CHAPTER 3
MATHEMATICAL MODEL

3.1 KINEMATIC MODEL

3.1.1 Introduction

The kinematic model of a mobile robot, represented by a set of equations, allows estimation of the robot’s evolution on its trajectory, determination of its position and orientation and elaboration of possible control strategies. All motions of the mobile robot can be divided into translation and rotation components. Translation is the displacement of the mobile robot's center. Rotation is concerned with the rotational movement of each wheel's axis.

The basic element of every mobile robot is the wheel, which can be simplified to the kinematics of a rolling disk. Kinematic parameters include radius of the wheel, length of the wheel axle, location of the mass center, etc. The kinematic model of this thesis is simplified and is similar to the model of a unicycle. Here, the wheeled robot is a 2-DOF mobile robot, a three wheeled robot with two drive wheels and one castor wheel. The states of the mobile robot with differential driving mechanism change according to the two wheel velocities, when the mobile robot moves from current location to where the robot is to be located.
3.1.2  Kinematics

3.1.2.1  Representing Robot’s Position

Kinematic model of the wheeled robot assumes that the robot is placed on a plane surface and the contacts between the wheels of the robot and the rigid horizontal plane have pure rolling and non slipping conditions during the motion. This nonholonomic constraint can be written as

\[ \dot{y}\cos\theta - \dot{x}\sin\theta = 0 \]  \hspace{1cm} (3.1)

The center position of the robot is expressed in the inertial coordinate frame \((x, y, \theta)\). Here \(x\) and \(y\) are the position of the robot and \(\theta\) is the orientation of the robot with respect to inertial frame. Suppose the robot moves on a plane with linear and angular velocities, the state vector can be expressed as \(\dot{q} = (\dot{x} \ \dot{y} \ \dot{\theta})^T\). The differential drive mobile robot has two drive wheels which are independently driven for getting desired path. The robot’s motion on linear trajectories is given by constant velocities of the wheels and the motion on circular trajectories is determined by the difference between the angular velocities of the two drive wheels. The state of the robot is defined by its position and orientation and by the speeds of the two drive wheels \((x, y, \theta, \phi_l, \phi_r)\). A simple structure of differentially driven three wheeled mobile robot is shown in Figure 3.1. For the robot’s movement, the linear velocity \(v\) and the angular velocity \(\omega\) are chosen by the path planner. These values are converted into the velocities of the left and right wheels. The kinematic model is formulated by using these wheel speeds and geometric constraints of the vehicle also.
3.1.2.2 Kinematic Wheel Model

To simplify the analysis of the rolling and contact constraints, the assumptions are made such that the wheels remain vertical to the plane of motion and have a point or line contact to the ground plane. Likewise, the model will assume that there is no sliding motion orthogonal to the rolling motion at the wheel’s single point of contact. The wheel coordinate frame is positioned at the center of the wheel, on the axle. At times it may be convenient to define this frame at the wheel contact point and it may be useful to have it rotate with the wheel. Each wheel has its own frame as shown in Figure 3.2.

When the wheel is rolling along a curved line, the linear velocity of its center \((x_i, y_i, \theta_i)\) in the base coordinate system OXY depends on the wheel orientation in the plane defined by the angle \(\theta_i\).

\[
\begin{align*}
\dot{x}_c - \omega \cos \theta_i &= 0 \\
\dot{y}_c - \omega \sin \theta_i &= 0
\end{align*}
\]
The above equations can not be integrated in order to define relations only between the wheel positions. In the plane motion, the wheel velocities are imposed restrictions; thus, the mobile robots from this type shown in Figure 3.2 are called nonholonomic WMR and represented by Equations (3.2) and (3.3). The angular speed of each wheel is calculated by $r\omega_i$.

