Chapter 4

Oscillating D-Strings from IIB Matrix Theory

One of the most challenging problems in string theory has been to understand its strong coupling aspects\cite{i, 2}, including its moduli space structure in full quantum theory at the non-perturbative level. One also hopes that such an investigation will lead to an understanding of supersymmetry breaking in these theories and will give the correct string theory vacuum describing the real world. As is well known, the unravelling of non-perturbative aspects includes an analysis of its soliton spectrum\cite{3}, and their moduli dependence. Investigations along these lines have also led to a better understanding of the confinement mechanism in supersymmetric gauge theories from string theory point of view\cite{4}.

A useful mechanism in studying the strong coupling aspects has been the D-brane constructions of string solitons\cite{5}. As a result, one can obtain the soliton spectrum and their interactions using open string Conformal Field Theory. Since the D-branes preserve a certain amount of supersymmetry, these are stable solitonic superstring vacua, around which a quantum field theory of the world-volume degrees of freedom can be formulated\cite{6}. Many such D-brane configurations have been obtained\cite{7} and the corresponding effective world-volume action have been analyzed. Among these, of particular interest have been some of the six\cite{8} and two\cite{9} dimensional supersymmetric gauge theories.

In the Matrix Theory\cite{10}\cite{6, 11, 12} proposal by Banks, Fischler, Shenker and Susskind (BFSS), the $SU(N)$ ($N \to \infty$) world-volume gauge theory of $N$ D-branes have themselves been conjectured to be a fundamental theory describing both the perturbative and non-perturbative aspects of string theory. In simplest cases, these are the dimensional reduction
of the $D = 10, N = 1$ Yang-Mills Theory to the relevant world-volume dimension. In this context, it has been shown that various brane solutions of string theory\[11, 6\], including their charges, can be obtained from classical solutions in such gauge theories.

We will concentrate on the IIB Matrix theory\[13\] which proposes that the 10-dimensional type IIB string theory is described by the dimensional reduction of the $D = 10, N = 1$ $SU(N)$ gauge theory to zero dimension. This possesses a manifest Lorentz invariance. The emergence of a D-string from such a Matrix theory has also been shown through an analysis of their interactions. A duality among Matrix theories proposed earlier for describing $M$-Theory and the one for the type IIB theory has also been argued.

In this chapter, we generalize some of the results in \[13\] and write down an infinite set of classical solutions of the IIB Matrix theory\[14\] by solving the field equations. These are classical gauge field configurations which correspond to D-strings with chiral (left-moving) oscillations. The existence of these solutions \[15\] follow from oscillating fundamental string solutions\[16\] in type IIB string theory and its $SL(2, Z)$ S-duality in 10-dimensions\[3\]. As in the case of fundamental strings, we show that the matrix theory solutions preserve 1/4 supersymmetry.

The BPS mass formula for type IIB string theory, when compactified to 9-dimensions, have been written down earlier. The mass formula is parameterized by integers $(m, n)$, namely internal momenta and winding in the compactified direction, as well as by the gauge charges $(p, q)$ corresponding to the NS-NS and R-R antisymmetric tensor fields in 10-dimensions. It is also known that this BPS formula is invariant under the $SL(2, Z)$ U-duality in 9-dimensions, which follows from the $S$-duality of the 10-dimensional type IIB strings. In this chapter, we mainly concentrate on the BPS formula for $(p = 0, q = 1)$ case which corresponds to a single D-string. An explicit form for the BPS formula for this case can be derived by using the $SL(2, Z)$ duality on the mass formula of the fundamental IIB string in 9-dimensions and by restricting to the supersymmetric ground states. The mass formula for the fundamental string has a form\[3\]:

\[
M^2 = \left( \frac{m}{R_B} \right)^2 + (2\pi R_B n T_q)^2 + 4\pi T_q (N_L + N_R),
\]

with

\[
N_R - N_L = mn,
\]
and $T_q$ is the string tension of the fundamental string. A general formula for the $(p, q)$ string involves a generalization in the definition of $T_q$, written in terms of the ten-dimensional axion-dilaton moduli as:

$$T_q^2 = [p^2 + e^{2\phi_0}(p\chi_0 + q)^2]e^{-\phi_0}T^2,$$

(3)

For a $(0,1)$ string we then have $T_q = e^{\frac{i}{2}\phi_0}T$.

The BPS states which preserve 1/2 supersymmetry and their interactions have already been analyzed in the Matrix Theory context[13]. These correspond to the supersymmetric ground states $N_L = N_R = 0$. We will examine the BPS configurations of the Matrix Theory preserving 1/4 supersymmetry. These are the supersymmetric ground states with either $N_L = 0$ or $N_R = 0$ and provide a rich spectra parameterized by integers $(m, n)$. The BPS mass then satisfies the relation:

$$M_{BPS} = (2\pi R_B nT_q + m/R_B).$$

(4)

The mass formula (4) is an exact expression which does not receive quantum corrections. In Matrix Theory we verify this by showing that the one-loop quantum effective action for our solution vanishes.

