INTRODUCTION

Let \( R \) be a ring with identity and let \( \text{Aut}(R) \) denote the group of all automorphisms of \( R \). Consider a group \( G \subseteq \text{Aut}(R) \). For any \( g \in G \), the action of \( g \) on \( R \) is denoted by \( x \mapsto x^g \) for \( x \in R \). For a subset \( S \subseteq R \), we have \( S^g = \{ s^g | s \in S \} \), \( g \in G \); \( S \) is said to be \( G \)-invariant if \( S^g = S \) for all \( g \in G \). The \( G \)-fixed subring of \( R \) is given by \( R^G = \{ x \in R | x^g = x, \text{ for all } g \in G \} \); in other words \( R^G \) is the set all of \( G \)-fixed elements of \( R \). Clearly \( R^G \) is non-empty as \( 0, 1 \in R^G \). The skew group ring \( R \ast G \) is a free \( R \) - module with basis \( \{ g | g \in G \} \), that is

\[
R \ast G = \left\{ \sum_{g \in G} r_g g \mid r_g \in R, \text{ and } r_g \neq 0 \text{ for only finitely many } g \in G \right\}
\]

\( R \ast G \) becomes a ring under component wise addition and multiplication

\[
(rg)(sh) = rs^{g' h},
\]

for \( r, s \in R \) and \( g, h \in G \).

The study of groups acting on rings and that of skew group rings was started initially to develop Galois theory for rings. The most interesting application of skew group ring methods to Galois theory is the correspondence obtained between the prime ideals of \( R \) and of \( R^G \) and between the \( G \)-prime ideals of \( R \) and prime ideals of \( R \ast G \). The relationships of the structure of a ring \( R \) to the structure of the fixed subring \( R^G \) and to the skew group ring \( R \ast G \) have been investigated by V.K. Karchenko, M. Cohen, S. Montgomery,
L.W. Small, M. Lorenz and D.S. Passman. The researchers in this direction have settled many questions of the following nature:

If a ring satisfies certain property "P" then does the fixed subring satisfy "P" and conversely? This approach followed by an immense amount of research has ultimately culminated in the development of a new Galois theory for semiprime rings by V.K. Karchenko. The skew group ring $R \ast G$ is an useful tool to study the relationships between $R$ and the $G$-fixed subring $R^G$. The main questions, Lorenz and Passman settled for a ring $R$ with finite group action on it in 1979 are the following:

1. Incomparability. If $P_1 \subseteq P_2$ are prime ideals of $R \ast G$, does it follow that $P_1 \cap R \subseteq P_2 \cap R$?

2. Going Down. Given $G$-prime ideals $A_1 \subseteq A_2$ of $R$ and a prime $P_2$ of $R \ast G$ with $P_2 \cap R = A_2$. Does there exists a prime $P_1$ of $R \ast G$ satisfying $P_1 \subseteq P_2$ and $P_1 \cap R = A_1$?

Indeed, they prove that the primeness of larger ideal $P_2$ in incomparability is unnecessarily. The following results were used to settle the above problem:

3. Let $Q$ be a $G$-prime ideal of $R$. Then there exists a prime ideal $P$ of $R$ such that $P^G = Q$. Further, $P$ is unique upto its $G$-orbit $\{P^g | g \in G\}$.

4. Let $P \subseteq I$ be ideals of $R \ast G$ with $G$ finite. If $P$ is prime and $P \neq I$, then $P \cap R \neq I \cap R$.

5. Let $R \ast G$ be given with $G$ finite and with $R$ a $G$-prime ring.

(i) An ideal $P$ of $R \ast G$ is minimal if and only if $P \cap R = 0$
(ii) There are finitely many such minimal primes, say \( P_1, P_2, \ldots, P_n \) and

\[ n \leq |G| . \]

(iii) \( J = P_1 \cap P_2 \cap \ldots \cap P_n \) is the unique largest nilpotent ideal of \( R \ast G \) and

\[ J^{[G]} = 0. \]

Motivated by the foregoing work of Lorenz and Passman [8], Sharma and Samriti [24], Sharma, Gupta and Arvind [19] and Arvind [1] extended group action to the fuzzy ideals of a ring with finite group action on it and established the above mentioned results in context of fuzzy ideals of \( R \) and \( R \ast G \). In order to mention some more results concerning fuzzy ideals of such rings relevant to the present thesis, we digress slightly for a brief discussion on fuzzy sets and fuzzy ideals in a ring.

