Bayesian Inference on Queueing Models
Using Bivariate Prior Distributions

4.1 Introduction

A major problem associated with the analysis of real life queueing data is the model selection. The great diversity of queueing problems gives rise to an enormous variety of models each with specific features. Incorporating more than one or two such features usually makes the model not only complicated but also analytically intractable. Therefore a substantial part of the literature deals with the models of a very simple structure. The Markovian assumption greatly simplifies the modeling and solution. In this paper we consider a single server model with FCFS queue discipline, involving Markovian assumption on inter arrival and service times. That is the inter arrival and service times are independently and exponentially distributed random variables with means \( \frac{1}{\lambda} \) and \( \frac{1}{\mu} \) respectively.

After selecting an appropriate model for the queueing data, the next step is the estimation of performance measures. The common performance measures are traffic

\[ \text{Traffic} = \frac{\text{arrival rate}}{\text{service rate}} \]

\[ = \frac{\lambda}{\mu} \]

\[ \begin{align*}
\text{1Some part of this chapter is based on Joby K. Jose and M. Manoharan (2011c)}
\end{align*} \]
intensity ($\rho$), expected queue size ($L_q$), expected system size ($L$), expected waiting time ($W$) and expected waiting time in the queue ($W_q$). All these quantities are functions of the queue parameters arrival rate ($\lambda$) and service rate ($\mu$).

The different types of queueing data available are observations on inter arrival time, service time, inter departure time and waiting time. Since different types of data are available, there are different estimates for queue parameters. In all the above data types the observations should be sufficiently spaced so that they are independent. Hence we require longer time for data collection. This makes the problem of data collection difficult. However Clark (1957) suggested a data selection procedure in which we observe the system during an interval of length $t$ and there are $n_a$ arrivals and $n_c$ departures during this interval. Let $n_0$ be the initial system size. Let $t_b$ be the length of the busy period and $t_e = t - t_b$ be the length of the ideal period. Based on these information Clark obtained the likelihood function $L(\lambda, \mu)$ for an M/M/1 queueing system. The likelihood function is made up of components which are formed based on the information on the intervals of length $x_b$ spent in a non zero state and ending in an arrival or departure, intervals of length $x_e$ spent in the zero state and ending in an arrival, the very last interval of (length $x_l$) observation, arrivals to a busy system, departures and the initial number of customers. The contributions to the likelihood of each of these are $(\lambda + \mu) e^{-\lambda + \mu}, \lambda e^{-\lambda x_e} e^{-\lambda x_l}$ or $e^{-(\lambda + \mu)x_e}, \frac{\lambda}{\lambda + \mu}, \frac{\mu}{\lambda + \mu}$ and $P[N(0) = n_0]$ respectively. Since $\sum x_b = t_b$ and $\sum x_e = t_e$ the likelihood function becomes

$$L(\lambda, \mu) = e^{-(\lambda + \mu)t_b} \lambda^{n_a} e^{-\lambda x_e} \mu^{n_c} p(n_0)$$

(4.1)

The earliest attempt of statistical inference on M/M/1 queue from the frequentist standpoint was made by Clarke (1957) yielding the maximum likelihood estimates of service rate, arrival rate and traffic intensity. Basawa and Prabhu (1988) obtained
asymptotic results for different queueing data. An illustration of statistical estimation technique applied to the queueing problem can be found in Rubin and Robson (1990). Basawa et al. (1996) studied maximum likelihood estimates of the parameters in the single server queue using waiting time data. Scruben and Kulkarni (1982) and Zheng and Seila (2000) showed that under frequentist setup the expected value and standard errors of the estimators of popular queue performance measures like expected system size \( L \), expected queue size \( L_q \), expected waiting time in the system \( W \), expected waiting time in the queue \( W_q \) do not exist. Zheng and Seila (2000) obtained a way out of this problem by restricting the upper bound of \( \rho \) to some value \( \rho_0 < 1 \).

Maximum likelihood estimates and confidence interval in an M/M/2 queue with heterogeneous servers were derived by Dave and Shah (1980) and Jain and Templeton (1991), respectively. Recently Wang et al. (2006) obtained maximum likelihood estimates and confidence interval of an M/M/R queue with heterogeneous servers.

Usually we assume that the queue parameters \( \lambda \) and \( \mu \) are constants. But in many real life situations they are found to be random variables. Combining the prior information about the variations in these parameters with the current data on the queueing system we can obtain better estimates of the characteristics. This can be done by the Bayesian analysis. Point estimates in Bayesian framework are termed as Bayes estimates. Assuming square error loss, these are means of posterior distributions. An important objective of the scientific investigation is the prediction of the future based on past and present data. Predictive distributions are designed to perform this role.

McGrath and Singpurwalla (1987) and McGrath et al. (1987) utilized the subjective Bayesian approach to the statistical inference in queues. Armero and Bayarri (1994a) developed the Bayesian prediction on M/M/1 queue. They have worked
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with a likelihood function constructed by observing independent inter arrival and service times. They showed that the moments of the predictive distribution do not exist. To overcome this undesirable situation Armero and Bayarri (1994b) proposed the Gauss Hypergeometric distribution as prior. Armero and Conesa (1995) studied Bayesian inference and prediction for Markovian queues with bulk arrivals. David et al. (1998) described Bayesian inference and prediction for some M/G/1 queueing models. Recently Choudhury and Borthakur (2008) discussed the Bayesian inference for an M/M/1 queue in detail using system size data. They obtained several closed form expressions on posterior inference and prediction. Transacting the problem of non-existence of the posterior moments, they suggested methods to obtain interval estimates and tests of hypothesis for performance measures.