As an application of the IIB Matrix Theory, we then obtain the world-volume gauge theory in the classical background of an oscillating D-string solution. It is known that the world-volume theory for a static D-string configuration of IIB matrix theory is a two dimensional gauge theory with $(8,8)$ supersymmetry[6, 17]. In this chapter, we obtain an explicit expression for the supersymmetric world-volume gauge theory action with $(8,0)$ supersymmetry from the Matrix Theory action. We show its Lorentz, gauge and supersymmetry invariance. The gauge and supersymmetry invariance are the residual symmetries of the original type IIB theory. The supersymmetry is a global symmetry in this case, as it originates from the global supersymmetry of the Green-Schwarz superstring action, in the Schild gauge, or from the supersymmetry of the $N = 1$ Yang-Mills theory in 10-dimensions. The gauge invariance of the $(8,0)$ world-volume action also follows from that of the gauge invariance of the 10-dimensional superYang-Mills theory. Although the final model does not possess an explicit left-right symmetry, we will argue in section-3, from Matrix Theory point of view, that the particle spectrum is anomaly free. We also argue that the worldsheet actions for the
static and oscillating strings define equivalent quantum field theories. This is demonstrated through a mapping of operators in the two cases. Physically this also implies that the static string is a quantum state of the world-volume theory in the classical background we have studied.

This work has been partly motivated by an analysis of BPS states in compactified M-Theory using BFSS model[18]. We have carried out this analysis in the framework of $S^1$ compactified IIB Matrix Theory[13]. The rest of the chapter is organized as follows. In section-1, we review the oscillating fundamental string solutions from supergravity point of view and mention how the corresponding D-strings can be obtained using the S-duality of the ten-dimensional type IIB string theory. In section-2, we obtain these solutions from IIB Matrix theory. We also show that the Matrix Theory solution preserves 1/4 supersymmetry. In this section, we also point out that the one-loop quantum effective action of the Matrix Theory, for this solution, vanishes. In section-3, we present the $(8,0)$ supersymmetric gauge theory of oscillating strings and show its connection with static strings. Conclusions and discussions are presented in section-4.

4.1 Oscillating String Solution

We now start with a review of oscillating string solution in string theory[16]. They were obtained as a generalization of the static fundamental strings found earlier[19] and are the solutions of the supergravity equations of motion. The singularity of the field configuration represents the position of the string. However unlike the static case, they correspond to the states preserving only 1/4 supersymmetry.

It is also known that the static fundamental string solutions can be identified with charged extremal black holes in one lower dimension. Similarly, the oscillating string solutions, after compactification along its length, can asymptotically be identified with the supersymmetric, stationary, rotating, charged black holes. In the context of our previous discussion, the static string is a supersymmetric ground state and the oscillator numbers are fixed to their minimum values $N_L = N_R = 0$. On the other hand, in the oscillating string configuration only $N_R = 0$ and $N_L$ is an arbitrary oscillator number. From the space-time point of view, the oscillating string solutions require the presence of a (large) compactified direction on which
the string is wrapped, as otherwise the only BPS configurations are those preserving 1/2 supersymmetry in ten noncompact dimensions. We take $x^1$ as the compactified coordinate of radius $R$.

The supergravity solution corresponding to the oscillating fundamental string is given as:

$$ds^2 = -e^{2\phi} du dv + [e^{2\phi} p(v) r^{-D+4} - (e^{2\phi} - 1) \tilde{F}(v)^2] dv^2 + 2(e^{2\phi} - 1) \tilde{F}(v). dxdv + dx.dx,$$

$$B_{uv} = \frac{1}{2} (e^{2\phi} - 1),$$

$$B_{ui} = \tilde{F}_i(u)(e^{2\phi} - 1),$$

$$e^{-2\phi} = 1 + \frac{Q}{|x-p|^D}.$$  \hspace{1cm} (5)

where, for a fundamental string solution, $B_{\mu\nu}$ is the NS-NS antisymmetric tensor field and $F_i(v)$ are functions of the light-cone coordinate $v = x^0 + x^1$ only. $u = x^0 - x^1$ is the other light-cone coordinate. Dots denote the derivative with respect to argument $v$ and bold-faced letters denote a vector in the transverse directions labeled by indices $i$'s. To match properly with a string source, one also requires $p(v) = 0$. The field configuration in eqns. (5) define an asymptotically flat space. As a result, one can properly define the ADM mass and charge for the supergravity background. It has also been pointed out that the supergravity solution as well as the ADM energy properly matches with a string source, written in terms of the worldsheet coordinates $\tau$ and $\sigma$ as:

$$V(\tau, \sigma) = 2Rn\sigma^+, \hspace{1cm} U(\tau, \sigma) = (2Rn + a)\sigma^- + f^V \tilde{F}^2, \hspace{1cm} X(\tau, \sigma) = F(V),$$ \hspace{1cm} (6)

where $\sigma^{\pm} = \tau \pm \sigma$ and $V, U$ are the space-time light-cone string coordinates: $U = X^0 - X^1$, $V = X^0 + X^1$ and $X^i$ are once again the string coordinates along the transverse directions. The constant $'a'$ is the zero mode of $\tilde{F}^2$:

$$a = \frac{1}{\pi} \int_0^{2\pi Rn} \tilde{F}^2, \hspace{1cm} (7)$$

and $F_i$'s have no zero modes. The oscillating string is specified by the left-moving wave profile $F_i(v)$. In [16] some specific wave profiles have been used to show the connection
of the oscillating string solution with the charged rotating black holes. For our purposes, however, we do not need their specific form. The worldsheet configuration (6) has been identified with a string source of momenta and winding

\[ p^\mu = (2\alpha')^{-1}(2Rn + a, -a, 0), \quad n^\mu = (0, n, 0), \]

along the directions \((X^0, X^1, X^2)\). The internal momenta \(m/R\) in the compact direction is then specified by integers

\[ m = -\frac{Ra}{2\alpha'}, \]

and the oscillator number, obtained by the level-matching condition is

\[ N_L = \frac{nRa}{2\alpha'}. \]

An oscillating D-string in the supergravity context can be obtained by applying an \(SL(2, Z)\) duality transformation on the fundamental string solution presented above. The general procedure, as well as the specific \(SL(2, Z)\) transformation matrix \((\lambda)\) is similar to the generation of a static \( (p = 0, q = 1) \) string solution from the \((1, 0)\) solution as described in [3]. We do not elaborate on them further, except to note that the fundamental string tension will be replaced appropriately by the one for a D-string.