![Figure 3.2](image_url)  
**Figure 3.2**  
Wheel moving in a plane

### 3.1.2.3 Kinematic Robot Model

The methodology of the kinematic model of the robot is to develop a model of the robot’s motion as a function of time that will predict the position and orientation of the robot due to wheel velocities and initial body pose. The kinematic model is conducted through a two step process that involves determining the position and velocity of the robot body followed by analysing the position, orientation and wheel velocities of the robot.
Each member of the robot is constructed on a two dimensional, Cartesian based coordinate system known as the global frame (Figure 3.3). For a robot with known dimensional constraints on the body, the position of the body can be defined in the global frame \((X, Y)\) by the location of the center of axle and the angle \(\theta\) between the two angular positions. A rotation matrix \(R_m\) is used to map the motion of the body in the global frame through a rotation about the Z-axis.

\[
R_m = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  
(3.4)

![Coordinate position of each member of the robot](image)

Figure 3.3  Coordinate position of each member of the robot

Given the rotational speeds of the left and right wheels, their respective radii \(r\) and the distance \(l/2\) to the center of the robot, the model can predict the velocity of the robot. For this differentially driven mobile robot,
the two drive wheels are fixed so that a positive angular rotation produces a positive displacement along the axis. Rolling and sliding constraints for a fixed standard wheel ensures that all rotational motion exerted by the wheels produce an accompanied translation and rotational motion of the robot. In case of differential drive, to avoid slippage and have only a pure rolling motion, the robot must rotate around a point that lies on the common axis of the two driving wheels. This point is known as the instantaneous center of rotation (I_{CR}). By changing the velocities of the two wheels, the instantaneous center of rotation will move and different trajectories will be followed (Figure 3.4).

Figure 3.4  Differential drive motion

From the model of each wheel, the robot translational velocity is the average linear velocity of the wheels

\[ \dot{x} = r \frac{\dot{\theta}_l + \dot{\theta}_r}{2} \]  

(3.5)
And the rotational motion of the robot is

\[
\dot{\theta} = \frac{\Phi_L - \Phi_R}{l} \tag{3.6}
\]

The kinematics behaviour of the robot depends on the control variables \( u \) and \( \omega \).

\[
u = \frac{(\omega_L + \omega_R) r}{2} \tag{3.7}
\]

\[
\omega = \frac{(\omega_L - \omega_R)}{l} \tag{3.8}
\]

\[
\begin{bmatrix}
\dot{y}_R \\
\dot{y}_r
\end{bmatrix} = \begin{bmatrix}
\cos\theta & \sin\theta & l/2 \\
\cos\theta & \sin\theta & -l/2
\end{bmatrix} \begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} \tag{3.9}
\]

where \( \omega_L \) and \( \omega_R \) are the angular velocities of the left and right wheels of the mobile robot respectively. For different values of \( \omega_L \) and \( \omega_R \) of the mobile robot, the trajectory is followed by the robot. \( \nu_L \) is the left wheel’s linear velocity along the ground, \( \nu_R \) is the right wheel’s linear velocity along the ground and \( R \) is the signed distance from the ICR to the midpoint between the two wheels.

If \( \nu_L = \nu_R \) then the radius \( R \) is infinite and the robot moves in a straight line (Figure 3.5 (a)). For different values of \( \nu_L \) and \( \nu_R \), the mobile robot does not move in a straight line but follows a curved trajectory around a point located at a distance \( R \) from the centre of rotation, changing both the robot’s position and orientation (Figure 3.5 (b)). If \( \nu_L = -\nu_R \) then the radius \( R \) is zero and the robot rotates around its center (Figure 3.5 (d)).
3.1.2.4 Geometric Model

The path planning under kinematic constraints is transformed into a pure geometric problem. There are many possible ways to describe the path. The resulting shortest path is composed of circular arcs and straight lines as shown in Figure 3.6. The robot motion control can be done providing wheel velocities \( v(t) \) and \( \omega(t) \) called control variables. The planned path consists of an arc of a circle followed by a tangent straight line segment. The
discontinuity of the curvature of the path at the tangent point can be solved by smoothing the path before computing the trajectory.

Figure 3.6  Path geometry (a) Trajectory of the vehicle and (b) Relationship of circles and tangent line
\[ \Phi = \langle a + <b \rangle \] 

\[ \tan a = \frac{c_{2y} - c_{1y}}{c_{2x} - c_{1x}} \] 

\[ \tan b = \frac{r_2 + r_1}{L} \] 

\[ \tan b = \frac{r_2 + r_1}{\sqrt{c_1 c_2^2 - (r_1 + r_2)^2}} \] 

\[ \alpha_1 = \varphi - \theta_d \] 

\[ \alpha_2 = \beta_d - (90 - \varphi) \] 

The mathematical model of this kinematic problem considers these two \((\psi(t), \omega(t))\) control variables and three state variables \((x(t), y(t), \theta(t))\).

