) = 3n + i, \]
\[ f(v_nv_1) = 4n, \]

For \( 2 \leq i \leq n - 1 \):

\[ f(e_ie_{i+1}) = i + 1, \]
\[ f(e_ne_1) = 2, \]
\[ f(e_1e_2) = 1, \]

For \( 1 \leq i \leq n \):

\[ f(v_ie_i) = n + 2i, \]

For \( 2 \leq i \leq n \):

\[ f(v_ie_{i-1}) = n - 1 + 2i, \]
\[ f(v_1e_n) = n + 1, \]

**Sub case 2**: \( n \equiv 2 \pmod{4} \)

For \( 3 \leq i \leq n - 1 \):

\[ f(v_1v_2) = 3n + 2, \]
\[ f(v_2v_3) = 3n + 1, \]
\[ f(v_iv_{i+1}) = 3n + i, \]

For \( 2 \leq i \leq n - 1 \):

\[ f(v_1e_2) = 3n + 1, \]
\[ f(e_2e_{i+1}) = i + 1, \]
\[ f(e_ne_1) = 2, \]
\[ f(e_1e_2) = 1, \]

For \( 1 \leq i \leq n \):

\[ f(v_ie_i) = n + 2i, \]

For \( 2 \leq i \leq n \):

\[ f(v_ie_{i-1}) = n - 1 + 2i, \]
\[ f(v_1e_n) = n + 1, \]
\[ f(e_i e_{i+1}) = i + 1, \]
\[ f(e_ne_1) = 2, \]
\[ f(e_1e_2) = 1, \]

For \( 1 \leq i \leq n \):

\[ f(v_ie_i) = n + 2i, \]

For \( 2 \leq i \leq n \):

\[ f(v_ie_{i-1}) = n - 1 + 2i, \]
\[ f(v_1e_n) = n + 1. \]

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling.

Hence \( T(C_n) \) is antimagic. \[\square\]

**Illustration 6.4.4.** \( T(C_6) \) and its antimagic labeling is shown in Figure 6.7.

![Figure 6.7: \( T(C_6) \) and its antimagic labeling.](image)

**Theorem 6.4.5.** \( S'(C_n) \) is antimagic.
Proof. Let \( v_1, v_2, \ldots, v_n \) be the vertices and \( e_1, e_2, \ldots, e_n \) be the edges of cycle \( C_n \). Let \( v'_1, v'_2, \ldots, v'_n \) be the newly added vertices to form the splitting graph of cycle \( C_n \). Let \( G = S'(C_n) \) be the splitting graph of cycle \( C_n \). \( V(S'(C_n)) = \{v_i\} \cup \{v'_i\}, 1 \leq i \leq n \) and \( E(S'(C_n)) = \{v'_iv_{i+1}; 1 \leq i \leq n-1, v'_nv_1, v'_1v_n, v'_iv_{i-1}; 2 \leq i \leq n, v_iv_{i+1}; 1 \leq i \leq n-1, v_nv_1\} \). Here \(|V(G)| = 2n\) and \(|E(G)| = 3n\).

To define \( f : E(G) \to \{1, 2, \ldots, 3n\} \) we consider following two cases.

**Case 1:** \( n \equiv 1(\text{mod} \ 2) \)

For \( 1 \leq i \leq n-1 \):

\[
\begin{align*}
    f(v_iv_{i+1}) &= 2n + 1 + i, \\
    f(v_nv_1) &= 2n, \\
    f(v'_iv_{i+1}) &= \begin{cases} 
    2i + 1, & i \equiv 1(\text{mod} \ 2) \\
    2i, & \text{otherwise}
    \end{cases} \\
    f(v'_nv_1) &= 2n + 1,
\end{align*}
\]

For \( 2 \leq i \leq n \):

\[
\begin{align*}
    f(v'_iv_{i-1}) &= \begin{cases} 
    2i - 1, & i \equiv 1(\text{mod} \ 2) \\
    2i - 2, & \text{otherwise}
    \end{cases} \\
    f(v'_1v_n) &= 1,
\end{align*}
\]