The string source (6) will play a crucial role in obtaining a Matrix Theory solution as these, with appropriate modifications of string tension will specify the gauge field configuration, which are the solution of the Matrix Theory field equations. So far we have only discussed a single fundamental \((1, 0)\) and D-string \((0, 1)\) solutions. The existence of multiple supersymmetric parallel string configuration has also been shown in [16]. These correspond to \((p, 0)\) and \((0, q)\) type BPS states preserving once again 1/4 supersymmetry. It may also be possible to obtain higher dimensional oscillating branes[20] and to obtain their parallel and orthogonal supersymmetric configuration.

In next section we obtain the oscillating string as a solution to the field equation in Matrix Theory and examine its properties. We also show the BPS nature of \((0, q)\) or multi D-string solutions of Matrix Theory from the results of one-loop effective action.
4.2 Type IIB Matrix Theory

We now obtain an infinite set of solutions of the Type IIB Matrix theory and show that these correspond to the oscillating D-strings discussed in the last section from the supergravity point of view. The IIB Matrix Theory action is obtained by the dimensional reduction of the D=10, N=1 SU(N) super Yang-Mills to zero dimension and is written as:

$$S = \alpha \left( -\frac{1}{4} Tr[A_\mu, A_\nu]^2 - \frac{1}{2} Tr(\bar{\psi} \Gamma^\mu [A_\mu, \psi]) \right) + \beta Tr 1,$$

where the last term in the action is a "chemical potential". A similar term in the Schild-type string action is necessary to show its equivalence with Nambu-Goto action. \(\alpha\) and \(\beta\) are constants with \(\sqrt{\alpha \beta}\) defining the D-string tension. Eqn. (11) without the chemical potential term is also referred as the D-instanton Matrix action. The constants \(\alpha\) and \(\beta\) can be determined by comparing the string interaction in Matrix Theory with those from open strings. The final results are:

$$\alpha = \frac{g_s^2}{\sqrt{3}\gamma^2} \frac{1}{g_s}, \quad \beta = \frac{2g_s^2}{\sqrt{3}\gamma^2} \frac{1}{g_s},$$

with \(\gamma\) being a numerical constant.

In [13], the target space metric, represented by the the indices \(\mu\), has been chosen as Euclidean, whereas the oscillating string solutions of [16] presented in the last section are in the Minkowski metric. We take care of this discrepancy by putting appropriate factors of 'i' in the solutions of section-2 while computing the one-loop effective action. For the moment, however we continue to work with the Minkowski metric.

The field equations of the Matrix theory are:

$$[A^\mu, [A_\mu, A_\nu]] = 0,$$

$$[A_\mu, (\Gamma^\mu \psi)_{\alpha}] = 0.$$  \hspace{1cm} (12)

As fermions do not have a classical background, only the first equation of (12) is considered for analyzing the classical solutions.

The action (11) is invariant under supersymmetry transformations

$$\delta^{(1)} \psi = \frac{i}{2} [A_\mu, A_\nu] \Gamma^{\mu \nu} \epsilon, \quad \delta^{(1)} A_\mu = i \epsilon \Gamma_\mu \psi,$$

and

$$\delta^{(2)} \psi = \xi, \quad \delta^{(2)} A_\mu = 0.$$  \hspace{1cm} (13)

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These are also referred as the “dyanamical” and “kinematic” supersymmetry transformations[6] and follow from the dimensional reduction of the worldsheet Green-Schwarz superstring action in Schild gauge to zero dimension. In addition, action (11) is invariant under a gauge transformation:

$$\delta_{\text{gauge}} A_\mu = i[A_\mu, \alpha], \quad \delta_{\text{gauge}} \psi = i[\psi, \alpha].$$  \hfill (15)

The field equations (12) are now solved by infinite dimensional hermitian matrices $A_\mu$'s. In turn, using the familiarity with the quantum mechanics, these matrices are represented by the canonical conjugate variables, $q_i$'s and $p_i$'s. The relationship of these solutions with those in string theory are established through an identification of the commutators with the Poisson bracket for the Schild action[13]:

$$\{X, Y\} = \frac{1}{\sqrt{g}} e^{ab} \partial_a X \partial_b Y, \quad \hfill (16)$$

where $a, b$ denote the worldsheet coordinates $\tau, \sigma$. Moreover one also identifies

$$-i[, ] \rightarrow \{, \}, \quad T\tau \rightarrow \int d^2\sigma \sqrt{g}, \quad \tau \rightarrow \frac{\rho}{\sqrt{2\pi N}}, \quad \sigma \rightarrow \frac{\rho}{\sqrt{2\pi N}}. \quad \hfill (17)$$

with the commutator $[q, p] = 2\pi i$. The static D-string:

$$X^0 = T\tau, \quad X^1 = \frac{L}{2\pi} \sigma, \quad X^i = 0, \quad \hfill (18)$$

can then be represented by the gauge field configuration:

$$A^0 = \frac{T}{\sqrt{2\pi N}} q, \quad A^1 = \frac{L}{\sqrt{2\pi N}} p, \quad A^i = 0. \quad \hfill (19)$$

and satisfies the fields equations (12). Similarly the oscillating string can be represented by a gauge field configuration which is obtained through the identifications in (17). Continuing to work in light-cone coordinates, the components $A^{a'}$'s are given as:

$$A^V = 2Rn\hat{\sigma}^+, \quad A^U = (2Rn + a)\hat{\sigma}^- + j\hat{V}, \quad A^i = F^i(\hat{V}), \quad \hfill (20)$$

where $\hat{\sigma}^\pm = q \pm p / \sqrt{2\pi N}$ and $\hat{V}$ denotes an operator replacement in the function $V$: $\hat{V}(\tau, \sigma) \rightarrow V(q / \sqrt{2\pi N}, p / \sqrt{2\pi N})$. Once again, the gauge field configuration for a static string (18) corresponds to $F^i = 0$ and $T = L / 2\pi = 2Rn$. 