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & 0 \\
\sin \theta & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\psi \\
\omega
\end{bmatrix}
\] 

An appropriate approach is implemented to produce the desired trajectory described by a sequence of coordinates. An arbitrary configuration \((x, y, \theta)\), is defined by the robot path planner and the robot is placed on the path. The path tracking controller has to ensure a geometrical convergence towards the path to be followed. The stability of a wheeled robot is analysed for a kinematical model using a linearised kinematical model. Here the analysis is done for the case of a straight line and a circle with the trajectories that can be desegmented into pieces of curvature. Differentially driven nonholonomic mobile robot tracks a simple geometric path which is the optimum path designing of the Fuzzy Logic Controller (FLC) for mobile robot trajectory tracking.
The motion of the robot could be described with two modes, either pure rotation or pure translation. The path consists of circular arcs with a specified radius and turning angle and a straight tangent line. The circular arc and straight tangent line are used to avoid stoppage and provide continuity for the robot. To achieve controlled trajectory, the linear and angular velocities $v$ and $\omega$ are calculated by the Equations (3.17) and (3.18) through fuzzy logic controller.

$$
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
\omega
\end{bmatrix}
$$

(3.17)

$$
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
v \cdot \cos \theta \\
v \cdot \sin \theta \\
0
\end{bmatrix}
$$

(3.18)

The radius of the circular path depends on the robot’s velocity and its position. The algorithm first calculates a suitable intersection point to be reached and then calculates a circular arc through the target position. The target position can then be reached with the desired orientation. So the trajectory is generated out of the target parameters and the present position, which consists of straight lines and circular arcs, linear and angular velocities.

3.2 Dynamic Model

Motion control is a path generation and the motion controller has to follow the plan as closely as possible. This divide and conquer approach has produced many sophisticated techniques for path planning and control. However these are very difficult to implement in a real system. The kinematics and dynamics of a WMR can be modeled based on the following design assumptions: i) the WMR does not contain flexible parts; ii) there is at the most one steering link per wheel; and iii) all steering axes are
perpendicular to the surface of travel. So, the equations that describe the
kinematics and the dynamics of a nonholonomic WMR are obtained knowing
that the motion of a mechanical system can be described by nonholonomic
constraint. In this research, a unicycle mobile robot is considered as a case
study. The robot body is symmetrical around the perpendicular axis and the
center of mass is at the coordinate center of the WMR. It has two driving
wheels fixed to the axis that passes through C and one castor wheel that is
placed in front of the axis and normal to it. The two fixed wheels are
controlled independently by motors, and the castor wheel prevents the robot
from tipping over as it moves on a plane. It is assumed that the motion of
caster wheel can be ignored in the dynamics of the mobile robot. The system
model of the mobile robot is shown in Figure 3.7 where l represents the
distance between two driving wheels, and r is the radius of the wheel.

In this kind of mobile robot, two kinematic constraints are imposed.
First, no lateral slip is allowed between the wheels and the ground and the
second is no longitudinal slip. According to these two constraints, the mobile
robot is subject to two kinds of nonholonomic constraints which can be
described by:

\[ \dot{x}_c \cos \theta - \dot{y}_c \sin \theta = 0 \]

\[ \dot{x}_c \cos \theta + \dot{y}_c \sin \theta = \frac{r}{2} (\dot{\phi}_r + \dot{\phi}_l) \]  

where \((x_c, y_c)\) are the coordinates of the center of the mobile robot, \(\theta\) is the
heading angle of the mobile robot measured from X axis and \(\phi_r\) and \(\phi_l\) are the
angular positions of the right wheel and the left wheel.
Figure 3.7  Forces on robot (a) Robot structure, (b) Forces on a wheel and (c) Dynamics for the robot
The general dynamic equation of the wheel robot is given below,

\[ M(q)\ddot{q} + C(q, \dot{q}) + G(q) + \tau_d = B(q)\tau + A^T(q)\lambda \]  

