Chapter 6

THREE-DIMENSIONAL VIBRATION ANALYSIS OF A THERMOELASTIC CYLINDRICAL PANEL WITH VOIDS

6.1 INTRODUCTION

In this Chapter, based on three-dimensional linear generalized thermoelasticity, an exact analysis of free vibration of a simply supported homogeneous isotropic, thermally conducting, cylindrical panel with voids initially at uniform temperature and undeformed state has been presented. Three displacement potential functions are introduced for solving the equations of motion, heat conduction and volume fraction field. The purely transverse wave gets decoupled from rest of motion and is not affected by thermal and volume fraction (voids) fields. After expanding the displacement potentials, volume fraction and temperature functions with orthogonal series, the equations of the considered vibration problem are reduced to five-second order coupled ordinary differential equations whose formal solution can be expressed by using Bessel functions with complex arguments. The corresponding results for thermoelastic panel without voids, elastic panel with and without voids have been deduced as special cases from the present analysis. In order to illustrate the analytical results, the numerical solutions of various relations and equations have been obtained to compute the lowest frequency as function of different cylindrical panel parameters. The computer simulated results have been presented graphically.
6.2 FORMULATION OF THE PROBLEM

We consider a homogeneous isotropic, thermally conducting, elastic cylindrical panel with voids in the undisturbed state and at uniform temperature $T_0$ initially. Let '$L'$ be the length; '$R_1'$ and '$R_2'$ are respectively the inner and outer radii of the cylindrical panel and $\eta$ is the central angle. The geometry of the problem is shown in Fig. 6.1 below.

Figure 6.1: Geometry of the problem
Upon employing Lord and Shulman (1967) model of linear generalized thermoelasticity, the governing field equations of motion, volume fraction field, and heat conduction (1.5.2)-(1.5.5) for such a solid with voids, in the absence of body forces and heat sources, become

\[
\sigma_{rr} + \frac{1}{r} \sigma_{r\theta} + \sigma_{r\varphi} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \ddot{u}_r \quad (6.2.1)
\]

\[
\sigma_{r\theta} + \frac{1}{r} \sigma_{\theta\theta} + \sigma_{\varphi\varphi} + \frac{2\sigma_{r\theta}}{r} = \rho \ddot{u}_\theta \quad (6.2.2)
\]

\[
\sigma_{r\varphi} + \frac{1}{r} \sigma_{\theta\varphi} + \sigma_{\varphi\varphi} + \frac{\sigma_{r\varphi}}{r} = \rho \ddot{u}_\varphi \quad (6.2.3)
\]

\[
\frac{\partial h_r}{\partial r} + \frac{h_r}{r} + \frac{1}{r} \frac{\partial h_\theta}{\partial \theta} + \frac{\partial h_\varphi}{\partial z} + g + mT = \rho \ddot{\varphi} \quad (6.2.4)
\]

\[
K \nabla^2 T - \rho C_v \left[ T + T_0 \right] = \beta T \left[ \frac{\partial}{\partial t} + T_z \frac{\partial^2}{\partial t^2} \right] \left[ e_{rr} + e_{\theta\theta} + e_{\varphi\varphi} \right] + m \tau_0 (\ddot{\phi} + T_0 \dot{\phi}) \quad (6.2.5)
\]

where

\[
\sigma_{ij} = \lambda \varepsilon_k \delta_{ij} \varepsilon_k + 2 \mu \varepsilon_{ij} + (b\phi - \beta T) \delta_{ij}, \quad i, j, k = r, \theta, z \quad (6.2.6)
\]

\[
e_{rr} = \frac{\partial u_r}{\partial r},
\]

\[
e_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r},
\]

\[
e_{\varphi\varphi} = \frac{\partial u_\varphi}{\partial z}
\]

\[
e_{r\varphi} = \frac{1}{2} \left[ \frac{\partial u_r}{\partial r} + \frac{\partial u_\varphi}{\partial r} \right] \quad (6.2.7)
\]

\[
e_{r\theta} = \frac{1}{2} \left[ \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right],
\]
Here $\bar{u}(r, \theta, z, t) = (u_r, u_\theta, u_z)$ is the displacement vector, $T(r, \theta, z, t)$ is the temperature change; $\phi$ is volume fraction change, $\mathcal{X}$ is equilibrated inertia, $h_i$ is equilibrated stress vector; $\lambda, \mu$ are Lamé parameters, $\beta = (3\lambda + 2\mu)\alpha_T$ is coupling parameter of thermal and mechanical fields, $\alpha_r$ and $K$ are respectively, the coefficients of linear thermal expansion and thermal conductivity; $\rho$ and $C_r$ are the mass density and specific heat at constant strain, respectively; $t_0$ is thermal relaxation time. $b$ is the coupling parameter of the mechanical and volume fraction field, $m$ is the coupling parameter of the thermal and volume fraction field. $\alpha$ is void parameter. The comma notation is used for spatial derivatives; the superposed dot represents time differentiation. The coefficients $\xi_1$, $\xi_2$ and equilibrated inertia $\mathcal{X}$ must be positive so as to satisfy dissipation inequality resulting from second law of thermodynamics.