The natural prior distribution for Exponential parameter is Gamma distribution. Invoking independence of ‘arrival’ and ‘service’ yields the product of the Gamma densities as the natural joint prior distribution of \((\lambda, \mu)\) which we name as prior 1. A stationary M/M/1 queue should satisfy the condition \(0 < \lambda \leq \mu < \infty\) always. Sometimes certain bivariate distributions of \((\lambda, \mu)\) with this inherent order restriction on \(\lambda\) and \(\mu\) perform better than the natural joint prior distribution. Two such bivariate prior distributions are

**Prior 2:**

\[
\pi(\lambda, \mu) = \alpha \beta e^{-[(\alpha - \beta)\lambda + \beta \mu]}, 0 < \lambda \leq \mu < \infty; \alpha, \beta > 0; \alpha \neq \beta. \tag{4.2}
\]

**Prior 3:**

\[
\pi(\lambda, \mu) = \frac{m \lambda^m}{\mu^{1+m}} e^{-\beta \lambda}, 0 < \lambda \leq \mu < \infty; m > 0, \beta > 0 \tag{4.3}
\]

In this paper we obtain the Bayes estimates of the queue parameters \(\lambda, \mu\) and \(\rho\)
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(traffic intensity) using these prior distribution of \((\lambda, \mu)\) and the data obtained by the experiment suggested by Clarke.

This chapter is organized as follows. In section 2 we obtain the Bayes estimate of the queue parameters using the natural joint prior distributions of \((\lambda, \mu)\). The posterior distribution and credible sets for traffic intensity and predictive distribution of number in the system are also derived in this section. In section 3 we obtain the Bayes estimate of the queue parameters using the bivariate prior distributions of \((\lambda, \mu)\) order restriction on \(\lambda\) and \(\mu\). The posterior distribution and credible sets for traffic intensity are also derived in this section. In the last section some numerical results are given.

4.2 Bayesian Estimation of Queue Parameters by Natural Prior (Prior 1)

Throughout this paper, we consider the experiment suggested by Clarke and use the likelihood function given in (4.1). The main drawback associated with all other commonly used data collection methods is that the independence of the observations can be guaranteed only if the observation points are sufficiently spaced. As a result the data collection becomes a more time consuming one. In contrast the Clarke’s likelihood function has a simple closed form expression which is analytically tractable. We suggest the product of the Gamma densities as the natural joint prior distribution of \((\lambda, \mu)\). So we have the density function of the prior distribution is

\[
\pi(\lambda, \mu) = \left[ \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta \lambda} \lambda^{\alpha - 1} \right] \left[ \frac{\delta^\gamma}{\Gamma(\gamma)} e^{-\delta \mu} \mu^{\gamma - 1} \right], \beta > 0, \delta > 0 \tag{4.4}
\]
4.2.1 Bayes Estimates

We observe a steady state M/M/1 queueing system during an interval of length $t$ and find the number arrivals $n_a$ and number of departures $n_c$ during this interval. Also calculate the length of the busy period $t_b$ and the length of the total period $t$. Then we have a set of observations $x = (t, t_b, n_a, n_c)$ on this queueing system. Then we have the joint likelihood function of $x = (t, t_b, n_a, n_c)$ and $(\lambda, \mu)$ is

$$h(x, \lambda, \mu) = L(x, \lambda, \mu) \pi(\lambda, \mu)$$

and the marginal likelihood function of $x = (t, t_b, n_a, n_c)$ is

$$m(x) = \int_0^\infty \int_0^\infty h(x, \lambda, \mu) d\lambda d\mu$$

where

$$m(x) = p(n_0) \beta^\alpha \Gamma(n_a + \alpha) \delta^\gamma \Gamma(n_c + \gamma) \Gamma(\alpha) (t + \beta)^{n_a + \alpha} \Gamma(n_c + \gamma) (t_b + \delta)^{n_c + \gamma} \Gamma(t + \beta)^{n_a + \alpha - 1} \Gamma(n_c + \gamma)^{n_c + \gamma - 1}$$

Hence the posterior joint distribution of $(\lambda, \mu)$ is

$$\pi(\lambda, \mu/x) = \left[ \frac{(t + \beta)^{n_a + \alpha}}{\Gamma(n_a + \alpha)} e^{-(t + \beta)\lambda} \lambda^{n_a + \alpha - 1} \right] \left[ \frac{(t_b + \delta)^{n_c + \gamma}}{\Gamma(n_c + \gamma)} e^{-(t_b + \delta)\mu} \mu^{n_c + \gamma - 1} \right]$$

Clearly the posterior distribution of $\lambda$ and $\mu$ are independent. The posterior marginal distribution of $\mu$ is

$$\pi(\mu/x) = \int_0^\infty \pi(\lambda, \mu/x) d\lambda$$

where

$$\pi(\mu/x) = \frac{(t_b + \delta)^{n_c + \gamma}}{\Gamma(n_c + \gamma)} e^{-(t_b + \delta)\mu} \mu^{n_c + \gamma - 1}$$
The posterior marginal distribution of $\lambda$ is

$$
\pi(\lambda/x) = \int_0^\infty \pi(\lambda, \mu/x).d\mu
$$

$$
= \frac{(t + \beta)^{n_a + \alpha}}{\Gamma(n_a + \alpha)} e^{-(t + \beta)\lambda} \lambda^{n_a + \alpha - 1} \quad (4.8)
$$

Using the posterior joint distribution of $(\lambda, \mu)$ posterior distribution and distribution function of the traffic intensity $\rho = \frac{\lambda}{\mu}$ can be easily obtained and are given by

$$
g(\rho/x) = c. \rho^{n_a + \alpha - 1} \frac{\rho^{n_a + \alpha - 1}}{((t + \beta)\rho + t_b + \delta)^{n_a + n_c + \alpha + \gamma}}, \quad 0 < \rho < \infty \quad (4.9)
$$

where

$$
c = \frac{(t + \beta)^{n_a + \alpha}(t_b + \delta)^{n_c + \gamma}}{\beta(n_c + \gamma, n_a + \alpha)}.
$$

$$
G(u) = \int_{\frac{1}{1+\frac{u}{t_b+\delta}}}^1 \frac{y^{n_c+\gamma-1}(1-y)^{n_a+\alpha-1}dy}{\beta(n_c+\gamma, n_a+\alpha)}. \quad (4.10)
$$