(3.21)

where \( M(q) \) is the inertia matrix, \( C(q, \dot{q}) \) is a matrix containing the centrifugal and coriolis terms, \( G(q) \) is the gravity force matrix, \( B(q) \) is the input transformation matrix, \( \tau \) is the input torque, \( A^T(q) \) is the Jacobian matrix associated with the constraints, \( \lambda \) is the constraint force vector and \( \dot{q} \) is the state vector representing the generalized coordinates. \( \tau_d \) denotes the bounded unknown external disturbance. For this, the dynamic model holds the properties,

\[
M(q) = \begin{bmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & I
\end{bmatrix}
\]

\[
A^T(q) = \begin{bmatrix}
-sin\theta \\
cos\theta \\
0
\end{bmatrix}
\]

\[
B(q) = \frac{1}{r} \begin{bmatrix}
cos\theta & cos\theta \\
sin\theta & sin\theta \\
l & -l
\end{bmatrix}
\]

\[
\tau = [\tau_l \tau_r]^T
\]

\[
\lambda = -m(\dot{x}_c\cos\theta + y_c\sin\theta)\dot{\theta}
\]

where \( m \) is the mass of the WMR, \( I \) is the moment of inertia of the WMR about its center. \( \tau_l \) and \( \tau_r \) are the torque control inputs generated by the left and right wheels respectively. It would be more suitable to express the dynamic equations of motion in terms of internal velocities. When the mobile robot is subject to nonholonomic constraints then,

\[ A(q)\dot{q} = 0 \]  

(3.22)
where $A(q)$ is a $m \times n$ full rank nonholonomic constraint matrix. Assuming that $S(q)$ is a $n \times p$ full rank matrix whose subset is a set of smooth and linearly independent vectors in the null space of $A(q)$, such that $A(q)S(q) = 0$.

where,

$$S(q) = \begin{bmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ 0 & 1 \end{bmatrix}$$

$$\dot{q} = S(q) \begin{bmatrix} u \\ \omega \end{bmatrix}$$

Imposing nonholonomic constraints in Equations (3.19) and (3.20) to the system, Equation (3.21) can be solved for each $\dot{q}$. The constraint forces do not work, which means there is no movement along the constraint force directions. As seen in Equation (3.22), the constraint forces are eliminated from the dynamic equations. The dynamic equations in Equation (3.21) can be written as

$$S^T(q) (M(q)\ddot{q} + C(q, \dot{q}) + G(q) + \tau_d) = S^T(q)(B(q)\tau + A^T(q)\lambda) \quad (3.23)$$

$$\bar{M}(q)\ddot{q} + \bar{C}(q, \dot{q}) + \bar{G}(q) + \bar{\tau}_d = \bar{B}(q)\tau \quad (3.24)$$

The gravity force $G(q)$ is ignored since the mobile robot is assumed to move on a horizontal plane. Since the distance between $C$ and the coordinate center of the WMR is zero, the effect of $C(q, \dot{q})$ can be eliminated from Equation (3.24). By considering the surface friction and the disturbance torque as the modeling uncertainties and disturbances, then the dynamic Equation (3.21) is

$$M(q)\ddot{q} + R(q, \dot{q}) = B(q)\tau \quad (3.25)$$
It is noted that \( v_d \) and \( \omega_d \) are the desired velocities to make the kinematic stable. The motion control is proposed based on \( e_x \), \( e_y \) and \( e_\theta \) of state error variables and the linear torque \( \tau_{li} \) and the angular torque \( \tau_{an} \) are the control signals.

\[
\tau_{li} = m\dot{v}_d + k_l(v_d - v) \quad (3.26)
\]

\[
\tau_{an} = I\dot{\omega}_d + k_a(\omega_d - \omega) \quad (3.27)
\]

\[
m\dot{u} = m\dot{v}_d + k_l(v_d - v) \quad (3.28)
\]

\[
I\dot{\omega} = I\dot{\omega}_d + k_a(\omega_d - \omega) \quad (3.29)
\]

\[
m\dot{\omega} = \frac{1}{r} \times (\tau_l + \tau_r) \quad (3.30)
\]

\[
I\dot{\omega} = \frac{1}{r} \times (\tau_l - \tau_r) \quad (3.31)
\]