We define the quantities
A STUDY OF WAVE PROPAGATION IN GENERALIZED THERMOELASTIC MATERIALS WITH VOIDS

\[ \begin{align*}
  r' &= \frac{\omega^* r}{c_1}, \quad z' = \frac{\omega^* z}{c_1}, \quad t' = \omega^* t, \quad u'_i = \frac{\rho c_i \omega^*}{\beta T_0} u_i, \\
  T' &= \frac{T}{T_0}, \quad \sigma'_{ij} = \frac{\sigma_{ij}}{\beta T_0}, \quad \phi' = \frac{\omega^*^2 \phi}{c_1^2}, \quad e'_{ij} = \frac{\rho c_1^2}{\beta T_0} e_{ij}, \\
  h'_i &= \frac{\omega^* \chi}{\alpha c_1} h_i, \quad \xi = \frac{\xi_2 \omega^*}{\xi_1}, \quad t'_0 = \omega^* t_0, \quad \omega' = \frac{\omega}{\omega^*}, \\
  \omega^* &= \frac{C_e(\lambda + 2 \mu)}{K}, \quad \epsilon_T = \frac{\beta^2 T_0}{\rho C_e(\lambda + 2 \mu)}, \quad \delta^2 = \frac{c_2^2}{c_1^2}, \\
  \delta_1^2 &= \frac{c_2^2}{c_1^2}, \quad a_1 = \frac{bc_1^2}{b \beta T_0 \chi \omega^*^2}, \quad a_2 = \frac{b \chi T_0^2}{\alpha \rho c_1^2}, \quad a_3 = \frac{\xi_1 c_1^2}{\alpha \omega^*^2}, \\
  a_4 &= \frac{m T_0 \chi}{\alpha}, \quad a_5 = \frac{m c_1^4}{\alpha K \chi \omega^*^3}, \quad c_1^2 = \frac{\lambda + 2 \mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}, \\
  c_3^2 &= \frac{\alpha}{\rho \chi}, \quad R_m = \frac{R_1 + R_2}{2}, \quad R'_m = \frac{\omega^*}{c_1} R_m, \quad L' = \frac{\omega^*}{c_1} L, \\
  R'_1 &= \frac{\omega^*}{c_1} R_1, \quad R'_2 = \frac{\omega^*}{c_1} R_2.
\end{align*} \]

Here $\epsilon_T$ is the thermoelastic coupling parameter, $c_1, c_2, c_3$ are respectively the velocities of longitudinal, transverse and volume fraction fields. Introducing quantities defined by (6.2.10) in equations (6.2.1) - (6.2.9), we obtain (on suppressing the primes for convenience)

\[ \begin{align*}
  \sigma_{rr,r} + \frac{1}{r} \sigma_{r,\theta} + \sigma_{rz,z} + \frac{\sigma_{rr} - \sigma_{\theta \theta}}{r} &= \ddot{u}_r, \\
  \sigma_{r\theta, r} + \frac{1}{r} \sigma_{\theta \theta, \theta} + \sigma_{\theta z, z} + \frac{2 \sigma_{r \theta}}{r} &= \ddot{u}_\theta, \\
  \sigma_{rz, r} + \frac{1}{r} \sigma_{\theta z, \theta} + \sigma_{zz, z} + \frac{\sigma_{rz}}{r} &= \ddot{u}_z, \\
  h_{r, r} + \frac{1}{r} h_{\theta, \theta} + h_{r, z} - a_2 e_{sk} - a_3 (1 + \xi \frac{\partial}{\partial t}) \phi + a_4 T = \frac{1}{\delta_1^2} \phi.
\end{align*} \]
A STUDY OF WAVE PROPAGATION IN GENERALIZED THERMOELASTIC MATERIALS WITH VOIDS

\[
\nabla^2 T - \left[ \partial_t + \partial_t \phi \right] = \varepsilon \left[ \frac{\partial}{\partial t} t + \frac{\partial^2}{\partial t^2} \right] \varepsilon_{kk} + a_s (\phi + \partial_t \phi) \tag{6.2.15}
\]

\[
\sigma_{ij} = \left( 1 - 2\delta^2 \right) \varepsilon_{kk} \delta_{ij} + 2\delta^2 \varepsilon_{ij} + \left( a_i \phi - T \right) \delta_{ij}, \quad i, j = r, \theta, z \tag{6.2.16}
\]

\[
h_r = \frac{\partial \phi}{\partial r},
\]

\[
h_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta},
\]

\[
h_z = \frac{\partial \phi}{\partial z}
\]

where

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.
\]