Therefore Bayes estimate of $\mu$, $\lambda$ and $\rho$ are

$$
\mu^* = \frac{n_c + \gamma}{t_b + \delta}
$$

$$
\lambda^* = \frac{n_a + \alpha}{t + \beta}
$$

$$
\rho^* = \frac{(t_b + \delta)(n_a + \alpha)}{(t + \beta)(n_c + \gamma - 1)}. \quad (4.11)
$$

The 100(1 − $\alpha$)% Credible region of $\rho$ is a region $C \subset R'$ such that

$$
\int_C \pi(\rho/x)d\rho = 1 - \alpha. \quad (4.12)
$$

Then the lower limit of the Credible region $a$ and upper limit of the Credible
region \( b \) can be determined such that

\[
\beta(n_c + \gamma, n_a + \alpha, a') = 1 - \frac{\alpha}{2}, \tag{4.13}
\]

and

\[
\beta(n_c + \gamma, n_a + \alpha, b') = \frac{\alpha}{2}, \tag{4.14}
\]

Where

\[
\beta(n_c + \gamma, n_a + \alpha, t) = \int_t^1 \frac{y^{n_c+\gamma-1}(1-y)^{n_a+\alpha-1}dy}{\beta(n_c + \gamma, n_a + \alpha)}, \tag{4.15}
\]

\[
a' = \frac{1}{1 + \frac{t + \beta}{t_a + \delta}}
\]

and

\[
b' = \frac{1}{1 + \frac{t + \beta}{t_b + \delta}}
\]

### 4.2.2 Conditional Bayes Estimates

Here we are considering a stationary M/M/1 queueing system and for such systems \( 0 < \rho < 1 \). So we now consider posterior conditional distribution of the traffic intensity \( \rho \) given \( \rho < 1 \). The conditional density function and conditional distribution function can be obtained as follows

\[
g(\rho, x) = k' \left( t + \beta \right)^{n_a + \alpha} \left( t_b + \delta \right)^{n_c + \gamma} \frac{\rho^{n_a + \alpha - 1}}{\beta(n_c + \gamma, n_a + \alpha) \left( (t + \beta) \rho + t_b + \delta \right)^{n_a + n_c + \alpha + \gamma}}, \tag{4.16}
\]

and

\[
G(u) = k' \int_{\frac{1}{1 + \frac{t + \beta}{t_a + \delta}}}^1 \frac{y^{n_c+\gamma-1}(1-y)^{n_a+\alpha-1}dy}{\beta(n_c + \gamma, n_a + \alpha)}, \tag{4.17}
\]
where
\[ k' = \frac{1}{\sum_{i=n_a+\alpha}^{N} \binom{N}{i} \left( \frac{t_b + \delta}{t + t_b + \beta + \delta} \right)^i \left( \frac{t + \beta}{t + t_b + \beta + \delta} \right)^{N-i}}. \]

and \( N = n_a + \alpha + n_c + \gamma - 1. \)

Hence Bayes estimate of \( \rho \) given \( \rho < 1 \) is

\[ \rho^{**} = \frac{(t_b + \delta)(n_a + \alpha)}{(t + t_b + \beta + \delta)(n_c + \gamma - 1)} \] (4.18)

where

\[ \phi = \frac{\sum_{i=n_a+\alpha+1}^{N=n_a+\alpha+n_c+\gamma-1} \binom{N}{i} \left( \frac{t_b + \delta}{t + t_b + \beta + \delta} \right)^i \left( \frac{t + \beta}{t + t_b + \beta + \delta} \right)^{N-i}}{\sum_{i=n_a+\alpha}^{N=n_a+\alpha+n_c+\gamma-1} \binom{N}{i} \left( \frac{t_b + \delta}{t + t_b + \beta + \delta} \right)^i \left( \frac{t + \beta}{t + t_b + \beta + \delta} \right)^{N-i}}. \]

Here the lower limit of the Credible region \( a \) and upper limit of the Credible region \( b \) can be determined such that

\[ k' \beta(n_c + \gamma, n_a + \alpha, a') = 1 - \frac{\alpha}{2}, \] (4.19)

and

\[ k' \beta(n_c + \gamma, n_a + \alpha, b') = \frac{\alpha}{2}. \] (4.20)

where \( k', a', b' \) and \( \beta(n_c + \gamma, n_a + \alpha, t) \) are as defined above.
4.2.3 Predictive Distribution

We have for an M/M/1 queueing system steady state system size distribution is 
\[ P[N = n/\lambda, \mu] = (1 - \rho)^n \rho^n, \ 0 < \rho < 1. \]
Hence the Predictive distribution of number in the system is

\[
P[N = n/x] = \int_0^1 (1 - \rho)^n \pi(\rho/x) d\rho,
\]

\[= I(n)
\]

\[
\begin{align*}
&\left[ \sum_{i=n_a+\alpha+n}^{N=n_a+\alpha+n} \binom{N}{i} \left( \frac{t + \beta}{t_b + \delta} \right)^i \right] - I(n + 1) \\
&\left[ \sum_{i=n_a+\alpha+n+1}^{N=n_a+\alpha+n+1} \binom{N}{i} \left( \frac{t + \beta}{t_b + \delta} \right)^i \right],
\end{align*}
\]

where

\[
I(n) = \left( \frac{t_b + \delta}{t + \beta} \right)^n \frac{\beta(n_c + \gamma - n, n_a + \alpha + n)}{\beta(n_c + \gamma, n_a + \alpha)}.
\]

4.3 Bayesian Estimation of Queue Parameters by Bivariate Prior with Order Restriction

A stationary M/M/1 queue should satisfy the condition \(0 < \lambda \leq \mu < \infty\) always.