**Case 2:** \( n \equiv 0(\text{mod} \ 2) \)

For \( 1 \leq i \leq n-1 \):

\[
\begin{align*}
    f(v_iv_{i+1}) &= 2n + 1 + i, \\
    f(v_nv_1) &= 3n,
\end{align*}
\]

**Sub Case 1:** \( n \equiv 0(\text{mod} \ 4), n \neq 4 \)

\[
\begin{align*}
    f(v'_2v_1) &= 4, \\
    f(v'_2v_3) &= 2,
\end{align*}
\]
For $1 \leq i \leq n$, ($i \neq 2$):

$$f(v'_i v_{i+1}) = \begin{cases} 
2i + 1, & i \equiv 1 \pmod{2} \\
2i, & \text{otherwise}
\end{cases}$$

$$f(v'_1 v_n) = 1,$$

For $3 \leq i \leq n$:

$$f(v'_i v_{i-1}) = \begin{cases} 
2i - 1, & i \equiv 1 \pmod{2} \\
2i - 2, & \text{otherwise}
\end{cases}$$

**Sub Case 2:** $n = 4$ and $n \equiv 1 \pmod{4}$

$$f(v'_1 v_n) = 3,$$

$$f(v'_1 v_2) = 1,$$

For $2 \leq i \leq n$:

$$f(v'_i v_{i+1}) = \begin{cases} 
2i + 1, & i \equiv 1 \pmod{2} \\
2i, & \text{otherwise}
\end{cases}$$

$$f(v'_i v_{i-1}) = \begin{cases} 
2i - 1, & i \equiv 1 \pmod{2} \\
2i - 2, & \text{otherwise}
\end{cases}$$

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling.

Hence $S'(C_n)$ is antimagic.

**Illustration 6.4.6.** $S'(C_4)$ and its antimagic labeling is shown in Figure 6.8.

**Theorem 6.4.7.** $D_2(C_n)$ is antimagic.
Proof. Let $C'_n, C''_n$ be two copies of cycle $C_n$. We denote the vertices of first copy of $C_n$ by $v'_1, v'_2, ..., v'_n$ and second copy by $v''_1, v''_2, ..., v''_n$. Let $G$ be $D_2(C_n)$ with $|V(G)| = 2n$ and $|E(G)| = 4n$.

To define $f: E(G) \rightarrow \{1, 2, ..., 4n\}$ we consider following three cases.

Case 1: $n \equiv 1( \text{mod } 2)$

For $1 \leq i \leq n - 1$:

\[
\begin{align*}
  f(v'_i v'_{i+1}) & = i, \\
  f(v''_i v''_{i+1}) & = n, \\
  f(v'_i v'_{i+1}) & = 3n + i, \\
  f(v''_i v''_{i+1}) & = 4n, \\
  f(v'_i v'_{i+1}) & = n + 2i - 1, \\
  f(v''_i v''_{i+1}) & = 3n - 1, \\
  f(v'_i v''_{i+1}) & = n + 2i, \\
  f(v''_i v''_{i+1}) & = 3n, 
\end{align*}
\]

Case 2: $n \equiv 0( \text{mod } 2), n \neq 6$
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\[ f(v''_n v'_1) = 1, \]
\[ f(v'_n v'_1) = 4n, \]
\[ f(v''_n v''_1) = n + 1, \]
\[ f(v'_1 v''_n) = 3n, \]

For \( 1 \leq i \leq n - 1 \):

\[ f(v''_i v''_{i+1}) = 3n + i, \]
\[ f(v'_i v'_{i+1}) = i + 1, \]
\[ f(v'_i v''_{i+1}) = n + 1 + 2i, \]

For \( 2 \leq i \leq n \):

\[ f(v'_i v'_{i-1}) = n + 2i. \]

**Case 3:** For \( n = 6 \), an antimagic labeling of \( D_2(C_6) \) is shown in Figure 6.9.

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling.