6.3 SOLUTION OF THE PROBLEM

To solve the system of equations (6.2.11)-(6.2.15), we introduce displacement potential functions through the relations

\[
u_r = \frac{1}{r} \psi_r - G_{r,r},
\]

\[
u_\theta = -\frac{1}{r} G_{r,\theta} - \psi_r,
\]

\[
u_z = w_z \tag{6.3.1}
\]

Upon using relations (6.3.1) in governing field equations (6.2.11)-(6.2.15), we find that

\[
G, \ w, \ \psi, \ \phi \ \text{and} \ T \ \text{satisfy the equations,}
\]

\[
\left[ \nabla^2 + \delta^2 \left( \frac{R_n^2}{L^2} \frac{\partial^2}{\partial z^2} - \frac{R_m^2}{L^2} \frac{\partial^2}{\partial t^2} \right) \right] G - \left( 1 - \delta^2 \right) \frac{R_m^2}{L^2} \frac{\partial^2 w}{\partial z^2} - a_1 R_n^2 \phi + R_m^2 T = 0 \tag{6.3.2}
\]
A STUDY OF WAVE PROPAGATION IN GENERALIZED THERMOELASTIC MATERIALS WITH VOIDS

\[ - \left(1 - \delta^2\right) \nabla_i^2 G + \left(\delta^2 \nabla_i^2 + \frac{R_m^2}{L^2} \frac{\partial^2}{\partial z^2} - R_n^2 \frac{\partial^2}{\partial t^2}\right) w + a_1 R_m^2 \phi - R_n^2 T = 0 \quad (6.3.3) \]

\[ \left( \nabla_i^2 + \frac{R_m^2}{L^2} \frac{\partial^2}{\partial z^2} \right) \phi - a_2 \left( - \nabla_i^2 G + \frac{R_m^2}{L^2} \frac{\partial^2 w}{\partial z^2} \right) - a_1 \left( 1 + \xi \frac{\partial}{\partial t} \right) R_m^2 \phi + a_4 R_m^2 T = \frac{R_m^2 \phi}{\delta^2} \quad (6.3.4) \]

\[ \left( \nabla_i^2 + \frac{R_m^2}{L^2} \frac{\partial^2}{\partial z^2} \right) T - R_m^2 \left( T + t_0 \dot{T} \right) \]

\[ \quad \varepsilon_T \left( \frac{\partial}{\partial t} + t_0 \frac{\partial^2}{\partial t^2} \right) \left( - \nabla_i^2 G + \frac{R_m^2}{L^2} \frac{\partial^2 w}{\partial z^2} \right) + a_2 R_m^2 (\phi + t_0 \dot{\phi}) \quad (6.3.5) \]

\[ \left( \nabla_i^2 + \frac{R_m^2}{L^2} \frac{\partial^2}{\partial z^2} - \frac{1}{\delta^2} R_m^2 \frac{\partial^2}{\partial t^2} \right) \psi = 0 \quad (6.3.6) \]

where \( \nabla_i^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \), \( r' = R_m^2 \), \( z' = \frac{z}{L} \). Here primes have been suppressed for convenience.

The equation (6.3.6) in the above system of equations in \( \psi \) represents purely transverse waves, which are not affected by the temperature change and volume fraction field. This wave motion is polarized in planes perpendicular to the z-axis and may be referred to as shear horizontal (SH) wave. We consider the free vibrations of a cylindrical panel (see Fig. 6.1) subjected to the traction free, thermally insulated or isothermal boundary conditions with simply supported edges. We can write displacement potential functions, volume fraction field and temperature change as below:

\[ \psi(r, \theta, z, t) = \psi(r) \sin(\rho \pi z) \cos(\nu \theta) \exp(i \omega t) \]

\[ G(r, \theta, z, t) = G(r) \sin(\rho \pi z) \sin(\nu \theta) \exp(i \omega t) \]

\[ \phi(r, \theta, z, t) = \phi(r) \sin(\rho \pi z) \sin(\nu \theta) \exp(i \omega t) \]

\[ w(r, \theta, z, t) = w(r) \sin(\rho \pi z) \sin(\nu \theta) \exp(i \omega t) \quad (6.3.7) \]
A STUDY OF WAVE PROPAGATION IN GENERALIZED THERMOELASTIC MATERIALS WITH VOIDS

\[ T(r, \theta, z, t) = \bar{T}(r) \sin(p \pi z) \sin(\nu \theta) \exp(i \omega t) \]

where \( \nu = \frac{n \pi}{\eta} \). Here \( n \) and \( p \) respectively define the circumferential and axial wave numbers.