In this section we consider certain bivariate distributions of \((\lambda, \mu)\) with this inherent order restriction on \(\lambda\) and \(\mu\) as the prior distribution and obtain Bayes estimates of the queue parameters.
4.3.1 Bayesian Estimation Using Prior 2

Here we take the joint prior distribution of \((\lambda, \mu)\) as

\[
\pi(\lambda, \mu) = \alpha \beta e^{-(\alpha - \beta)\lambda + \beta \mu}, 0 < \lambda \leq \mu < \infty; \alpha, \beta > 0; \alpha \neq \beta.
\]  

(4.23)

Now using the result \(\int_\alpha^{\infty} \theta^s e^{-\beta \theta} d\theta = \frac{s!}{\beta^{s+1}} e^{-\beta \alpha} \sum_{i=0}^{s} \frac{(\beta \alpha)^i}{i!} \) we can show that the marginal likelihood function of \(x = (t, t_b, n_a, n_c)\) is

\[
m(x) = p_{na} \alpha \beta \frac{n_c!}{(t_b + \beta)^{n_c+1}} \frac{n_a!}{(t + t_b + \alpha)^{n_a+1}} \sum_{i=0}^{n_c} \frac{n_a+i}{i!} \left( \frac{t_b + \beta}{t + t_b + \alpha} \right)^i
\]  

(4.24)

Hence the posterior joint distribution of \((\lambda, \mu)\) is

\[
\pi(\lambda, \mu/x) = \frac{e^{-(t_b+\beta)\mu} \mu^{n_c} e^{-(\alpha - \beta + t)\lambda} \lambda^{n_a}}{\frac{n_c!}{(t_b + \beta)^{n_c+1}} \frac{n_a!}{(t + t_b + \alpha)^{n_a+1}} \sum_{i=0}^{n_c} \frac{n_a+i}{i!} \left( \frac{t_b + \beta}{t + t_b + \alpha} \right)^i}
\]  

(4.25)

Proceeding as in section 2.1 we can obtain easily obtain the posterior distribution of \(\lambda, \mu\) and \(\rho\) as,

\[
\pi(\mu/x) = \frac{e^{-(t_b+\beta)\mu} \mu^{n_c} \left[ 1 - \frac{n_a!}{(\alpha - \beta + t)^{n_a+1}} e^{-(\alpha - \beta + t)\mu} \sum_{i=0}^{n_a} \frac{[(\alpha - \beta + t)\mu]^i}{i!} \right]}{\frac{n_c!}{(t_b + \beta)^{n_c+1}} \frac{n_a!}{(t + t_b + \alpha)^{n_a+1}} \sum_{i=0}^{n_c} \frac{n_a+i}{i!} \left( \frac{t_b + \beta}{t + t_b + \alpha} \right)^i}
\]  

(4.26)
\[
\pi(\lambda/x) = e^{-\frac{\beta + t}{t + \alpha}} (\lambda)^n c \left[ \frac{n_c!}{(t + \beta)^{n_c+1}} e^{-(t + \beta)\lambda} \sum_{i=0}^{n_c} \frac{[t + \beta]^{i}}{i!} \right] \]

where 0 < \lambda < 1, and

\[
g(\rho/x) = c'' \frac{\rho^n}{[(\alpha - \beta + t)\rho + (t + \beta)]^{n_a + n_c + 2}} \]

Therefore the posterior distribution function of the traffic intensity \( \rho \) is

\[
G(u) = \int_0^u g(\rho) d\rho
\]

\[
= \nu' \int_0^1 \frac{y^{n_c} (1-y)^{n_a} dy}{1 + \frac{\alpha - \beta + t}{t + \alpha} u}
\]

where \( \nu' = \frac{1}{\sum_{i=0}^{n_c} \binom{n_a + i}{i} \left( \frac{t + \beta}{t + \alpha} \right)^i \binom{\alpha - \beta + t}{t + \alpha}^{n_a+1}} \)

The Bayes estimates of \( \lambda, \mu \) and \( \rho \) are,

\[
\mu^* = \frac{t + \beta}{n_c + 1} \sum_{i=0}^{n_a} \binom{n_a + i}{i} \left( \frac{t + \beta}{t + \alpha} \right)^i \binom{\alpha - \beta + t}{t + \alpha}^{n_a+1}.
\]

(4.29)
\[ \lambda^* = \frac{n_a + 1}{t + t_b + \alpha} \left[ \sum_{i=0}^{n_c} \binom{n_a + i + 1}{i} \left( \frac{t_b + \beta}{t + t_b + \alpha} \right)^i \right] \left[ \sum_{i=0}^{n_c} \binom{n_a + i}{i} \left( \frac{t_b + \beta}{t + t_b + \alpha} \right)^i \right] \] (4.30)

\[ \rho^* = \frac{(n_a + 1)(t_b + \beta)}{n_c(t + t_b + \alpha)} \left[ \sum_{i=0}^{n_c-1} \binom{n_a + i + 1}{i} \left( \frac{t_b + \beta}{t + t_b + \alpha} \right)^i \right] \left[ \sum_{i=0}^{n_c} \binom{n_a + i}{i} \left( \frac{t_b + \beta}{t + t_b + \alpha} \right)^i \right] . \] (4.31)