Hence,

\[
\tau_{li} = \frac{1}{r} \times (\tau_l + \tau_r)
\]

\[
\tau_{an} = \frac{1}{r} \times (\tau_l - \tau_r)
\]

\[
Ku = m\dot{v} + Bu
\]

\[
u = [e_l \ e_r]^T
\]

The free body diagram of forces and velocities is shown in Figure 3.8, with the robot wheel having instantaneous positive velocity components \( \dot{x} \) and \( \dot{\delta} \) and negative velocity \( \dot{y} \). The rolling motion generates a longitudinal reactive force \( R_x \) and a lateral reactive force \( R_y \), while the twisting motion generates a pure reactive turning moment \( M_z \) in the vertical direction.
Figure 3.8  Free body diagram of forces

Assuming that the ground is flat and does not deform, the above three quantities are defined in vehicle dynamics as,

\[ R_x = \mu \frac{mg}{2} \text{sgn}(\dot{x}_i) \]  (3.32)

\[ R_y = \mu_t \frac{mg}{2} \text{sgn}(\dot{y}_i) \]  (3.33)

\[ M_x = \mu \frac{mg}{2} \text{sgn}(\dot{x}_i)r_c \]  (3.34)

\[ M_y = \mu \frac{mg}{2} \left( \text{sgn}(\dot{z}_i) \right)^b \]  (3.35)

where b is the width of the wheel, \( \mu \) - longitudinal friction coefficient and \( \mu_t \) - lateral friction coefficient.

As shown in Figure 3.9, the dynamic equations of left and right wheel are expressed as

\[ \tau_i = I_w \ddot{\omega}_i + r_c R_{x_l} + c\omega_i \]  (3.36)

\[ \tau_r = I_w \ddot{\omega}_r + r_c R_{x_r} + c\omega_r \]  (3.37)
where $I_w$ is the inertia of the wheel system, $R_{xl}, R_{xr}$ are the rolling resistances in left and right wheels, $r_e$ effective radius of the wheel and $c$-viscous coefficient.

\[ I \omega \]

\[ c \omega \]

\[ R_{xl} = \mu \frac{mg}{2} (sgn(x_1) + sgn(x_2)) \]  

\[ R_{xr} = \mu \frac{mg}{2} (sgn(y_1) + sgn(y_2)) \]  

\[ M_r = \left[ \mu \frac{mg}{2} (sgn(x_1) - sgn(x_2)) \right] \frac{l}{2} + \left[ \mu \frac{mg}{2} (sgn(y_1) - sgn(y_2)) \right] d \]

**Figure 3.9 Wheel ground interactions**

The dynamic model is obtained from dynamic properties of mass, inertia force, moments, friction force, gravitation and wheel ground interaction. The orthogonal force components are vertical, longitudinal and lateral. The lateral frictional forces also prevent the vehicle from sliding to unwanted directions. Several parameters of the terrain are used to estimate normal, lateral and longitudinal forces at the wheel contact patch. If the frictional force is less than the maximum value, the wheel position is not changed; if it is greater than or equal to maximum value, the wheel is pulled in direction opposite to the friction force from the wheel position. The total resistive quantities are defined in vehicle dynamics as:
The model sought here is a relatively simple model that captures the resistive forces of the robot. It is thus chosen to model one wheel at a time and then construct the model of the robot motion based on the forces and torques with which the wheels affect the main body.

The dynamic model is derived from the law of physics that govern the several robot subsystems, including the actuator dynamics (electric and mechanical characteristics of the motors), friction and robot dynamics (movement equations). For most robots, the modeling process generates a second order model expressed by Equation (3.41).

\[
K u = M \dot{u} + B u
\]  

where \(u = [v \ \omega]^T\) represents the robot linear and angular velocities, \(u = [e_r \ e_l]^T\) contains the input signals applied to the right and left motors, \(K\) is the matrix which transforms the electrical signals \(u\) into forces to be generated by the robot wheels, \(M\) is the generalized inertia matrix and \(B\) is the generalized damping matrix, which includes terms of dynamic and static frictions and electric resistance. Now the dynamic equation of the wheeled robot can be written as,

\[
M(q) \ddot{q} + R(q, \dot{q}) = B(q) \tau
\]