On using solution (6.3.7) in equations (6.3.2)- (6.3.6), we obtain

\[
\left( \nabla^2 + k_1^2 \right) \overline{\psi} = 0 \tag{6.3.8}
\]

\[
\left( \nabla^2 + g_1 \right) \overline{G} + g_2 \overline{\psi} - a_1 R_m^2 \overline{\phi} + R_m^2 \overline{T} = 0 \tag{6.3.9}
\]

\[
-(1 - \delta^2) \nabla^2 \overline{G} + \delta^2 \left( \nabla^2 + g_3 \right) \overline{\psi} + a_1 R_m^2 \overline{\phi} - R_m^2 \overline{T} = 0 \tag{6.3.10}
\]

\[
\left( \nabla^2 + g_4 \right) \overline{T} + i \omega r_0 \in_R \left( \nabla^2 \overline{G} + \tau_L^2 \overline{\psi} \right) - i \omega r_0 R_m^2 a_2 \overline{\phi} = 0 \tag{6.3.11}
\]

\[
\left( \nabla^2 + g_5 \right) \overline{\phi} + a_2 \nabla^2 \overline{G} + a_4 \tau_L^2 \overline{\psi} + a_4 R_m^2 \overline{T} = 0 \tag{6.3.12}
\]

where

\[
\nabla^2 = \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) - \frac{\nu^2}{r^2}
\]

\[
\tau_s = 1 + i \omega \tau_s, \quad \xi^s = 1 + i \omega \xi^s, \quad \Omega = \omega R_m, \quad \tau_L = \frac{p \pi R_m}{L} \tag{6.3.13}
\]

\[
k_1^2 = \frac{\Omega^2}{\delta^2} - \tau_L^2, \quad g_1 = \delta^2 k_1^2, \quad g_2 = (1 - \delta^2) \tau_L^2, \quad g_3 = \frac{1}{\delta^2} (\Omega^2 - \tau_L^2)
\]

\[
g_4 = i \omega r_0 \left( \Omega^2 - \frac{i \omega \tau_L^2}{\tau_s} \right), \quad g_5 = \left( \frac{\Omega^2}{\delta^2} - \tau_L^2 - a_3 R_m^2 \xi^s \right) \tag{6.3.14}
\]

The equation (6.3.8) is a Bessel equation with its possible solution as

\[
\overline{\psi} = \begin{cases} A_3 J_\nu(k_1 r) + B_3 Y_\nu(k_1 r), \quad k_1^2 > 0 \\ A_3 r^\nu + B_3 r^{-\nu}, \quad k_1^2 = 0 \\ A_3 I_\nu(k_1 r) + B_3 K_\nu(k_1 r), \quad k_1^2 < 0. \end{cases} \tag{6.3.15}
\]

where \( k_1^2 = -k_2^2 \). Here \( J_\nu \) and \( Y_\nu \) (\( I_\nu \) and \( K_\nu \)) are respectively the Bessel functions (modified Bessel functions) of the first and second kind, \( A_3 \) and \( B_3 \) are arbitrary.
constants. Generally \( k_i^2 \neq 0 \), so the specific situation \( k_i^2 = 0 \) will not be discussed in the following analysis. From equations (6.3.9) – (6.3.12), we obtain

\[
(y_1 + m_1)(y_1 + m_2)(y_2 + m_1)(y_2 + m_2)\bar{F} = 0
\]  

(6.3.16)

where \( \bar{F} \) may be any one of the functions \( G, \phi, \bar{w}, \) or \( \bar{T} \) and the quantities \( m_i^2 \), \( i = 1, 2, 3, 4 \) are the roots of the complex algebraic equation

\[
Z^4 + A Z^3 + B Z^2 + C Z + D = 0
\]  

(6.3.17)

Here the coefficients \( A, B, C \) and \( D \) are given by

\[
A = g_1 + g_3 - g_4 + g_5 + \frac{1-\delta^2}{\delta^2} g_2 + a_1 a_4 R_m^2 - i \omega \tau_0 \in \Omega^2
\]  

(6.3.18)

\[
B = g_1 g_3 - g_4 \left( g_1 + g_3 + \frac{1-\delta^2}{\delta^2} g_2 \right)
+ g_5 \left( g_1 + g_3 - g_4 + \frac{1-\delta^2}{\delta^2} g_2 \right) + a_1 a_4 R_m^2 \left( g_3 - g_4 + \frac{g_2}{\delta^2} - \tau_0^2 \right)
- i \omega \tau_0 \Omega^2 \left[ \left( g_3 + g_5 + \frac{g_2}{\delta^2} - \tau_0^2 + a_1 a_4 R_m^2 \right) - a_1 (a_4 - a_5) R_m^2 \right]
\]  