The notion of a fuzzy set was formulated by Zadeh [30]. The perspicacity involved in the introduction of the concept of a fuzzy set by Zadeh [30] was so strong that the initial opposition to it, on account of its causing ripples in the otherwise smooth and placid waters of probability theory based on Aristotelian two valued logic, soon died down and the theory of fuzzy sets has become a vigorous area of research with manifold applications. For a nice and an excellent exposition to the major developments of the theory as well as to some of the most successful applications, one may be referred to Klir and Yuan [4]. Fuzzy extensions of some mathematical theories include, for example, fuzzy topological spaces [Chang 1968; Wang 1975; Lowen 1976], fuzzy algebraic concepts such as fuzzy subgroups [17], fuzzy ideals [6], prime fuzzy ideals [14], G-prime fuzzy ideals [18, 24] and completely G-prime fuzzy ideals [19].

Since the formulation of fuzzy set developed by Zadeh [30], this theory has evoked great interest among the researchers working in different branches of
Mathematics and Computer Sciences. Rosenfeld [17] applied this concept to the theory of groupoids and groups. Liu [6] introduced the notion of a fuzzy ideal of a ring and studied their properties. Mukherjee and Sen [15] and Malik and Mordeson [9] have carried a study of prime fuzzy ideals of a ring. The latter authors, namely, Malik and Mordeson have also carried a study of primary fuzzy ideals in a commutative ring with identity [10]. We present some definitions and results relevant to this thesis, from [6],[9],[15],[17],[18],[19],[24] and [30] in Chapter I.

The group theory and ring theory play significant role in several areas of mathematical sciences such as practical physics, coding theory and engineering etc. Therefore the interaction of group theory and ring theory with the theory of fuzzy sets is a very desirable feature in the domain of fuzzy mathematics. Rosenfeld introduced the concept of fuzzy subgroup of a group in 1971. He showed how some basic notions of group theory could be extended to develop the theory of fuzzy groups. Since then a lot of work has been done in fuzzy group theory and fuzzy ring theory.

Let $R$ be ring with identity and $G$ be a finite group acting on $R$. We, now, mention some results concerning the $G$-prime fuzzy ideals of $R$ and prime fuzzy ideals of $R \star G$ from [18] and [24].

6. Let $\mu$ be a prime fuzzy ideal of $R$. Then $\mu^G$ is $G$-prime fuzzy ideal of $R$.

Conversely, if $\lambda$ is a $G$ - prime fuzzy ideal of $R$, then there exists a prime fuzzy ideal $\mu$ of $R$ such that $\mu^G = \lambda$, $\mu$ is unique upto its $G$- orbit.

7. Let $P$ be a fuzzy ideal of $R$. Then $P$ is a $G$ - prime fuzzy ideal of $R$ if and only if there exists a non-empty $G$ - prime ideal $\pi$ of $R$ such that $P_r \in \{\pi, R\}$ for all $r \in [0, 1]$. 
The fuzzy analogues of the Incomparability and Going Down problems are proved in [1] for $R$ and $R \ast G$ as follows:

8. Let $\chi_\sigma$ (the characteristic function of $\{0\}$) be prime in $R \ast G$ and $\sigma \neq \chi_\sigma$ be another prime fuzzy ideal in $R \ast G$. Then $\sigma \cap \chi_R \neq \chi_\sigma$.

9. Let $\mu$ and $\sigma$ be prime fuzzy ideals of $R \ast G$, $\mu \nsubseteq \sigma$. Then $\sigma \cap \chi_R \nsubseteq \mu \cap \chi_R$.

10. Let $R$ be a $G$-prime ring. Let $\chi_\sigma \neq \sigma$ be a $G$-prime fuzzy ideal of $R$ and $\lambda$ be a prime fuzzy ideal of $R \ast G$ such that $\lambda \cap \chi_R = \sigma$. Then there exists a prime fuzzy ideal $\mu_i$ of $R \ast G$ satisfying $\mu_i \nsubseteq \lambda$ and $\mu_i \cap \chi_R = \chi_\sigma$.