Dynamic model for backstepping approach is

\[
M(q) \ddot{q} + R(v, \omega) = B(q) \tau
\]

where

\[
M(q) = \begin{bmatrix}
m & 0 \\
0 & 1
\end{bmatrix}
\]
\[ R(q, q) = \begin{bmatrix} F_{\text{lan}} & 0 \\ 0 & F_{\text{lat}} \end{bmatrix} \begin{bmatrix} \text{sgn} (v) \\ \text{sgn} (\omega) \end{bmatrix} \]

\[ B(q) = \frac{1}{r} \begin{bmatrix} 1/2 & 1 \\ -1/2 & 1 \end{bmatrix} \]

Now the dynamical model is

\[
\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} + \begin{bmatrix} F_{\text{lan}} & 0 \\ 0 & F_{\text{lat}} \end{bmatrix} \begin{bmatrix} \text{sgn} (v) \\ \text{sgn} (\omega) \end{bmatrix} = \frac{1}{r} \begin{bmatrix} 1/2 & 1 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} \tau_l \\ \tau_r \end{bmatrix} \quad (3.43)
\]

where

\[ F_{\text{lan}} \] - longitudinal force acting on the robot

\[ F_{\text{lat}} \] - lateral force acting on the robot

Traction force, \( F = F_c + F_f + F_m + F_i \)

where

\[ F_c \] - contact force, \( F_f \) - Friction force

\[ F_m \] - Moment force, \( F_i \) - Inertia force

This is one of the consequences of the presence of nonholonomic constraints in the dynamic model of mechanical systems in which, because of Newton’s second law, usually up to the second derivative of position vector should appear in the model based control algorithm. This leads to a dynamically extended input-output feedback linearization for kinematically modeled mobile robots. Static input-output feedback linearization requires only taking up to the second derivatives of the outputs.
At low speeds, the lateral load transfer due to centrifugal forces on curved paths can be neglected. But in the case of high speeds, centrifugal forces would be taken into account. And also in case of hard ground, the contact patch between wheel and ground is rectangular and that the vertical load produces an uniform pressure distribution along the width of the wheel which produces resistance to motion are considered. In this condition, \( \mu \) is the coefficient of rolling resistance, then total longitudinal resistive force \( R_X \) is,

\[
R_X = 2 \mu \frac{mg}{z} (\text{sgn}(x_1) + \text{sgn}(x_2)) \quad (3.44)
\]

Introducing a lateral friction coefficient \( \mu_t \), the total lateral resistive force \( R_Y \) acting on wheels will be,

\[
R_Y = 2 \mu_t \frac{mg}{z} (\text{sgn}(y_1) + \text{sgn}(y_2)) \quad (3.45)
\]

The dynamic model can be rewritten in the coordinate frame, introducing the generalized coordinates \( q=(x, y, \theta) \) and the matrix notation is,

\[
M(q)\ddot{q} + R_{eq}(q, \dot{q}) = B(q)\tau
\]

### 3.3 CONTACT MODEL
#### 3.3.1 Introduction

While the mobile robots are designed, more weightage is given to designing of mechanical structure. The contact between the robot wheel and the terrain for the contact force is important since it influences the stability of the robot. This wheel contact force is defined through a surface load spreads over the contact surface according to the Hertz’s contact theory. The contact forces will directly affect the motor torque and the actual wheel ground interaction is to be considered in order to improve the robot motion control. Here the terrain is assumed to be rigid and the wheel is deformable.
3.3.2 Hertz Theory for Contact of Elastic Solids

The origin of the theory of rolling of two linearly elastic bodies in contact is founded on the law of friction of coulomb-amontons and the analytic models of deformation of a three dimensional half-space elastic body due to a concentrated load of Cerruti and Hertz’s theory of two elastic surfaces with curvature in contact. In particular, the two linearly elastic bodies in rolling contact are assumed to follow the following hypothesis:

i The rolling bodies are linearly elastic.

ii Quasi-identity relation on the elastic properties of the two bodies in contact holds. (This includes the case when the two bodies are elastically similar and approximates the situation when body, say a polyurethene wheel is incompressible and the other body, say the concrete ground is relatively rigid).

iii The area of contact between the two bodies is symmetric about the direction of the rolling of the wheels.