(6.3.19)

\[
C = -g_1 g_3 g_4 + g_5 \left( g_1 g_3 - g_1 g_4 - g_3 g_4 - \frac{1-\delta^2}{\delta^2} g_2 g_4 \right)
- a_1 a_4 R_m^2 \left( g_4 \tau_0^2 + g_3 g_4 + \frac{1}{\delta^2} (g_4 \tau_0^2 + g_2 g_4) \right)
\]
Upon solving equations (6.3.9) - (6.3.12) and noting that the roots \( m_i^2 \), \( i = 1, 2, 3, 4 \) of equation (6.3.17) are complex, the functions \( \overline{G}, \overline{\phi}, \overline{w} \) and \( \overline{T} \) are obtained as

\[
\overline{G}(r) = \sum_{i=1}^{4} \left[ A_i J_v(m_i, r) + B_i Y_v(m_i, r) \right]
\]

\[
\overline{w}(r) = \sum_{i=1}^{4} \overline{W}_i \left[ A_i J_v(m_i, r) + B_i Y_v(m_i, r) \right]
\]

\[
\overline{T}(r) = \sum_{i=1}^{4} \frac{S_i}{R_i} \left[ A_i J_v(m_i, r) + B_i Y_v(m_i, r) \right]
\]

\[
\overline{\phi}(r) = \sum_{i=1}^{4} \overline{W}_i \left[ A_i J_v(m_i, r) + B_i Y_v(m_i, r) \right]
\]

(6.3.22)

where

\[
\overline{W}_i = \frac{D_i^j}{D_j}, \quad W_i = \frac{-D_i^j}{D_j}, \quad S_i = \frac{D_i^3}{D_i}
\]

\[
D_i = \begin{vmatrix}
  g_2 & a_i R_m^2 & 1 \\
  \delta^2 (m_i^2 - g_3) & a_i R_m^2 & 1 \\
  a_i t_i^2 & (m_i^2 - g_3) & a_i
\end{vmatrix}, \quad i = 1, 2, 3, 4
\]

(6.3.23)
and \( D_i^j, j = 1, 2, 3 \) can be obtained from \( D_i \) by replacing \( j^{th} \) column with

\[
\begin{pmatrix}
m_i^2 - g_{i1}, & (1 - \delta^2) m_i^2, & a_2 m_i^2
\end{pmatrix}
\]

(6.3.24)

It is noticed that the characteristics equation (6.3.17), in general possesses complex roots and hence Bessel functions \( J_\nu \) and \( Y_\nu \) constitute the solution of equation (6.3.8)-(6.3.12). Also for homogeneity, we proceed with our derivation by taking \( k_i^2 > 0 \) (the derivation for \( k_i^2 < 0 \) is obviously similar) in equation (6.3.8), which corresponds to

\[
\bar{\psi}(r) = A_3 J_{\nu}(k_1 r) + B_3 Y_{\nu}(k_1 r)
\]

(6.3.25)

### 6.4 FREQUENCY EQUATION

In this section we shall derive the secular equation for three dimensional vibrations of cylindrical panel subjected to traction free and thermally insulated/isothermal boundary conditions at the lower and upper surfaces \( r = R_1, R_2 \) of the thermoelastic cylindrical panel with voids. Upon using solutions (6.3.22) and (6.3.25) in (6.3.7) and then in relations (6.2.16), and (6.3.1), the displacements, volume fraction field, temperature change and stresses are obtained as

\[
\bar{u}_r = \frac{1}{R_m} \left( - \overline{G}' - \frac{V \overline{\psi}}{r} \right) \sin(p \pi z) \sin(v \vartheta) \exp(i \omega \tau)
\]

\[
\bar{u}_\theta = \frac{1}{R_m} \left( - \overline{G}' - \frac{V \overline{\psi}}{r} \right) \sin(p \pi z) \cos(v \vartheta) \exp(i \omega \tau)
\]

\[
\bar{u}_z = \frac{t_m}{R_m} \overline{w}(r) \cos(p \pi z) \sin(v \vartheta) \exp(i \omega \tau)
\]

\[ \phi = \overline{\phi}(r) \sin(p \pi z) \sin(v \vartheta) \exp(i \omega \tau) \]
A STUDY OF WAVE PROPAGATION IN GENERALIZED THERMOELASTIC MATERIALS WITH VOIDS

\[ T = T(r) \sin(p \pi z) \sin(\nu \theta) \exp(i\omega t) \]  

(6.4.1)

\[
\sigma_{rr} = \frac{1}{R_m^2} \left\{ \left( \overline{G'} - \frac{(1 - 2\delta^2)}{r} \overline{G} + \frac{\nu^2}{r^2} \left( 1 - 2\delta^3 \right) \overline{G} \right) - 2\delta^2 \nu \left( \frac{\overline{\psi'}}{r} - \frac{\overline{\psi}}{r^2} \right) \right. 

- \left( 1 - 2\delta^2 \right) \overline{\psi} t_i^2 + (a_i \overline{\Phi} - \overline{T}) R_m^{-2} \left\} \sin(p \pi z) \sin(\nu \theta) \exp(i\omega t) \right.