Hence \( D_2(C_n) \) is antimagic. ■

**Illustration 6.4.8.** \( D_2(C_5) \) and its antimagic labeling is shown in Figure 6.10.
6.5 Switching of a vertex and Antimagic Labeling

**Theorem 6.5.1.** The graph obtained by switching of a pendant vertex in $P_n$ admits antimagic labeling.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the vertices of $P_n$ and $G_v$ denotes the graph obtained by switching of a pendant vertex $v$ of $G = P_n$. Without loss of generality let the switched vertex be $v_1$. We note that $|V(G_{v_1})| = n$ and $|E(G_{v_1})| = 2n - 4$.

We define $f : E(G_{v_1}) \to \{1, 2, \ldots, 2n - 4\}$ as follows:

For $2 \leq i \leq n - 1$:

$$f(v_iv_{i+1}) = i - 1,$$

For $3 \leq i \leq n$:

$$f(v_1v_i) = n + i - 4.$$

Above defined edge labeling function will generate all distinct vertex labels as per the definition of antimagic labeling.
Hence the graph obtained by switching of a pendant vertex in a path $P_n$ is antimagic. ■

**Illustration 6.5.2.** The graph obtained by switching of a vertex $v_1$ in path $P_5$ and its antimagic labeling is shown in Figure 6.11.

![Figure 6.11](image)

**Theorem 6.5.3.** The graph obtained by switching of a vertex in cycle $C_n$ admits antimagic labeling.

**Proof.** Let $v_1, v_2, \ldots, v_n$ be the successive vertices of $C_n$ and $G_v$ denotes graph obtained by switching of vertex $v$ of $G = C_n$. Without loss of generality let the switched vertex be $v_1$. We note that $|V(G_{v_1})| = n$ and $|E(G_{v_1})| = 2n - 5$.

We define $f : E(G_{v_1}) \rightarrow \{1, 2, \ldots, 2n - 5\}$ as follows.

For $3 \leq i \leq n$:

\[
\begin{align*}
    f(v_1 v_i) & = 2(i - 2), \\
    f(v_{i-1} v_i) & = 2i - 5.
\end{align*}
\]

Above defined edge labeling function will generate all distinct vertex labels as per the definition of an antimagic labeling.

Hence the graph obtained by switching of a vertex in a cycle $C_n$ is antimagic. ■

**Illustration 6.5.4.** The graph obtained by switching of a vertex $v_1$ in cycle $C_7$ and its antimagic labeling is shown in Figure 6.12.
**Theorem 6.5.5.** The graph obtained by switching of a rim vertex in a wheel $W_n$ admits antimagic labeling.

**Proof.** Let $v$ as the apex vertex and $v_1, v_2, \ldots, v_n$ be the rim vertices of wheel $W_n$. Let $G_{v_1}$ denotes graph obtained by switching of a rim vertex $v_1$ of $G = W_n$. We note that $|V(G_{v_1})| = n + 1$ and $|E(G_{v_1})| = 3n - 6$.

We define $f : E(G_{v_1}) \to \{1, 2, \ldots, 3n - 6\}$ as follows.

For $2 \leq i \leq n - 1$:

$$f(v_i v_{i+1}) = i - 1,$$

For $2 \leq i \leq n$:

$$f(v_i v_{i}) = 2n - i - 1,$$

For $3 \leq i \leq n - 1$:

$$f(v_1 v_{i}) = 2n + i - 5.$$
Above defined edge labeling function will generate all distinct vertex labels as per the definition of an antimagic labeling.

Hence the graph obtained by switching of a rim vertex in a wheel $W_n$ is antimagic. ■

**Illustration 6.5.6.** The graph obtained by switching of a vertex $v_1$ in wheel $W_8$ and its antimagic labeling is shown in Figure 6.13.