The analogy between rings graded by a finite group $G$ and rings on which $G$ acts as automorphisms led mathematicians like M. Cohen and S. Montgomery [2] to extend the results of Lorenz and Passman [8] to the rings graded by finite group $G$. Let $R$ be a $K$-algebra with $I$, over a commutative ring $K$ with $I$ and $G$ be finite group whose identity is also denoted by $1$. If $R$ is graded by $G$, then $R$ is a $K[G]^*$-module algebra. Conversely, if $R$ is a $K[G]^*$-module algebra, then $R$ is graded by $G$. Thus when $R$ is graded by $G$, we can construct an algebra (smash product) $R \# K[G]^*$. This algebra plays the same role for graded rings that the skew group algebra plays for group actions. For $a, b \in R$ and basis elements $p_g \in K[G]^*$, the product in $R \# K[G]^*$ is given by

$$(a \# p_g) \cdot (b \# p_h) = \sum_{g \in G} a(p_{g^{-1}}b) \# (p_i p_h) = a \cdot b \cdot p_{g\cdot h^{-1}} \# p_h.$$  

For $(a \# 1) \cdot (1 \# p_h) = (a \# \sum_{g} p_g) \cdot (1 \# p_h) = a \cdot (\sum_{g} p_{g^{-1}}) \# p_h = a \cdot p_h$.

That is, $R$ may be identified with $R \# 1$ and $K[G]^*$ with $1 \# K[G]^*$ in $R \# K[G]^*$. For more details about $R \# K[G]^*$ one may be referred to [2].
Cohen and Montgomery have extended the results of group action to the graded rings as follows:

11. Consider \( R \# K[G]^* \), where \( R \) is graded by \( G \).
   
   (i) If \( P \) is a prime ideal of \( R \), then there exists a prime \( \mathcal{P} \) of \( R \# K[G]^* \) such that
   
   \[ \mathcal{P} \cap R = P_G . \mathcal{P} \text{ is unique up to its } G \text{-orbit \{ \mathcal{P}^g \}, and } P_G \# K[G]^* = \bigcap_g \mathcal{P}^g, \]
   
   \( G \)-prime ideal of \( R \# K[G]^* \), \( (P_G \) is the largest graded ideal of \( R \) contained in \( P \)).
   
   (ii) If \( \mathcal{P} \) is any prime ideal of \( R \# K[G]^* \), then \( \mathcal{P} \cap R = P_G \), for some prime \( P \) of \( R \).

12. If \( P \) is a prime ideal of the prime ring \( R \) such that \( P \neq 0 \), then \( P \cap R_I \neq 0 \), \( R_I \) is the identity component of \( R \).

13. If \( A_1 \) and \( A_2 \) are prime ideals of \( R \) with \( A_1 \subseteq A_2 \) and \( P_2 \) be a prime ideal of \( R \# K[G]^* \) such that \( (P_2) \cap R_I = A_2 \), then there exists a prime ideal \( P_I \) of \( R \# K[G]^* \) satisfying
   
   \[ P_I \subseteq P_2 \text{ and } (P_I) \cap R_I = A_1. \]

We started studying the fuzzy ideals of \( R \) and \( R \# K[G]^* \) with the aim of extending the Orbit problem, Incomparability and Going Down problem for the fuzzy ideals of \( R \) and \( R \# K[G]^* \).

Let \( R \) be a \( K \)-algebra with \( I \) over a commutative ring \( K \) with \( I \) and \( G \) be a finite group whose identity is also denoted by \( I \). The group \( G \) acts on \( R \# K[G]^* \). An action of \( G \) on \( R \# K[G]^* \) is given by \( (r p_h)^g = r p_{hg}, \) for \( r \in R, \) \( p_h \in K[G]^*, \) \( g \in G \).

Thus, the known results for fuzzy ideals of rings with finite group actions on them mention earlier can be applied to the fuzzy ideals of \( R \# K[G]^* \). We use these results to study the fuzzy ideals of \( R \). We are able to prove the fuzzy analogues of the above mentioned problems in Chapter II as follows:
14. Orbit Problem. If $\lambda$ is a prime fuzzy ideal of $R$, then there exists a prime fuzzy ideal $\mu$ of $R \# K[G]^*$ such that $\mu \cap \chi_R = \lambda_G$, where $\chi_R$ is the characteristic function of $R$. Further, $\mu$ is unique up to its $G$-orbit, $\{\mu^g \mid g \in G\}$. Conversely, if $\mu$ is a prime fuzzy ideal of $R \# K[G]^*$, then there exists a prime fuzzy ideal $\lambda$ of $R$ such that $\mu \cap \chi_R = \lambda_G$.