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Now, to verify that $A^s$ in eqns. (20) are solutions of (12), we evaluate their commutators. The nonzero ones are:

\[
\begin{align*}
[A^V, A^U] &= -\frac{2}{R} (2Rn)(2Rn + a), \\
[A^U, A^i] &= \frac{2}{R} (2Rn)(2Rn + a)\hat{F}^i.
\end{align*}
\] (21)

These imply that the field equations are once again satisfied. We have therefore found a class of solutions of the Matrix Theory field equations specified by the wave-profile $F(V)$.

We now examine the BPS and supersymmetry property of the solution (20). In the background of static string configuration, the dynamical supersymmetry transformation is given as:

\[
\delta^{(1)} \psi = -\frac{TL}{2\pi N} \Gamma^{01} \epsilon, \quad \delta^{(1)} A_\mu = 0.
\] (22)

As a result, only way to preserve some amount of supersymmetry is to cancel the dynamical supersymmetry transformation with the kinematic one by defining $\xi = \pm \frac{TL}{2\pi N} \Gamma^{01} \epsilon$. We then have $(\delta^1 \pm \delta^2)\psi = 0$ and $(\delta^1 \pm \delta^2)A_\mu = 0$, which implies that the solution preserves 1/2 supersymmetry.

Now, for the oscillating string background, the dynamical supersymmetry transformation can be written as:

\[
\delta^{(1)} \psi = \frac{1}{2N} (2Rn)(2Rn + a) \left[ \Gamma^{UV} \epsilon + \hat{F}^i \Gamma^{Vi} \epsilon \right],
\]

\[
\delta^{(1)} A_\mu = 0,
\] (23)

Since the transformation $\delta^{(2)}$ is still given by eqn.(14), hence to make sure that a certain amount of supersymmetry, namely $\delta^{(1)} \pm \delta^{(2)}$, is preserved, one also has to impose the condition,

\[
\hat{F}_i \Gamma^{Vi} \epsilon = 0.
\] (24)

Before solving this equation explicitly, we notice that eqn.(24) is a chirality condition on $\epsilon$ in the light-cone directions, namely $(1 + \Gamma^0 \Gamma^1)\epsilon = 0$. Since the string worldsheet is identified with light-cone coordinates, eqn.(24) implies a chirality condition in the world-volume directions. More explicitly, by choosing ten-dimensional Gamma matrices in the Majorana representation as:

\[
\Gamma^0 = i \begin{pmatrix} 0 & -I_8 \\ I_8 & 0 \end{pmatrix}, \quad \Gamma^1 = -i \begin{pmatrix} 0 & I_8 \\ I_8 & 0 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} \gamma^i & 0 \\ 0 & -\gamma^i \end{pmatrix},
\] (25)
and by decomposing the ten-dimensional spinor $\epsilon$ in terms of the eight-dimensional ones as $\epsilon = \begin{pmatrix} \epsilon_L \\ \epsilon_R \end{pmatrix}$, the the condition (24) implies $\epsilon_R = 0$. To summarize this part of the discussion, we have shown that a cancellation between the “dynamical” and “kinematic” supersymmetry transformations can occur in the Matrix background (20) provided half the components of the dynamical supersymmetry transformations are zero. This, in turn, implies that our solution preserves only 1/4 supersymmetry, as expected of an oscillating string.

To further identify the solution of the Matrix Theory (20) with the oscillating string solution we evaluate the classical action for this configuration. We have:

$$S_B = \frac{\alpha}{2} \left( \frac{(2Rn)(2Rn + a)}{N} \right)^2 N + \beta N$$

An extremization with respect to $N$ and the identification $\sqrt{\alpha \beta} = 2\pi \rho$, with $\rho$ being the string tension now gives

$$S_B = 2\pi \rho(2Rn)(2Rn + a).$$

To verify that the action (27) is proportional to the area of the worldsheet, we have directly evaluated the Polyakov action, acting as the source for the supergravity background, for the oscillating string solution and shown that it again gives the same value as in (27). Since the solutions in [16] also satisfy the Virasoro condition, the evaluation of the Nambu-Goto action in this background also gives the same value. These results once again confirm that the Yang-Mills field configuration do indeed represent the oscillating strings, and in turn the infinite hierarchy of BPS states. In this context, we notice that the BPS mass formula (4) also follows from the time component of the target space momentum for these strings written in (8). It is also interesting to note that the Matrix solution represents a string with well defined string tension for generic oscillations $F_i(v)$. The change in the value of the action with respect to the static string is by an amount:

$$\Delta S_B = 2\pi \rho(2Rn).a = \frac{2}{\pi} N_L$$

where the last equality follows from the relation $a = -(2\pi^2 \rho)^{-1} \frac{m}{\hbar}$ for a D-string which is analogous to (9) for a fundamental string through a replacement: $\frac{1}{2\pi\alpha'} \rightarrow 2\pi \rho$. Solitons (20) can then be interpreted as an excitation over the static string state by an amount $N_L$ from this point of view as well.
We now analyze the one-loop effective action of the Matrix Theory for the classical background (20) and show that the effective action vanishes. The effective action in a general background $A_\mu = p_\mu$ has a form [13]:

$$ReW = \frac{1}{2}Tr\log(P_\lambda^2\delta_{\mu\nu} - 2iF_{\mu\nu}) - \frac{1}{4}Tr\log((P_\lambda^2 + \frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu})(\frac{1 + \Gamma_{11}}{2})) - Tr\log(P_\lambda^2).$$  (29)

where $P_\mu$ and $F_{\mu\nu}$ are operators acting on the space of matrices as:

$$P_\mu X = [p_\mu, X], \quad F_{\mu\nu} X = [f_{\mu\nu}, X],$$  (30)

with $f_{\mu\nu} = i[P_\mu, p_\nu]$, $p_\mu$ being the operator replacement for variables $A_\mu$. The terms in (29) correspond to the contributions from the bosons $A_\mu$, the fermions $\psi$ and the Fadeev-Popov ghosts respectively. It has also been noticed in [13] that the imaginary part of $W$ vanishes when $P_\lambda$ is zero along at least one of the transverse directions $i$ and implies the absence of anomaly in the world-volume action. This holds in our case, provided $F_i = 0$ for this index $i$. However it is likely that $ImW = 0$ in generic cases as well.

To evaluate the effective action in our case, we rewrite the gauge field commutators in eqn.(21) in Euclidean metric and notice that only nonzero components of $F_{\mu\nu}$, namely $F_{0i}$ and $F_{ii}$ satisfy a relation: $F_{0i} = -iF_{ii}$. The form of the matrix $P_\lambda^2\delta_{\mu\nu} - 2iF_{\mu\nu}$:

$$P_\lambda^2\delta_{\mu\nu} - 2iF_{\mu\nu} = \begin{pmatrix}
0 & 0 & -2iF_{02} \\
0 & P_\lambda^2 & -2iF_{12} \\
2iF_{02} & 2iF_{12} & P_\lambda^2 \\
0 & 0 & 0
\end{pmatrix}$$  (31)

and property of operators $P_\lambda^2$, $F_{\mu\nu}$ in our case: $[P_\lambda^2, F_{\mu\nu}] = 0$ then implies that $F_{\mu\nu}$'s cancel out in the expression of the determinant of the matrix. To show this in another way, we expand

$$Tr\log(P_\lambda^2\delta_{\mu\nu} - 2iF_{\mu\nu}) = Tr\log(P_\lambda^2) + Tr(2iF_{\mu\nu}/P_\lambda^2) + \frac{1}{2}(2i)^2Tr(F_{\mu\alpha}F_{\nu}^\alpha/(P_\lambda^2)^2) + ...$$  (32)

and use the fact that only non-vanishing components of $\delta_{\mu\nu}$, $F_{\mu\nu}$ in the $u, v$ coordinates are: $\delta^{uw} = 1$ and $F^{ui} = -2F_{ui}$. It can then be shown that all the higher-order terms vanish, as one can not form invariants out of the above non-vanishing components. A similar property of
certain classical field configurations, namely chiral-null models, have been used to show that they are an exact solution of the first quantized string theory[22]. We interestingly observe the reappearance of this property in the context of Matrix Theory.

Similarly, they all cancel in the trace of the matrix \((P^2_x + \frac{1}{2} F_{\mu \nu} \Gamma^{\mu \nu})\) as well. Various terms in the effective action (29) then cancel out as in the static case and imply that the one-loop contribution to the effective action for the oscillating case vanishes as well. This confirms the exactness of the BPS formula (4) argued on the basis of supersymmetric ground earlier. One can also examine the status of the multi-string solution. The parallel configuration of oscillating strings from Matrix Theory can be obtained as block-diagonal matrices. Then the cancellations in \(W\) occur within each block in an identical fashion and they once again vanish, showing that they are BPS configurations as well.

### 4.3 World-Volume Action

In this section we obtain the world-volume gauge theory from IIB Matrix Theory for the classical configuration corresponding to an oscillating string. We also analyse this world-volume gauge theory action in some detail and show its connection with static strings upon quantization.

It is known that the zero modes of a static D-string give rise to an \(N = 8\) \(U(1)\) vector multiplet in two dimensions. We will now see that the zero modes of an oscillating string are the \((8, 0)\) \(U(1)\) vector multiplets together with 8 scalar multiplets containing the world-sheet fermions of opposite chirality. Similarly zero modes of \(N\) coinciding D-branes[6] are now expected to give rise to an \((8, 0)\) \(SU(N)\) gauge theory. Two dimensional worldsheet action with \((8, 0)\) and \((4, 0)\) supersymmetry have been written in other contexts[21, 12] earlier and it may be interesting to show the exact connection among these actions.

The world-volume action describing the dynamics of these fields[6] can also be obtained by adding the quantum fluctuations to the classical backgrounds and then by expanding the Matrix Theory action. The one-loop effective action of the Matrix Theory (29) is in fact the quantum effective action of these gauge theories. Thus in the static case we have[17]:

\[
A_0 = -\sigma + \alpha' \tilde{A}_0(\tau, \sigma), \quad A_1 = \tau + \alpha' \tilde{A}_1(\tau, \sigma),
\]

(33)
\[ A_i = \alpha' \phi_i(\tau, \sigma), \quad \psi = \alpha' \psi(\tau, \sigma), \] 

where \( \tilde{A}_\alpha \) (\( \alpha = 0,1 \)) now are the gauge fields on the world-volume whereas the transverse components (\( \phi_i \)'s) are the scalar fluctuations. \( \psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \) are the worldsheet fermions which also transform as a spinor under an internal \( SO(8) \) symmetry. These are, as expected, the degrees of freedom for an \( N = 8 \) vector multiplet in two dimensions and are identified as the bosonic and fermionic zero modes of a static string.