![Figure 6.13: The graph obtained by switching of a vertex $v_1$ in wheel $W_8$ and its antimagic labeling.](image)

**Theorem 6.5.7.** The graph obtained by switching of an apex vertex in helm $H_n$ admits antimagic labeling.

**Proof.** Let $H_n$ be a helm with $v$ as the apex vertex, $v_1, v_2, \ldots, v_n$ be the vertices of cycle and $u_1, u_2, \ldots, u_n$ be the pendant vertices. Let $G_v$ denotes graph obtained by switching of an apex vertex $v$ of $G = H_n$. We note that $|V(G_v)| = 2n + 1$ and $|E(G_v)| = 3n$.

We define $f : E(G_v) \to \{1, 2, \ldots, 3n\}$ as follows.

**Case 1:** $n \equiv 0 \pmod{3}, n \neq 3$

\[
\begin{align*}
  f(vu_1) &= 2, \\
  f(v_1u_1) &= 1, \\
  f(v_1v_2) &= 3,
\end{align*}
\]
For $2 \leq i \leq n$:

\[
\begin{align*}
 f(vu_i) &= 3i - 2, \\
 f(v_iu_i) &= 3i - 1, \\
 f(v_iu_{i+1}) &= 3i, \text{ where } (v_{i+1} = v_1).
\end{align*}
\]

The case when $n = 3$ is to be deal separately and the graph is labeled as shown in Figure 6.14.

![Figure 6.14: The graph obtained by switching of an apex vertex in helm $H_3$ and its antimagic labeling.](image)

**Case 2: $n \equiv 1, 2 \pmod{3}$**

For $1 \leq i \leq n$:

\[
\begin{align*}
 f(vu_i) &= 3i - 2, \\
 f(v_iu_i) &= 3i - 1, \\
 f(v_iu_{i+1}) &= 3i, \text{ where } v_{n+1} = v_1.
\end{align*}
\]

Above defined edge labeling function will generate all distinct vertex labels as per the definition of an antimagic labeling.

Hence the graph obtained by switching of an apex vertex in a helm $H_n$ is antimagic. ■
Illustration 6.5.8. The graph obtained by switching of an apex vertex \( v \) in helm \( H_5 \) and its antimagic labeling is shown in Figure 6.15.

Theorem 6.5.9. The graph obtained by switching of a vertex with degree 2 in fan \( f_n \) admits antimagic labeling.

Proof. Let \( v \) as the apex vertex and \( v_1, v_2, \ldots, v_n \) be the vertices of fan \( f_n \). Let \( G_{v_1} \) denotes graph obtained by switching of a vertex \( v_1 \) having degree 2 of \( G = f_n \). We note that \( |V(G_{v_1})| = n + 1 \) and \( |E(G_{v_1})| = 3n - 5 \).

We define \( f : E(G_{v_1}) \to \{1, 2, \ldots, 3n - 5\} \) as follows.

For \( 2 \leq i \leq n \):

\[
f(v_i v_i) = i - 1,\]

For \( 2 \leq i \leq n - 1 \):

\[
f(v_i v_{i+1}) = n + i - 2,\]

For \( 3 \leq i \leq n \):
\( f(v_1 v_i) = 2n + i - 5. \)

Above defined edge labeling function will generate all distinct vertex labels as per the definition of an antimagic labeling.

Hence the graph obtained by switching of a vertex having degree 2 in fan \( f_n \) is antimagic.

**Illustration 6.5.10.** The graph obtained by switching of a vertex \( v_1 \) having degree 2 in fan \( f_5 \) and its antimagic labeling is shown in *Figure* 6.16.

![Graph](image)

**Figure 6.16:** The graph obtained by switching of a vertex \( v_1 \) having degree 2 in fan \( f_5 \) and its antimagic labeling.

### 6.6 Conclusion and Scope of Further Research

This chapter was targeted to discuss antimagic labeling of graphs. We investigate antimagic labeling for some path and cycle related graphs. We have also discussed antimagic labeling in the context of switching of a vertex.

It is also possible to investigate similar results for the graphs obtained by path union of cycles, wheels, Petersen graph etc. To identify the interrelations of other labeling schemes with antimagic labeling is also an open area of research.