\sigma_{r\theta} = \delta^2 \frac{t_i}{R_m^2} \left[ - \overline{G'} - \frac{\nu \overline{G}}{r} + \frac{\overline{G}}{r^2} \right] \cos(p \pi z) \sin(\nu \theta) \exp(i\omega t)

(6.4.2)

\[
\sigma_{r\phi} = \delta^2 \frac{t_i}{R_m^2} \left[ - \frac{2\nu \overline{G'}}{r} + \frac{2\nu \overline{G}}{r^2} - \overline{\psi'} + \frac{\overline{\psi}}{r^2} \right] \sin(p \pi z) \cos(\nu \theta) \exp(i\omega t) (6.4.3)

Here prime denotes differentiation with respect to radial co-ordinate \( r \). Considering the traction free, thermally insulated and isothermal conditions at the lower and upper surfaces \( r = t_1, t_2 \) and making use of constitutive relations (6.4.2) along with (6.3.22), (6.3.25) and (6.4.1), one can obtain the free vibration secular equation as below:

\[ |E_{ij}| = 0, \quad (i, j = 1, 2, ..., 10) \]

(6.4.3)

where

\[
E_{11} = -m_1^2 J_{\nu}^* (m_1 t_1) - \frac{1 - 2\delta^2}{t_1} m_1 J_{\nu} (m_1 t_1)

+ \frac{(1 - 2\delta^2) \nu^2}{t_1^2} J_{\nu} (m_1 t_1) - \left[ (1 - 2\delta^2) \nu^2 \overline{W}_1 - a_1 \overline{W}_1 R_m^2 + S_1 \right] J_{\nu} (m_1 t_1)

E_{13} = -m_2^2 J_{\nu}^* (m_2 t_1) - \frac{1 - 2\delta^2}{t_1} m_2 J_{\nu} (m_2 t_1)

+ \frac{(1 - 2\delta^2) \nu^2}{t_1^2} J_{\nu} (m_2 t_1) - \left[ (1 - 2\delta^2) \nu^2 \overline{W}_2 - a_1 \overline{W}_2 R_m^2 + S_2 \right] J_{\nu} (m_2 t_1)

E_{15} = -m_3^2 J_{\nu}^* (m_3 t_1) - \frac{1 - 2\delta^2}{t_1} m_3 J_{\nu} (m_3 t_1)

+ \frac{(1 - 2\delta^2) \nu^2}{t_1^2} J_{\nu} (m_3 t_1) - \left[ (1 - 2\delta^2) \nu^2 \overline{W}_3 - a_1 \overline{W}_3 R_m^2 + S_3 \right] J_{\nu} (m_3 t_1)

155
A STUDY OF WAVE PROPAGATION IN GENERALIZED THERMOELASTIC MATERIALS WITH VOIDS

\[ E_{17} = -m_4^2 J_\nu'(m_4 t_1) - \frac{1 - 2 \delta^2}{t_1} m_4 J_\nu'(m_4 t_1) \]
\[ + \frac{(1 - 2 \delta^2) \nu^2}{t_1^2} J_\nu(m_4 t_1) - \left[ (1 - 2 \delta^2) t_1^2 \bar{W}_4 - a_i W_i R^2 + S_4 \right] J_\nu(m_4 t_1) \]

\[ E_{19} = -2 \delta^2 \left( \frac{1}{t_1} J_\nu'(k_1 t_1) - \frac{1}{t_1^2} J_\nu(k_1 t_1) \right) , \]

\[ E_{31} = m_1 (\bar{W}_1 - 1) J_\nu'(m_1 t_1) , \]

\[ E_{33} = m_2 (\bar{W}_2 - 1) J_\nu'(m_2 t_1) \]

\[ E_{35} = m_3 (\bar{W}_3 - 1) J_\nu'(m_3 t_1) , \]

\[ E_{37} = m_4 (\bar{W}_4 - 1) J_\nu'(m_4 t_1) \]

\[ E_{39} = -\frac{\nu}{t_1} J_\nu(k_1 t_1) \]

\[ E_{51} = -\frac{2 \nu m_1}{t_1} J_\nu'(m_1 t_1) + \frac{2 \nu}{t_1^2} J_\nu(m_1 t_1) , \]

\[ E_{53} = -\frac{2 \nu m_2}{t_1} J_\nu'(m_2 t_1) + \frac{2 \nu}{t_1^2} J_\nu(m_2 t_1) \]

\[ E_{55} = -\frac{2 \nu m_3}{t_1} J_\nu'(m_3 t_1) + \frac{2 \nu}{t_1^2} J_\nu(m_3 t_1) \]

\[ E_{57} = -\frac{2 \nu m_4}{t_1} J_\nu'(m_4 t_1) + \frac{2 \nu}{t_1^2} J_\nu(m_4 t_1) \]

\[ E_{59} = -k_1^2 J_\nu'(k_1 t_1) + \frac{k_1}{t_1} J_\nu'(k_1 t_1) - \frac{\nu^2}{t_1^2} J_\nu(k_1 t_1) , \]

\[ E_{71} = W_1 m_1 J_\nu'(m_1 t_1) \]

\[ E_{73} = W_2 m_2 J_\nu'(m_2 t_1) \]
\[ E_{75} = W_2 m_2 J'_v (m_3 t_1), \]
\[ E_{77} = W_4 m_4 J'_v (m_4 t_1), \]
\[ E_{79} = 0 \]
\[ E_{91} = S_1 [m_1 J'_v (m_1 t_1) + h J_v (m_1 t_1)], \]
\[ E_{93} = S_2 [m_2 J'_v (m_2 t_1) + h J_v (m_2 t_1)], \]
\[ E_{95} = S_3 [m_3 J'_v (m_3 t_1) + h J_v (m_3 t_1)], \]
\[ E_{97} = S_4 [m_4 J'_v (m_4 t_1) + h J_v (m_4 t_1)]. \]
\[ E_{99} = 0 \] (6.4.4)