The commutators of Matrix variables, including the fluctuations (in the static case) have a form[17]:

\[ [A_0, A_1] = i\alpha' (1 + \alpha' F_{01}), \quad [A_\alpha, A_i] = i\alpha' D_\alpha \phi_i, \quad [A_\alpha, \psi] = i\alpha' D_\alpha \psi, \] 

where we have used the identification (17) to replace the commutators with Poisson Bracket and \( F_{01} = \partial_0 \tilde{A}_1 - \partial_1 \tilde{A}_0 + \alpha' \{ \tilde{A}_0, \tilde{A}_1 \} \) and \( D_\alpha \phi^i = \partial_\alpha \phi^i + \alpha' \{ \tilde{A}_\alpha, \phi^i \} \). Then, for a single D-string, the action (11) reduces to a \( U(1) \) gauge theory in two dimensions with \( N = 8 \) supersymmetry. The bosonic part of the gauge theory action for the \( U(1) \) case has the form:

\[ S_B = \frac{1}{2\pi \alpha' g_s} \int d^2 \sigma \left( 1 + \alpha'^2 F_{01}^2 - \alpha'^2 D_\alpha \phi^i D_\alpha \phi^i \right). \] 

The first (constant) term in (36) is the contribution of the classical background. In Born-Infeld action, they correspond to the term involving the induced world-volume metric. The forms of \( F_{01} \) and \( D_\alpha \phi_i \) also imply the existence of higher (than two) derivative terms in the action. These have been identified with the higher order terms in the expansion of the Born-Infeld action[17]. The two derivative terms are the standard gauge theory action of the bosonic part of an \( N = 8 \) abelian gauge theory.

We now obtain the worldvolume gauge theory action for the oscillating configuration from the Matrix Theory and show that they now correspond to an (8,0) supersymmetric gauge theory in two dimensions. The fact that the solution preserves 1/4 supersymmetry has already been pointed out. However this leaves us with two possibilities for the worldvolume supersymmetry. One can either have a (4,4) or an (8,0) supersymmetric gauge theory in two dimensions. The later possibility is more natural in our case, as the oscillating string solution discussed above is left-right asymmetric. We have however already shown the breaking of \( N = 8 \) or (8,8) supersymmetric gauge theory in two dimensions to an (8,0) theory explicitly in eqn.(24).
Once again, for writing down the action in two dimensions, we expand the Matrix theory coordinates around the classical background mentioned above in (20). We now have:

\[
\begin{align*}
A^V &= 2Rn\dot{\sigma}^+ + \alpha'\tilde{A}^V, \\
A^U &= (2Rn + a)\dot{\sigma}^- + j^V F^2 + \alpha'\tilde{A}^U, \\
A^i &= F^i(\tilde{V}) + \alpha'\phi^i.
\end{align*}
\] (37)

The supersymmetry breaking from \( N = 8 \) or \((8,8)\) gauge theory to an \((8,0)\) gauge theory can now also be seen from the background configuration in eqn. (37). It is known that the R-symmetry for an \( N = 8 \) supersymmetric theory is an \( SO(8)_L \times SO(8)_R \) global symmetry group which transforms the supercharges as: \((8_v, 1) + (1, 8_v)\). Then, due to the background configuration for the scalars in eqn. (37), the left-moving part of the world volume scalars acquire vacuum expectation value. This breaks the \( SO(8)_L \times SO(8)_R \) R-symmetry to \( SO(8)_L \) and the final worldvolume theory has an \((8,0)\) supersymmetry only.

We now derive this worldvolume action and show its invariance under gauge and supersymmetry transformations. To write down the worldvolume action, we once again compute the commutators appearing in the action (11) and make the identifications (17). The nonzero ones are:

\[
\begin{align*}
[A^U, A^V] &\rightarrow 2(2Rn)(2Rn + a) + 2\alpha'(2Rn)\partial_-\tilde{A}^U + 2\alpha'(2Rn + a)\partial_-\tilde{A}^V - (2Rn)\tilde{F}^2\partial_-\tilde{A}^V + \alpha'^2\{\tilde{A}^U, \tilde{A}^V\} \equiv 2(2Rn)(2Rn + a) + \alpha'\tilde{F}^{UV}, \\
[A^U, A^i] &\rightarrow 2(2Rn)(2Rn + a)\tilde{F}^i + 2\alpha'(2Rn + a)\partial_+\phi^i - (2Rn)\tilde{F}^2\partial_-\phi^i + (2Rn)\tilde{F}^i\partial_-\tilde{A}^V + \alpha'^2\{\tilde{A}^U, \phi^i\} \equiv 2(2Rn)(2Rn + a)\tilde{F}^i + \alpha'\tilde{D}_+\phi^i, \\
[A^V, A^i] &\rightarrow \alpha'[-2(2Rn)\partial_-\phi^i + 2(2Rn)\tilde{F}^i\partial_-\tilde{A}^V] + \alpha'^2\{\tilde{A}^V, \phi^i\} \equiv \alpha'\tilde{D}_-\phi^i, \\
[A^i, A^j] &\rightarrow 2\alpha'(2Rn)(\tilde{F}^j\partial_-\phi^i - \tilde{F}^i\partial_-\phi^j) + \alpha'^2\{\phi^i, \phi^j\} \equiv \alpha'\Phi^{ij}.
\end{align*}
\]