In this case the lower limit of the Credible region \( a \) and upper limit of the Credible region \( b \) can be determined such that

\[ \nu' \beta(n_c + 1, n_a + 1, a') = 1 - \frac{\alpha}{2}, \] (4.32)

and

\[ \nu' \beta(n_c + \gamma, n_a + \alpha, b') = \frac{\alpha}{2}. \] (4.33)

where

\[ \nu' = \frac{1}{\sum_{i=0}^{n_c} \binom{n_a + i}{i} \left( \frac{t_b + \beta}{t + t_b + \alpha} \right)^i \left( \frac{\alpha - \beta + t}{t + t_b + \alpha} \right)^{n_a + 1}} \]

\[ a' = \frac{1}{1 + \frac{\alpha - \beta + t}{t_b + \beta}} \]

and

\[ b' = \frac{1}{1 + \frac{\alpha - \beta + t}{t_b + \beta}} \]
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The predictive distribution of number in the system is

\[
P[N = n/x] = \phi.(n_c - n)
\]

\[
\frac{\sum_{i=0}^{n_c-n} \left( \begin{array}{c} n_a + n + i \\ i \end{array} \right) \left( \frac{t_b + \beta}{t + t_b + \alpha} \right)^{n+i}}{\sum_{i=0}^{n_c} \left( \begin{array}{c} n_a + i \\ i \end{array} \right) \left( \frac{t_b + \beta}{t + t_b + \alpha} \right)^{i}}
\]

\[
-\phi.(n_a + n + 1)
\]

\[
\frac{\sum_{i=0}^{n_c-n-1} \left( \begin{array}{c} n_a + n + i + 1 \\ i \end{array} \right) \left( \frac{t_b + \beta}{t + t_b + \alpha} \right)^{n+i+1}}{\sum_{i=0}^{n_c} \left( \begin{array}{c} n_a + i \\ i \end{array} \right) \left( \frac{t_b + \beta}{t + t_b + \alpha} \right)^{i}}
\]

(4.34)

where

\[
\phi = \frac{(n_a + n)!(n_c - n - 1)!}{n_a!n_c!}
\]

### 4.3.2 Bayesian Estimation Using Prior 3

Here we take the joint prior distribution of \((\lambda, \mu)\) as

\[
\pi(\lambda, \mu) = \frac{m\beta \lambda^m}{\mu^{1+m}} e^{-\beta \lambda}, 0 < \lambda \leq \mu < \infty; m > 0, \beta > 0
\]

(4.35)

Now proceeding as in the case of section 3.1 we can easily show that the marginal likelihood function of \(x = (t, t_b, n_a, n_c)\) is

\[
m(x) = m\beta p_{(n_0)} \frac{(n_c - m - 1)!}{t_b^{n_c-m}} \sum_{i=0}^{n_c-m-1} \frac{t_b^i}{i!} \frac{(m + n_a + i)!}{(t + t_b + \beta)^{m+n_a+i+1}}
\]

(4.36)

Hence the posterior joint distribution of
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\[ \pi(\lambda, \mu/x) = \frac{t_b^{n_c-m}(t + t_b + \beta)^m n_a + 1}{(n_c - m - 1)!(m + n_a)!} \sum_{i=0}^{n_c-m-1} \binom{m + n_a + i}{i} \left( \frac{t_b}{t + t_b + \beta} \right)^i \]

Proceeding as in section 4.3.1 we can obtain easily the posterior distribution of \( \lambda, \mu \) and \( \rho \) as,

\[ \pi(\mu/x) = \frac{t_b^{n_c-m}(t + t_b + \beta)^m n_a + 1}{(n_c - m + 1)!(m + n_a)!} \sum_{i=0}^{n_c-m-1} \binom{m + n_a + i}{i} \left( \frac{t_b}{t + t_b + \beta} \right)^i \]

(4.37)

\[ \pi(\lambda/x) = \frac{(t + t_b + \beta)^m n_a + 1}{(m + n_a)!} \sum_{i=0}^{n_c-m-1} \binom{m + n_a + i}{i} \left( \frac{t_b \lambda}{t + t_b + \beta} \right)^i \]

\[ g(\rho/x) = \tau' \frac{\rho^{m + n_a}}{[t + \beta \rho + t_b^{n_a + n_c + 1}]} \]

(4.38)

where \( 0 < \rho < 1 \) and

\[ \tau' = \frac{(t_b^{n_c-m})(t + t_b + \beta)^m n_a + 1}{(n_c - m + 1)!(m + n_a)!} \sum_{i=0}^{n_c-m-1} \binom{m + n_a + i}{i} \left( \frac{t_b}{t + t_b + \beta} \right)^i \]
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Therefor the posterior distribution function of the traffic intensity $\rho$ is

$$G(u) = \eta' \int_{1}^{1} \frac{g^{n_c-m-1}(1-y)^{m+n_a} dy}{\beta(n_c - m, m + n_a + 1)}.$$  

(4.39)

where

$$\eta' = \frac{1}{\sum_{i=0}^{n_c-m-1} \left( \binom{m+n_a+i}{i} \right) \left( \frac{t_b}{t+t_b+\beta} \right)^i \left( \frac{t+\beta}{t+t_b+\beta} \right)^{m+n_a+1}}.$$  

(4.40)

Proceeding as in section 2.1 we can obtain easily obtain Bayes estimates of $\lambda, \mu$ and $\rho$ as,

$$\mu^* = \frac{1 - \sum_{i=0}^{n_a+m} \left( \binom{n_c-m+i}{i} \right) \left( \frac{t+\beta}{t+t_b+\beta} \right)^i \left( \frac{t_b}{t+t_b+\beta} \right)^{n_c-m-1}}{\frac{t_b}{n_c-m} \sum_{i=0}^{n_c-m-1} \left( \binom{m+n_a+i}{i} \right) \left( \frac{t_b}{t+t_b+\beta} \right)^i \left( \frac{t+\beta}{t+t_b+\beta} \right)^{m+n_a+1}}.$$  