3.3.3 Contact in Single Wheel

For a simple dynamic model of the wheel, a thin cylinder that represents the middle cross section of the wheel and the linear velocity of the wheel lies in the plane of the wheel. For the continuous nature of the deformation and contact, the non linear finite element method is selected for the best model. The contact force is measured from the built in geometric model of a wheel and a terrain.

In a simple contact model, it is assumed as single line contact between the wheel and ground. Under this assumption, the kinematic and
dynamic models are derived based on the pure rolling without slipping condition. Such a condition may not be true in real situations; the wheel motion may suffer significant slip which can be removed using control algorithms based on the simplified models. This is the condition for the wheel to satisfy the pure rolling without slipping. The contact frame is defined as follows: \( \mathbf{n}_i \) is the contact normal vector, \( \mathbf{l}_i \) is the longitudinal vector and \( \mathbf{t}_i \) is the lateral vector as shown in Figure 3.10.

![Figure 3.10](image)

**Figure 3.10** Wheel ground contact frame

The vector \( \mathbf{p}_i \) is the projection of vector \( \mathbf{y}_i \) on the contact plane which depends on the system kinematic configuration:

\[
\mathbf{p}_i = \frac{\mathbf{y}_i \times \mathbf{n}_i}{|\mathbf{y}_i \times \mathbf{n}_i|}
\]

The center point of the contact area \( \mathbf{P}_i \) is defined as a projection of the center of the wheel on the mean contact plane. It depends only on the wheel radius \( r_i \) and the contact normal: \( \mathbf{C}_i \mathbf{P}_i = r_i \mathbf{n}_i \)
3.3.4 Contact Model of the Robot

The resulting frictional forces can be defined by integration of all forces acting on the contact surface. The pressure distribution resulting from the normal contact can be calculated in the local reference. As a consequence, the tangential and the normal forces in the global reference can be calculated by integrating the contact pressures on contact of the X and Y axes for the tangential forces and for the normal force on the Z axis.

\[
F_X = \iint p_x \, dx\, dy \quad (3.46)
\]

\[
F_Y = \iint p_y \, dx\, dy \quad (3.47)
\]

\[
M_Z = \iint (xp_y - yp_x) \, dx\, dy \quad (3.48)
\]

At the point of the contact, the projected force \( F_Y \) on the Y axis is zero due to the symmetry of the vehicle structure. As a result, contact friction leads not only to a resultant force applied to the center of the contact area, but also to a non-vanishing moment about the normal axis through the center of that contact area. This moment, \( M_Z \) is the function of the size of the contact area \( A_c \), wheel material, type of wheel ground contact, weight of the vehicle, etc. Since \( M_Z \) opposes the steering motion, it should be added to Equation (3.25) using a sign function. At the contact point, the contact force can be desegmented into normal and tangential components. \( F_X \) is the horizontal component of contact force and \( F_Z \) be the normal component of contact force. It is assumed that the coordinate frame and centre of gravity are lying in symmetry axis of the wheels, so that the contact forces \( F_Y = 0 \) and \( F_Z \) is expressed as the function of contact pressure.
This resultant frictional force is still acting, but the new distribution of the normal forces creates a net torque opposing the rotational contribution of the friction and causing an overall deceleration of the wheel’s forward velocity. The lateral wheel friction is a coulomb friction, whose force takes two sign opposite values depending on the direction of the turning of the robot. Therefore $M_Z$ can be rewritten as,

$$M_Z = 2 \left( - \int y p_x \, dx \, dy \right) \frac{1}{2} \text{sgn}(\dot{z}) = -F_c \frac{1}{2} \text{sgn}(\dot{\omega})$$

$$m\ddot{\mu} = (F_{xl} + F_{xr}) - R_x \text{sgn}(\dot{x})$$  \hspace{0.5cm} (3.49)