The bosonic part of the world-volume gauge theory action is then obtained by substituting the above commutators into the bosonic part of the Matrix Theory action (11) and by the identifications in eqns. (38)-(41). For example, in variables \( A_U, A_V \) and \( A_i \), the first term in (11) has a form:

\[
S_B = -\frac{\alpha}{4} Tr \left( \frac{1}{2}[A^U, A^V]^2 + 2[A^U, A^i][A^V, A_i] + [A^i, A^j]^2 \right) .
\] (42)
By ignoring the constant and total derivative terms, the bosonic world-volume action is then written as:

$$-\frac{\alpha}{4} \int d^2 x \frac{\alpha'^3}{2\pi} \left[ -\frac{1}{2} F^{UV} - 2D_+ \phi^i D_- \phi^i + \phi^{ij} - 4(2R_n)(2R_n + a) \hat{F}^i \{A^V, \phi^i\} \right]$$  \hspace{1cm} (43)

Last term in the above action comes from an expression $4(2R_n)(2R_n + a) \hat{F}^i D_- \phi^i$, by dropping the total derivative terms. The gauge transformations, derived from eqn.(15), in two dimensional gauge theory have a form:

$$\delta_g \tilde{A}^U = 2i(2R_n + a) \partial_+ \epsilon - 2i(2R_n) \hat{F}^2 \partial_- \epsilon + i\alpha' \{ \tilde{A}^U, \epsilon \},$$

$$\delta_g \tilde{A}^V = -2i(2R_n) \partial_- \epsilon + i\alpha' \{ \tilde{A}^V, \epsilon \},$$

$$\delta_g \phi^i = -2i(2R_n) \hat{F}^i \partial_- \epsilon + i\alpha' \{ \phi^i, \epsilon \},$$  \hspace{1cm} (44)

and imply the following transformations for quantities $F^{UV}$, $D_+ \phi^i$ and $D_- \phi^i$:

$$\delta_g F^{UV} = i\alpha' \{ F^{UV}, \epsilon \},$$

$$\delta_g D_+ \phi^i = -4i(2R_n)(2R_n + a) \partial_+ \hat{F}^i \partial_- \epsilon + i\alpha' \{ D_+ \phi^i, \epsilon \},$$

$$\delta_g D_- \phi^i = i\alpha' \{ D_- \phi^i, \epsilon \}.$$  \hspace{1cm} (45)

The gauge invarinace of the action $S_B$ then follows from these transformation rules. The fermionic part of the gauge theory action can also be written as:

$$S_F = -\frac{\alpha}{2} Tr \left( \bar{\psi} \Gamma^U[A_U, \psi] + \bar{\psi} \Gamma^V[A_V, \psi] + \bar{\psi} \Gamma^i[A_i, \psi] \right).$$  \hspace{1cm} (46)

By using commutators,

$$[A^U, \psi] = 2(2R_n + a) \partial_+ \psi - 2(2R_n) \hat{F}^2 \partial_- \psi + \alpha' \{ \tilde{A}^U, \psi \},$$

$$[A^V, \psi] = -2(2R_n) \partial_- \psi + \alpha' \{ \tilde{A}^V, \psi \},$$

$$[A^i, \psi] = -2(2R_n) \hat{F}^i \partial_- \psi + \alpha' \{ \phi^i, \psi \},$$  \hspace{1cm} (47)

and expanding in terms of the left and right-moving worldsheer fermions, we have an explicit form:

$$S_F = -\frac{\alpha}{2} \int d^2 x \frac{\alpha'^3}{2\pi} \left[ 2(2R_n + a) \psi_R^U \partial_+ \psi_R - 2(2R_n) \hat{F}^2 \psi_R^U \partial_- \psi_R + \alpha' \psi_R^U \{ \tilde{A}^U, \psi_R \} - 2(2R_n) \psi_L^U \partial_- \psi_L + \alpha' \psi_L^U \{ \tilde{A}^V, \psi_L \} + 2i(2R_n) \hat{F}^i (\psi_R^U \gamma^i \partial_- \psi_L + \psi_R^L \gamma^i \partial_- \psi_R) - i\alpha' \{ \psi_R^U \gamma^i (\phi^i, \psi_R) \} \right].$$  \hspace{1cm} (48)

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To obtain the supersymmetry transformations for the two dimensional gauge theory action, given by $S = S_B + S_F$, from Matrix Theory, we use the condition $\epsilon_R = 0$ which follows from (24). For supersymmetry transformation $\delta = \delta^1 - \delta^2$ we have

$$\delta A^U = 2i\epsilon_L^T\psi_L, \quad \delta A^V = 0, \quad \delta \phi^i = -\epsilon_L^T \gamma^i \psi_R$$

and

$$\delta \psi_L = \frac{i}{2} (F^{UV} + \Phi^{ij}\gamma^i) \epsilon_L, \quad \delta \psi_R = D_\gamma \phi^i \epsilon_L.$$

The supersymmetry invariance of the action can then be verified explicitly. In a compact (covariant) form, the supersymmetry transformations have an explicit form:

$$\delta S_B = -\delta S_F = -i Tr[A_\alpha, A_\beta][A^\alpha, \epsilon^\beta \psi].$$

A more explicit form of these transformations in terms of field variables $A^U$, $A^V$, $\phi^i$ and $\psi$ can be written down by using eqns. (38)-(41) and identifications (17). Finally, Lorentz invariance of the worldvolume action can be seen from the scaling transformations:

$$\sigma^+ \rightarrow \lambda \sigma^+, \quad \sigma^- \rightarrow \lambda^{-1} \sigma^-,$$

together with the transformation of the bosonic fields:

$$A^U \rightarrow \lambda^{-1} A^U, \quad A^V \rightarrow \lambda A^V, \quad \phi^i \rightarrow \phi^i, \quad F^i \rightarrow \lambda^{-1} F^i$$

and those of fermions:

$$\psi_R \rightarrow \lambda^{\frac{1}{2}} \psi_R, \quad \psi_L \rightarrow \lambda^{-\frac{1}{2}} \psi_L.$$