(4.41)

$$\lambda^* = \frac{\sum_{i=0}^{n_c-m-1} \left( \binom{m+n_a+i+1}{i} \right) \left( \frac{t_b}{t+t_b+\beta} \right)^i}{\sum_{i=0}^{n_c-m-1} \left( \binom{m+n_a+i}{i} \right) \left( \frac{t_b}{t+t_b+\beta} \right)^i}.$$  

$$\rho^* = \frac{(m+n_a+1) t_b}{(n_c-m-1)(t+t_b+\beta)} \frac{\sum_{i=0}^{n_c-m-2} \left( \binom{m+n_a+i+1}{i} \right) \left( \frac{t_b}{t+t_b+\beta} \right)^i}{\sum_{i=0}^{n_c-m-1} \left( \binom{m+n_a+i}{i} \right) \left( \frac{t_b}{t+t_b+\beta} \right)^i}.$$  

(4.42)
In this case the lower limit of the Credible region \(a\) and upper limit of the Credible region \(b\) can be determined such that

\[
\eta' \beta(n_c - m, n_a + m + 1, a') = 1 - \frac{\alpha}{2}, \tag{4.43}
\]

and

\[
\eta' \beta(n_c - m, n_a + m + 1, b') = \frac{\alpha}{2}. \tag{4.44}
\]

where

\[
\eta' = \frac{1}{\sum_{i=0}^{n_c-m-1} \left( \begin{array}{c} m + n_a + i \\ i \end{array} \right) \left( \frac{t_b}{t + t_b + \beta} \right)^i \left( \frac{t + \beta}{t + t_b + \beta} \right)^{m+n_a+1}}
\]

\[
a' = \frac{1}{1 + \frac{t + \beta}{t_b}} \quad \text{and} \quad b' = \frac{1}{1 + \frac{t + \beta}{t_b}}.
\]

The predictive distribution of number in the system is

\[
P[N = n/x] = \varphi(n_c - m - n - 1) \left[ \sum_{i=0}^{n_c-m-n-1} \left( \begin{array}{c} m + n_a + n + i \\ i \end{array} \right) \left( \frac{t_b}{t + t_b + \beta} \right)^{n+i} \right] \left[ \sum_{i=0}^{n_c-m-n-1} \left( \begin{array}{c} m + n_a + n + i \\ i \end{array} \right) \left( \frac{t_b+}{t + t_b + \beta} \right)^i \right] \]

\[-\varphi.(m+n_a+n+1) \left[ \sum_{i=0}^{n_c-m-n-2} \left( \begin{array}{c} m + n_a + n + i + 1 \\ i \end{array} \right) \left( \frac{t_b}{t + t_b + \beta} \right)^{n+i+1} \right] \left[ \sum_{i=0}^{n_c-m-n-1} \left( \begin{array}{c} m + n_a + n + i \\ i \end{array} \right) \left( \frac{t_b+}{t + t_b + \beta} \right)^i \right] \tag{4.45}
\]

where

\[
\varphi = \frac{(n_c - m - n - 2)!(m + n_a + n)!}{(n_c - m - 1)!(m + n_a)!}.
\]
4.4 Numerical Results

4.4.1 Example 1

We obtained a simulated data set for the Markovian queueing system with $\lambda \sim G(5, 2), \mu \sim G(6, 2)$. Here we take the observations at the time points corresponding to the 200th, 400th, 600th, 800th and 1000th departures from the system. At these points we determine the values of total time, busy time, idle time, number of arrivals and number of departures during the period of time. The values obtained are given in Table-4.1.

Table 4.1:

<table>
<thead>
<tr>
<th>sample</th>
<th>total time</th>
<th>busy time</th>
<th>idle time</th>
<th>arrivals</th>
<th>departures</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>110.3340001</td>
<td>81.64258569</td>
<td>28.69141531</td>
<td>218</td>
<td>200</td>
</tr>
<tr>
<td>2</td>
<td>185.1366352</td>
<td>146.9626385</td>
<td>38.17399673</td>
<td>405</td>
<td>400</td>
</tr>
<tr>
<td>3</td>
<td>275.3817516</td>
<td>227.7611080</td>
<td>47.62064362</td>
<td>603</td>
<td>600</td>
</tr>
<tr>
<td>4</td>
<td>372.2504290</td>
<td>311.4785219</td>
<td>60.77190704</td>
<td>802</td>
<td>800</td>
</tr>
<tr>
<td>5</td>
<td>471.9978722</td>
<td>393.8518412</td>
<td>78.14603103</td>
<td>1002</td>
<td>1000</td>
</tr>
</tbody>
</table>

Table 4.2:

<table>
<thead>
<tr>
<th>sample</th>
<th>$\hat{\lambda}$</th>
<th>$\hat{\mu}$</th>
<th>$\hat{\rho}$</th>
<th>$\lambda^*$</th>
<th>$\mu^*$</th>
<th>$\rho^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.975819</td>
<td>2.449702</td>
<td>0.806555</td>
<td>1.985151</td>
<td>2.462860</td>
<td>0.809967</td>
</tr>
<tr>
<td>2</td>
<td>2.187574</td>
<td>2.721780</td>
<td>0.803729</td>
<td>2.190913</td>
<td>2.725516</td>
<td>0.805837</td>
</tr>
<tr>
<td>3</td>
<td>2.189688</td>
<td>2.634339</td>
<td>0.831209</td>
<td>2.191925</td>
<td>2.637522</td>
<td>0.832428</td>
</tr>
<tr>
<td>4</td>
<td>2.154464</td>
<td>2.568395</td>
<td>0.838836</td>
<td>2.156310</td>
<td>2.571149</td>
<td>0.839698</td>
</tr>
<tr>
<td>5</td>
<td>2.122891</td>
<td>2.539026</td>
<td>0.836104</td>
<td>2.124482</td>
<td>2.541355</td>
<td>0.836796</td>
</tr>
</tbody>
</table>

Using the data sets given in Table-4.1 we calculate the Bayes and maximum likelihood estimates of the queue parameters $\lambda$, $\mu$ and $\rho$ and are given in table2. There is difference between the Bayes and maximum likelihood estimates if the total
time is small and as the time duration increases the the difference between the estimates becomes negligible as we expected.