15. Incomparability. If $\lambda$ is a prime fuzzy ideal of the prime ring $R$ such that $\lambda \neq \chi_0$, then $\lambda \cap \chi_{R,1} \neq \chi_0$, where $\chi_0$ is the characteristic function of $(0)$.

16. Going Down. If $\sigma_1$ and $\sigma_2$ are graded prime fuzzy ideals of $R$ with $\sigma_1 \subseteq \sigma_2$ and $\lambda_2$ be a prime fuzzy ideal of $R \# K[G]^*$ such that $(\lambda_2) \cap \chi_R = \sigma_2$, then there exists a prime fuzzy ideal $\lambda_1$ of $R \# K[G]^*$ satisfying $\lambda_1 \subseteq \lambda_2$ and $(\lambda_1) \cap \chi_R = \sigma_1$.

In Chapter III we study primary fuzzy ideals of noncommutative rings. In noncommutative rings primary ideals are defined through the associated primes under the assumption that the ring is Noetherian, for otherwise the set of associated primes may be empty (c.f. [27]). However, an alternate way to generalize primary ideals from commutative rings to noncommutative rings could be to replace the role of elements by that of ideals. This approach enables us, as shown in this chapter, to generalize some basic results for primary ideals in commutative rings to the noncommutative setting without resorting to the Noetherian restriction. Having derived these results we achieve their fuzzyfication by proving a theorem that characterizes primary fuzzy ideals in terms of primary ideals of noncommutative rings. It is pertinent to note here that while our presentation is geared towards the noncommutative case, the results we obtain are, of course, also valid for the commutative case.
The Chapter IV is in continuation of Chapter III and uses the results of Chapter II to carry over the study of primary ideals and fuzzy ideals to a ring with finite group action and to a group graded ring. In this Chapter, we study the effect of group action on primary ideals and primary fuzzy ideals of a ring $R$ with finite group action on it and show that the results derived in Chapter III concerning these ideals are of wider generality. We introduce here the notion of $G$-primary/ $G$-maximal ideals and fuzzy ideals and $G$-semiprime ideals in $R$, derive some basic results for such ideals and then prove their fuzzy analogues.

After studying the $G$-primary ideals ($G$-primary fuzzy ideals) of a ring with group action, we use these results together with the results of Chapter III to study the graded primary ideals (fuzzy ideals) of a group graded ring $R$ and its smash product $R \# K[G]^*$ with the help of the maps

$$(\cdot)_u : R \rightarrow R \# K[G]^*$$

and

$$(\cdot)^* : R \# K[G]^* \rightarrow R$$

defined in Chapter II.

The motivation for Chapter V is [19] wherein Sharma et. al. introduced the concept of a completely $G$-prime fuzzy ideal of a ring $R$ and characterized a completely prime / completely $G$-prime fuzzy ideals in terms of completely prime / completely $G$-prime ideals of $R$. This characterization provided a necessary and sufficient condition for a fuzzy ideal to be a completely prime / completely $G$-prime fuzzy ideal of $R$. Further, it is shown that there exists a subclass of completely $G$-prime fuzzy ideals whose members can be exactly characterized in terms of completely prime fuzzy ideals of $R$. 

8
The analogy between rings graded by a finite group $G$ and rings on which $G$ acts as automorphism once again motivates us to investigate the relationships established for completely $G$-prime fuzzy ideals and completely prime fuzzy ideals in the present context.

It is pertinent to note here that the concept of $G$-prime ideals (fuzzy ideals) and completely $G$-prime ideals (fuzzy ideals) in a ring are independent of one another in the sense that neither of these implies or is implied by the other. However, in this thesis we observe that a completely graded prime ideal (fuzzy ideal) is a graded prime ideal (fuzzy ideal) and the two concepts coalesce in a commutative ring. Moreover, for a completely $G$-prime fuzzy ideal $\mu$ of $R$, it is not necessary that $\mu_r \in \pi, R$ for all $r \in [0,1]$, where $\pi$ is a completely $G$-prime ideal of $R$. However, this result is true in the context of completely graded prime fuzzy ideals.