In the elements \( E_{91}, E_{93}, E_{95}, E_{97} \), \( h \to 0 \) corresponds to thermally insulated surfaces and \( h \to \infty \) refers to isothermal boundary conditions.

Here \( E_{ij} \ (j=2, 4, 6, 8, 10) \) can be obtained by just replacing Bessel functions of first kind in \( E_{ij} \ (i = 1, 3, 5, 7, 9) \) with that of the second kind, respectively; while \( E_{ij} \ (i = 2, 4, 6, 8, 10) \) can be obtained by just replacing \( t_i \) in \( E_{ij} \), \( (j = 1, 3, 5, 7, 9) \) with \( t_2 \), respectively.

Here \[ t_1 = \frac{R_{1 \perp}}{R_m} = 1 - \frac{t^*}{2}, \quad t_2 = \frac{R_{2 \perp}}{R_m} = 1 + \frac{t^*}{2} \] and \[ t^* = \frac{R_2 - R_1}{R_m} \] is the thickness to mean radius ratio of the panel. The results and various relations in dimensional form can be obtained from the above analysis via quantities (6.2.10).
6.5 SPECIAL CASES

The results for homogeneous isotropic, generalized thermoelastic cylindrical panel (see Sharma and Sharma (2002)) and coupled homogeneous isotropic thermoelastic cylindrical panel (see Sharma (2001a)) can be recovered from the present analysis by setting \( b = 0 = m \) and \( t_0 = 0, \ b = 0 = m \) respectively, in addition to substituting

\[
c_1 = 1, \quad c_2 = \frac{\mu}{\lambda + 2\mu} = c_4, \quad c_3 = 1 - c_2, \quad \bar{\rho} = 1, \quad \bar{K} = 1
\]

therein. If we assume that the thermal and mechanical fields are not coupled with each other \((e_\gamma = 0 = m)\), the results for elastic cylindrical panel with voids can be obtained from the present analysis. The results for elastic panel without voids can further be obtained by setting \( b = 0 \) in the resulting relations.

6.6 NUMERICAL RESULTS AND DISCUSSION

In order to illustrate and verify the analytical results obtained in the previous sections we present some numerical simulation results. For the purpose of numerical computations we have considered magnesium crystal-like material whose physical data is given in Table 2.1.

The value of thermal relaxation time \( t_0 \) is computed on the basis of equation (2.5) of Chandrasekhar (1986b). For closed cylindrical shell, the central angle \( \eta = 2\pi \) and the integer ‘n’ must be even, since shell vibrates in the circumferential full wave. Therefore the frequency equation for closed cylindrical shell can be written by setting

\[
\nu = \frac{n\pi}{\eta}, \ (\nu = 1, 2, 3,...), \text{ where } \nu \text{ is circumferential wave number. The computations}
\]
have been done for thermoelastic panel with voids (TEV) and thermoelastic panel without voids (TE) in order to make comparative study.

For two different values \( \nu = 1, 2 \) of circumferential wave number the variations of lowest frequency \( Q = \omega R_m \) with respect to the parameter \( t_L = p \pi R_m / L \) of a simply supported cylindrical shell of magnesium crystal-like material are shown in Figs.6.2 and 6.3 respectively for different values \( \tau^* = 0.01, 0.1, 0.25, 0.5 \) of thickness to mean radius of the shell. The effect of porosity (voids) is noticed to be quite prominent with increasing values of \( t_L \) and \( \tau^* \) as can be seen from lowest frequency profile in Fig. 6.2. From Fig. 6.3, it is observed that the lowest frequency is affected due to the presence of voids in the range \( 0.5 \leq t_L \leq 1.5 \) for all values of \( \tau^* \). The lowest frequency increases monotonically with increasing values of \( t_L \) for \( \nu = 1 \), and \( \nu = 2 \), which becomes stable and smooth at higher values of \( t_L \) and small values of \( \tau^* \). The lowest frequency is also affected by thickness to mean radius ratio \( (\tau^*) \) of panel for \( t_L \geq 0.5 \), \( \nu = 1 \) and for \( t_L \leq 1.5 \), \( \nu = 2 \) both in the presence and in absence of voids, as can be observed in Figs 6.2 and 6.3. The effect of voids is definite, though small, on the lowest frequency.