Table 4.3 and 4.4 give the confidence intervals (CI), credible regions (CR) and conditional credible regions \( CR^* \) for traffic intensity \( \rho \) at significance levels 0.05.
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Finally we compute the predictive distribution of the system size and its graph is displayed in figure 4.1. We select 1048 departure points from this system and observe the system size at these points. Based on this data we compute empirical system size distribution and is plotted in figure 4.2 along with predictive distribution.

4.4.2 Example 2

Using the inter-arrival times and service times in a teletraffic data we simulated a single server Markovian queueing system and observed the system for a period of 49927 unit time in the steady state. During this period there are 252 arrivals and 249 departures. The system was busy for 18675 unit time and idle for 31252 unit time. Hence the maximum likelihood estimates of the arrival rate ($\lambda$), service rate ($\mu$) and traffic intensity($\rho$) are 0.005048, 0.013601 and 0.372589 respectively. Since the time period is sufficiently large, maximum likelihood and Bayes estimates of
queue parameters are coincides.

We divide this period into 55 non-overlapping intervals and computed total time (t), busy time (t_b), number of arrivals (n_a) and number of departures (n_c) for each interval. Based on these observations the maximum likelihood estimates of λ, µ and ρ for each interval are obtained. Using this estimated values of λ and µ we fit three prior distributions and obtained estimates of the parameters of prior distributions. The estimates of the prior1 parameter values are α = 3.649761, β = 712.956761, γ = 3.597131, δ = 211.376920, prior2 parameter values are α = 195.343412, β = 84.044724 and prior2 parameter values are m = 0.6160681, β = 195.3434123

Table 4.5:

<table>
<thead>
<tr>
<th>sample</th>
<th>total time</th>
<th>busy time</th>
<th>idle time</th>
<th>arrivals</th>
<th>departures</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>765</td>
<td>198</td>
<td>567</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>783</td>
<td>223</td>
<td>560</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>947</td>
<td>261</td>
<td>686</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>1471</td>
<td>419</td>
<td>1052</td>
<td>10</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 4.6:

<table>
<thead>
<tr>
<th>sample</th>
<th>λ</th>
<th>̂λ</th>
<th>̂ρ</th>
<th>Err(λ)</th>
<th>Err(̂λ)</th>
<th>Err(̂ρ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.006536</td>
<td>0.025253</td>
<td>0.258824</td>
<td>0.001488</td>
<td>0.011651</td>
<td>-0.113765</td>
</tr>
<tr>
<td>2</td>
<td>0.008940</td>
<td>0.03139</td>
<td>0.284802</td>
<td>0.003892</td>
<td>0.017789</td>
<td>-0.087787</td>
</tr>
<tr>
<td>3</td>
<td>0.006336</td>
<td>0.022989</td>
<td>0.275607</td>
<td>0.001288</td>
<td>0.009387</td>
<td>-0.096982</td>
</tr>
<tr>
<td>4</td>
<td>0.006798</td>
<td>0.02148</td>
<td>0.316489</td>
<td>0.001750</td>
<td>0.007879</td>
<td>-0.056100</td>
</tr>
</tbody>
</table>

Table 4.7:

<table>
<thead>
<tr>
<th>sample</th>
<th>λ*</th>
<th>̂μ*</th>
<th>̂ρ**</th>
<th>Err(λ*)</th>
<th>Err(̂μ*)</th>
<th>Err(̂ρ**)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.005853</td>
<td>0.021001</td>
<td>0.31024</td>
<td>0.000805</td>
<td>0.007399</td>
<td>-0.06235</td>
</tr>
<tr>
<td>2</td>
<td>0.007119</td>
<td>0.024396</td>
<td>0.319451</td>
<td>0.002071</td>
<td>0.010795</td>
<td>-0.05314</td>
</tr>
<tr>
<td>3</td>
<td>0.005813</td>
<td>0.023596</td>
<td>0.319415</td>
<td>0.000766</td>
<td>0.009995</td>
<td>-0.05317</td>
</tr>
<tr>
<td>4</td>
<td>0.00625</td>
<td>0.019984</td>
<td>0.337928</td>
<td>0.001202</td>
<td>0.006382</td>
<td>-0.03466</td>
</tr>
</tbody>
</table>
Chapter 4: Bayesian Inference on Queueing Models

Table 4.8:

<table>
<thead>
<tr>
<th>sample</th>
<th>$\lambda^*$</th>
<th>$\mu^*$</th>
<th>$\rho^*$</th>
<th>Err($\lambda^*$)</th>
<th>Err($\mu^*$)</th>
<th>Err($\rho^*$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00672</td>
<td>0.018058</td>
<td>0.356708</td>
<td>0.001672</td>
<td>0.004457</td>
<td>-0.01588</td>
</tr>
<tr>
<td>2</td>
<td>0.008883</td>
<td>0.023077</td>
<td>0.375866</td>
<td>0.003835</td>
<td>0.009476</td>
<td>0.003277</td>
</tr>
<tr>
<td>3</td>
<td>0.006163</td>
<td>0.017457</td>
<td>0.348607</td>
<td>0.001115</td>
<td>0.003856</td>
<td>-0.02398</td>
</tr>
<tr>
<td>4</td>
<td>0.006914</td>
<td>0.017998</td>
<td>0.379811</td>
<td>0.001866</td>
<td>0.004397</td>
<td>0.007222</td>
</tr>
</tbody>
</table>