We have already pointed out in section-3 that the world-volume action is anomaly free. This is essentially due to the fact that spectrum contains equal number of left and right moving fermions. Moreover, as in the case of gauge theory with $(8,8)$ supersymmetry and the corresponding Abelian Born-Infeld action, all the fermions as well as matter scalars are neutral under gauge symmetry on the world-volume for the oscillating D-string as well. As a result they do not contribute to the anomaly. In section-3 we have argued this more concretely by pointing out that $ImW = 0$ whenever transverse oscillation is absent along one of the directions.
We now discuss the connection of our solution with static D-strings by arguing that one can obtain the particle spectrum of static strings, from that of the classical configuration discussed above, after quantization. In two dimensions, this implies the quantum equivalence of the worldvolume action (36) for the static case with the oscillating one (42) and (48). This is true in spite of the fact that the manifest symmetries of the two actions are quite different. However this is expected from a different angle, namely the absence of spontaneous symmetry breaking in two dimensions.

To analyse the quantum spectrum corresponding to the action (42) and (48), we set the gauge field fluctuations to zero and make the choice: $F^i = 0$, for $i \neq 1$ and $F^1 = F$. Furthermore, we restrict the analysis to the bosonic sector only, as the fermionic part can also be analysed in a similar manner. After these simplifying assumptions, the bosonic action has a form:

$$ S_B = \left(-\frac{\alpha}{4}\right)(2Rn) \int d^2 \sigma \left[ (2Rn + a)\partial_+ \phi \partial_- \phi - (2Rn)\hat{F}^2(\partial_- \phi)^2 \right] $$  

The equation of motion corresponding to the action (55) for close strings can be solved as:

$$ \phi = p_L \sigma^+ + p_R (\sigma^- + \frac{1}{(2Rn + a)} \int^\nu \hat{F}^2) + \sum_{m \neq 0} \frac{(\alpha - m)}{m} e^{2im\sigma^+} + \sum_{m \neq 0} \frac{\bar{\alpha} - m}{m} e^{-2i(2Rn)(2Rn + a)\sigma^- + \int^\nu \hat{F}^2}. $$  

The canonical formulation can be applied to the time and space-dependent Lagrangian (55) in a standard way and leads to a Hamiltonian density of the form:

$$ \mathcal{H} = \left(-\frac{\alpha}{4}\right) \frac{(2Rn)}{2(2Rn + a)} (J^-)^2 \left[ (2Rn + a)^2 - (2Rn)^2 \hat{F}^4 \right] \left(\partial_- \phi\right)^2, $$

where

$$ J^- = (2Rn + a)\partial_+ \phi - (2Rn)\hat{F}^2 \partial_- \phi $$

is a chiral conserved current satisfying $\partial_+ J^- = 0$.

To show the mapping of the spectrum of oscillating string, specified by the oscillators $\alpha$ and $\bar{\alpha}$ with that of the static strings, specified by $\alpha$, $\bar{\alpha}$, we choose a specific wave-profile $F(v)$ of the classical oscillating string solution. This is $F(v) = \sqrt{a(2Rn)} \left[ \sin(V/2Rn) + \cos(V/2Rn) \right] + \frac{2}{2\pi Rn}$.

This choice satisfies the condition that $F(v)$ has no zero mode, namely $f^{2\pi Rn} F(v) = 0$. Then, by we represent the oscillators $\alpha$ and $\bar{\alpha}$ as Fourier components of two commuting operators.
$J^-$ and $\partial_\phi$ respectively. Then we compare those with the Fourier components of similar operators in the static case: $F = 0$. It can be shown that, at $\tau = 0$, the oscillators are mapped as:

$$\tilde{\alpha}_m = \frac{(2Rn + a)}{(2Rn)} \alpha_m, \quad \tilde{\alpha}_m = \frac{(2Rn + a)}{(2Rn)} \sum_q J_{m-q} \left( \frac{2a}{Rn} \right) \tilde{\alpha}_q,$$

(59)

where $J_m(x)$ are the Bessel functions. Equation (59) gives a mapping between operators in the spectrum of oscillating and static strings. The mapping at $\tau \neq 0$ is given by the time-evolution of these operators with respect to the corresponding Hamiltonians. A similar mapping should be possible for the fermionic oscillators as well.

### 4.4 Conclusions

We have presented a class of solutions of the IIB Matrix theory and shown that the solutions preserve 1/4 supersymmetry. The supersymmetry that is preserved is chiral in nature in terms of the wave motion on the D-string. We have confirmed the BPS nature of these solutions by computing the one-loop effective action and derived the world-volume gauge theory. It was also shown that the world-volume action in the classical background of oscillating string is anomaly free for a large class of models. However, it should be possible to show this property without making any assumption about the form of the transverse oscillations.

There can be several applications and generalizations of these results. First, it will be interesting to extend the results of this chapter to other extended objects with oscillations. A membrane solution of this type has already been known[20] and implies that a similar analysis in the BFSS Matrix Theory should be possible. Another interesting aspect of this analysis may be to examine the gauge theories that might arise through other oscillating D-brane configurations. The BPS states of strings have been analyzed using the results in four dimensional gauge theories[18] in the context of BFSS Matrix Theory. However it should be possible to carry out our analysis in similar circumstances. One may also be able to apply the results of this chapter to study black holes in the IIB Matrix Theory picture. This can be done through the identification of the compactified oscillating strings with extremal black holes.
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