$$I\ddot{\omega} = (F_{xl} - F_{xr}) l/2 - M_Z$$ \hspace{0.5cm} (3.50)

$$\tau_{li} = m\ddot{\mu}_d + k_l (u_d - u)$$ \hspace{0.5cm} (3.51)

$$\tau_{an} = I\ddot{\omega}_d + k_a (\omega_d - \omega)$$ \hspace{0.5cm} (3.52)

$$m\ddot{\mu} = \frac{1}{r} (\tau_l + \tau_r) = m\ddot{\mu}_d + k_l (u_d - u)$$

$$I\ddot{\omega} = \frac{1}{r} (\tau_l - \tau_r) = I\ddot{\omega}_d + k_a (\omega_d - \omega)$$

Considering the motion resistances, the dynamic model of the robot Equation (3.21) is rewritten as:

$$M(q)\ddot{q} + C(q, \dot{q}) + G(q) = B(q)\tau + \tau_c$$ \hspace{0.5cm} (3.53)

where $\tau_c$ is the torque generated by the contact forces. The contact torque can be written under the following equation,

$$\tau_c = J(q)^T R_{eq}(q, \dot{q})$$
where $J(q)$ is the Jacobian matrix of the constraint on the position of the points on which these contact forces are applied. In real situations, motion resistance generated by the wheel ground interaction always exists, so the actual governing dynamic equations of the motion of the robot are given by Equation (3.55) rather than Equation (3.24).

$$M(q)\ddot{q} + R(v, \omega) = B(q)\tau$$  \hspace{1cm} (3.54)

where

$$M(q) = \begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix}$$

$$R(v, \omega) = \begin{bmatrix} R_x & 0 \\ 0 & M_z \end{bmatrix} \begin{bmatrix} \text{sgn}(v) \\ \text{sgn}(\omega) \end{bmatrix}$$

$$B(q) = \frac{1}{r} \begin{bmatrix} 1 \\ l/2 \\ -l/2 \end{bmatrix}$$

In the case of trajectory tracking, control algorithms that consider wheel ground interaction are expected to demonstrate better tracking performance than those that do not consider the wheel ground interaction.

The general form of the dynamics of a robot subject to contact efforts can be written as

$$M(q)\ddot{q} + \mathcal{C}(q, \dot{q}) + \mathcal{G}(q) + J(q)\tau_c = B(q)\tau$$  \hspace{1cm} (3.55)

When a robot is stationary, there is no input torque to the system. This is a result of the simplifying assumption that the contact between wheel and ground takes place at a point or a line (contact length is equal to the width of the wheel). The reality is that, because of the weight of the wheel and the fact that the wheel is not completely rigid; there is a contact surface which is known as the Hertzian surface. As a result, contact friction leads not only to a
resultant force applied to the center of the contact area but also to a non
vanishing moment about the normal axis through the center of that contact
area. Now the dynamic equation of the wheel robot with resistance due to
contact force can be written as,

\[ M(q)\ddot{q} + R_{eq}(q, \dot{q}) = B(q)\tau \]  \hspace{1cm} (3.56)

where \( R_{eq}(q, \dot{q}) = [ R_X, M_Z]^T \)

\( R_X \)- rolling resistance, \( M_Z \)- turning resistance

In this case, the moments are acting against the motion of the
wheels. The resulting model that is used for motion control includes
consideration of wheel resistance and its moments. In real situations, motion
resistance generated by the wheel ground interaction always exists, so the
actual governing dynamic equations of the motion of the robot are given by
Equation (3.55), rather than Equation (3.24). In the case of trajectory tracking,
control algorithms that consider wheel ground interaction Equation (3.56) are
expected to demonstrate better tracking performance than those that do not
consider the wheel ground interaction.

3.4 SUMMARY

This chapter provides the details of the mathematical model related to
kinematic and dynamics of MWRs. This research work is organised in
following steps  (i) Complete derivation of the kinematic equations of a
wheeled mobile robots in terms of physical variables useful for motion
planning and control; (ii) Study of the wheel ground interaction influenced to
dynamic control; (iii) Derivation of the dynamic equations of a wheeled
mobile robots which needs to be done to make the controller design robust
with respect to wheel-ground contact parameter uncertainties.