The effect of length to mean radius ratio \( (L/R_m) \) of the panel on the lowest frequency \( (Q) \) has been shown in Fig. 6.4 for the values of central angle \( \eta = \pi / 4 \), \( \tau^* = 0.2 \) and \( p = n = 1 \). It is observed that the lowest frequency \( (Q) \) decreases monotonically with increasing values of \( L/R_m \) and ultimately it becomes stable and smooth for \( (L/R_m) \geq 1.5 \). The presence of the voids exhibits its effect, though small, on the magnitude of lowest frequency at all values of \( L/R_m \) as can be seen from the profiles in Fig. 6.4.
The effect of thickness to mean radius ratio ($r^*$) of the panel on the lowest frequency ($\Omega$) has been plotted in Fig. 6.5 for the values of central angle $\eta=\pi/4$, $(L/R_m)=1$ and $p=n=1$. It is observed that the lowest frequency ($\Omega$) increases monotonically with increasing values of $r^*$. It is observed that the presence of voids results in decrease, though small, in the magnitude of lowest frequency ($\Omega$) at all values of $r^*$. Fig. 6.6 reveals that the lowest frequency ($\Omega$) decreases with increasing values of the central angle ($\eta$) when $r^* = 0.2$, $(L/R_m) = 1$ and $p=n=1$. The decrease is rapid up to $\eta = 45^0$ and steady for $\eta > 45^0$. It is also noticed that the presence of voids results in decrease in the magnitude of lowest frequency ($\Omega$) at all values of the central angle ($\eta$). Thus from Figures 6.4 to 6.6 it is observed that the voids have similar effects on the lowest frequency with length to mean radius ratio, central angle and thickness to mean radius ratio of the panel.

6.7 CONCLUSIONS

In this analysis the Bessel functions with complex arguments have been directly used to study the three dimensional vibrations of homogeneous, isotropic thermally conducting cylindrical panel with voids. Three displacement potential functions are introduced so that the equations of motion, volume fraction field and heat conduction are uncoupled and simplified. It is noticed that a purely transverse mode is independent of rest of the motion, thermal change and volume fraction field. It is also noticed that the presence of voids results in definite decrease, though small, in magnitude of lowest frequency ($\Omega$) at all values of length to mean radius ratio $(L/R_m)$, $t_L = p \pi R_m / L$. 

160
central angle ($\eta$) and the ratio ($t^* = (R_2 - R_1)/R_m$) of thickness to mean radius of the shell. The analysis reported in this article is most general and exhaustive till date because the result of previous investigations can be obtained as particular cases from this analysis. Moreover, the work reported here is applicable to circular cylindrical ‘panels’ of arbitrary thickness, from thin shell to extremely thick ones. The solutions obtained are also applicable to both closed hollow cylinders and open ones (panels), depending upon whether $v = n\pi/\eta$ is an integer or not. This may be used in applications involving aerospace, offshore, submarine structures and even may be helpful to check and verify the results obtained by FEM and BEM for such problems.
A STUDY OF WAVE PROPAGATION IN GENERALIZED THERMOELASTIC MATERIALS WITH VOIDS

Fig. 6.2: Variation of lowest frequency (Ω) with respect to parameter (t_r) for ν = 1 and different values of thickness to mean radius ratio of panel (t*)

Fig. 6.3: Variation of lowest frequency (Ω) with respect to parameter (t_r) for ν = 2 and different values of thickness to mean radius ratio of panel (t*)
Fig. 6.4: Variation of lowest frequency (Ω) with respect to length to mean radius ratio for $\eta = \frac{\pi}{4}$, $p = n = 1$ and $t^* = 0.2$.

Figure 6.5: Variations of lowest frequency (Ω) with respect to thickness to mean radius ratio ($t^*$) for $\eta = \frac{\pi}{4}$, $\frac{L}{R_m} = 1$ and $p = n = 1$. 

163
**Figure 6.6:** Variations of lowest frequency ($\Omega$) with respect to central angle ($\eta$) for

\[ t^* = 0.2, \quad p = n = 1 \quad \text{and} \quad \frac{L}{R_m} = 1. \]
SUGGESTIONS

FOR

FUTURE WORK

The following suggestions are being made for further theoretical, computer simulations and possible extension of this work.

There is a scope to investigate the effect of anisotropy and non linearity on the thermoelastic materials with voids. These studies can be extended to the continua with voids under viscous/ inviscid fluid loadings. The effect of size and distribution of voids on the considered material models can also be investigated. The reflection and transmission phenomenon from the boundaries of thermoelastic materials with voids may be explored. There is a scope to investigate the forced vibrations and waves in the considered materials under mechanical loads and thermal sources. Some useful computer aided designs can be proposed in reference to these problems which can be used in the construction and design of ultrasonic equipments, sensors and the manufacture of building materials such as concrete, bricks and plasterboard in addition to airport runways, highways, railway tracks and similar other composite structures.