Table 4.9:

<table>
<thead>
<tr>
<th>sample</th>
<th>$\lambda^*$</th>
<th>$\mu^*$</th>
<th>$\rho^*$</th>
<th>Err($\lambda^*$)</th>
<th>Err($\mu^*$)</th>
<th>Err($\rho^*$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.005768</td>
<td>0.023444</td>
<td>0.33752</td>
<td>0.000721</td>
<td>0.009843</td>
<td>-0.03507</td>
</tr>
<tr>
<td>2</td>
<td>0.008229</td>
<td>0.029278</td>
<td>0.340859</td>
<td>0.003182</td>
<td>0.015677</td>
<td>-0.03173</td>
</tr>
<tr>
<td>3</td>
<td>0.005923</td>
<td>0.021478</td>
<td>0.352608</td>
<td>0.000875</td>
<td>0.007877</td>
<td>-0.01998</td>
</tr>
<tr>
<td>4</td>
<td>0.006676</td>
<td>0.020339</td>
<td>0.378842</td>
<td>0.001629</td>
<td>0.006738</td>
<td>0.006253</td>
</tr>
</tbody>
</table>

Now for comparing the different estimates we consider 4 intervals and computed total time ($t$), busy time ($t_b$), number of arrivals ($n_a$) and number of departures ($n_c$) for each interval. The data obtained are given in Table 4.5.

The maximum likelihood estimates and Bayes estimates of $\lambda$, $\mu$ and $\rho$ using various prior distributions are given in Tables 4.6 -4.9. The errors with respect to the corresponding estimates based on whole data set are also given in the tables. It can be observed that the error is minimum for Bayes estimates using prior 2 and prior 3.

We also calculate 95% and 99% credible regions for the traffic intensity using three prior distributions and are given in Table 4.10 and Table 4.11. The credible regions are calculated using the beta distribution and since the degrees of freedom of the beta distribution corresponding to the natural prior is higher, the credible regions obtained using this prior have shortest length.

The predictive distributions are also determined using various prior distributions for the entire data set and are almost identical. The predictive distributions are given in the Table 4.12.
Table 4.10:

<table>
<thead>
<tr>
<th>Sample</th>
<th>N C Prior: C R (.01)</th>
<th>Prior2: C R(.01)</th>
<th>Prior3: C R (.01)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.075877, 0.805059)</td>
<td>(0.065133, 0.95889)</td>
<td>(0.062392, 0.848148)</td>
</tr>
<tr>
<td>2</td>
<td>(0.091393, 0.829437)</td>
<td>(0.088268, 0.950997)</td>
<td>(0.078648, 0.872760)</td>
</tr>
<tr>
<td>3</td>
<td>(0.084060, 0.824012)</td>
<td>(0.075477, 0.951746)</td>
<td>(0.074183, 0.869945)</td>
</tr>
<tr>
<td>4</td>
<td>(0.111775, 0.835507)</td>
<td>(0.109905, 0.932212)</td>
<td>(0.108139, 0.893728)</td>
</tr>
</tbody>
</table>

Table 4.11:

<table>
<thead>
<tr>
<th>Sample</th>
<th>N C Prior: C R (.05)</th>
<th>Prior2: C R(.05)</th>
<th>Prior3: C R (.05)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.104833, 0.678258)</td>
<td>(0.097388, 0.842003)</td>
<td>(0.092492, 0.776189)</td>
</tr>
<tr>
<td>2</td>
<td>(0.121748, 0.675077)</td>
<td>(0.123765, 0.826324)</td>
<td>(0.109919, 0.766748)</td>
</tr>
<tr>
<td>3</td>
<td>(0.113858, 0.680290)</td>
<td>(0.108851, 0.825384)</td>
<td>(0.106390, 0.788259)</td>
</tr>
<tr>
<td>4</td>
<td>(0.143807, 0.676212)</td>
<td>(0.146078, 0.792053)</td>
<td>(0.143723, 0.786582)</td>
</tr>
</tbody>
</table>

Table 4.12:

<table>
<thead>
<tr>
<th>System size : n</th>
<th>N C Prior : $p_n$</th>
<th>Prior 2 : $p_n$</th>
<th>Prior 3 : $p_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.6284229</td>
<td>0.62658222</td>
<td>0.62740217</td>
</tr>
<tr>
<td>1</td>
<td>0.2324251</td>
<td>0.23432348</td>
<td>0.23266686</td>
</tr>
<tr>
<td>2</td>
<td>0.0866323</td>
<td>0.08722849</td>
<td>0.08696212</td>
</tr>
<tr>
<td>3</td>
<td>0.0325418</td>
<td>0.03293050</td>
<td>0.03275912</td>
</tr>
<tr>
<td>4</td>
<td>0.0123189</td>
<td>0.01252961</td>
<td>0.01243771</td>
</tr>
<tr>
<td>5</td>
<td>0.0046996</td>
<td>0.00480480</td>
<td>0.00475943</td>
</tr>
<tr>
<td>6</td>
<td>0.0018068</td>
<td>0.00185700</td>
<td>0.00183558</td>
</tr>
<tr>
<td>7</td>
<td>0.0007001</td>
<td>0.00072335</td>
<td>0.00071351</td>
</tr>
<tr>
<td>8</td>
<td>0.0002734</td>
<td>0.00028397</td>
<td>0.00027953</td>
</tr>
<tr>
<td>9</td>
<td>0.0001075</td>
<td>0.00011236</td>
<td>0.00011037</td>
</tr>
<tr>
<td>10</td>
<td>0.0000520</td>
<td>0.00004481</td>
<td>0.0000439</td>
</tr>
</tbody>
</table>