APPENDIX
Some Important Mathematical Concepts and Definitions

This appendix provides some important mathematical terms and definitions of mathematical concept used in the thesis.

1. Convolution

Let \( f(t) \) and \( g(t) \) be two functions of non-negative variable \( T \) then the convolution (or ordinary convolution) of the functions \( f(t) \) and \( g(t) \) is given by

Case I – When \( T \) is a non-negative continuous variable

\[
 f(t) \ast g(t) = \int_{0}^{t} f(t-u) g(u) \, du \text{ or } \int_{0}^{t} g(t-u) f(u) \, du
\]

Case II – When \( T \) is a non-negative integer valued variable

\[
 f(t) \ast g(t) = \sum_{n=0}^{t} f(t-u) g(u) \text{ or } \sum_{n=0}^{t} g(t-u) f(u)
\]

2. Laplace Transform

A transform is merely a mapping of a function from one space to another. While it may be very difficult to solve certain equations directly for a particular function of interest, it is often easier to solve a corresponding equation in terms of a transform of the function and then invert the transform to obtain the function. One particular transform that is quite useful for solving some types of differential equations as well as certain integral equations is the Laplace transform.

2.1 Definition

Let \( f(t) \) be a function of non-negative real variable \( T \), then the Laplace transform (L.T.) of \( f(t) \) is defined by,

\[
 L[f(t)] = \int_{0}^{\infty} e^{-st} f(t) \, dt = f^*(s)
\]

for the range of values \( 's' \) for which the integral exists.

2.2 Properties of Laplace Transform

Some important properties of the Laplace transform of a function are as follows:
(i) **Linearity Property:**

The Laplace transform of the sum of $n$ functions is equal to the sum of their Laplace transforms. Mathematically, if

$$ f(t) = \sum_{i=1}^{n} f_i(t) $$

then, $$ L[f(t)] = \sum_{i=1}^{n} f_i^*(s) $$

More generally, if $C_1, C_2, \ldots, C_n$ are the constants, and

$$ f(t) = \sum_{i=1}^{n} C_i f_i(t) $$

then, $$ L[f(t)] = \sum_{i=1}^{n} C_i f_i^*(s) $$

(ii) **Laplace Transform of Derivative of a Function $f(t)$:**

The Laplace transform of the first derivative of a function $f(t)$ is obtained by multiplying the Laplace transform of $f(t)$ by the argument 's' and subtracting the value of the function at $t = 0$ from this product i.e.

$$ L[f'(t)] = \int_{0}^{\infty} e^{-st} f(t) \, dt $$

$$ = s f^*(s) - f(0) $$

Similarly, the L.T. of second derivative is,

$$ L[f''(t)] = s^2 f^*(s) - s f(0) - f'(0) $$

and that of the $n^{th}$ derivative is given by,

$$ L[f^{(n)}(t)] = s^n f^*(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0) $$

(iii) **Laplace Transform of the Integral of a Function $f(t)$:**

The Laplace Transform of the integral $\int_{0}^{t} f(u) \, du$ is equal to the Laplace transform of the function $f(t)$ multiplied by the inverse of the argument 's', i.e.
\[ L \left[ \int_{0}^{t} f(u) \, du \right] = \int_{0}^{\infty} e^{-st} \int_{0}^{t} f(u) \, du \, dt \]
\[ = \frac{f^* (s)}{s} \]

(iv) **Shifting Theorem:**

Let us assume that
\[ g(t) = \begin{cases} 0 & \text{if } t < u \\ f(t - u) & \text{if } t \geq u \end{cases} \]
then,
\[ L[g(t)] = \int_{0}^{\infty} e^{-st} g(t) \, dt \]
\[ = \int_{0}^{\infty} e^{-st} f(t-u) \, dt = e^{-su} f^* (s) \]
i.e. if the graph of the function \( f(t) \) is shifted to the right by an amount \( u \), its Laplace transform is multiplied by \( e^{-su} \).

(v) **Change of Scale Property:**

This property is given as
\[ L[f(at)] = \int_{0}^{\infty} e^{-st} f(at) \, dt = \frac{1}{a} f^* \left( \frac{s}{a} \right) \]

(vi) **Displacement theorem:**

Suppose the function is multiplied by \( e^{-\lambda t} \), then
\[ L[e^{-\lambda t} f(t)] = \int_{0}^{\infty} e^{-st} e^{-\lambda t} f(t) \, dt = f^* (s + \lambda) \]
i.e. multiplication of a function by \( e^{-\lambda t} \) increases the argument of its Laplace transform by \( e^{-\lambda t} \).

(vii) **Limit Properties:**

Some limit properties are as follows:

(a) \( \lim_{t \to 0} f(t) = \lim_{s \to \infty} s f^* (s) \)
(b) \( \lim_{t \to \infty} f(t) = \lim_{s \to 0} s f^* (s) \)
(c) \( \lim_{t \to 0} \frac{f(t)}{t} = \lim_{s \to 0} s^2 \mathcal{L}^{-1} f(s) \)

(d) \( \lim_{s \to 0} \mathcal{L}^{-1} f(s) = 0 \)

(viii) Laplace Transform of the Convolution:

The Laplace transform of an ordinary convolution of two functions \( f(t) \) and \( g(t) \) is given by,

\[
\mathcal{L}[f(t) \ast g(t)] = \int_0^\infty e^{-st} \left[ \int_0^t f(t-u)g(u) \, du \right] \, dt
\]

\[
= \int_0^\infty g(u) \, du \int_u^\infty e^{-st} f(t-u) \, dt
\]

(on changing the order of integration)

\[
= \int_0^\infty e^{-su} g(u) \, du \int_0^u e^{-sv} f(v) \, dv
\]

\[
= \mathcal{L}^{-1}g(s) \mathcal{L}^{-1}f(s)
\]

i.e. the Laplace transform of the convolution of two functions is the product of their Laplace transform.

3. Laplace Stieltjes Transform

Let \( F(t) \) and \( f(t) \) be the c.d.f. and p.d.f. of r.v. ‘t’ respectively, then

\[
F^{**}(s) = \int_0^\infty e^{-st} \, dF(t) = s \int_0^\infty e^{-st} F(t) \, dt = s F^{*}(s)
\]

\[
= \int_0^\infty e^{-st} f(t) \, dt = f^{*}(s)
\]

i.e. Laplace Stieltjes transform of c.d.f. is equal to the Laplace transform of the corresponding p.d.f.

4. Laplace Stieltjes Transform Convolution

If \( F(t) \) and \( G(t) \) be two real valued distribution functions defined for \( t \geq 0 \), the resulting convolution is again distribution function and the integral

\[
\int_0^t F(t-u) \, dG(u) = \int_0^t G(t-u) \, dF(u) = F(t) \ast G(t)
\]

i.e. \( \text{L.S.T.} [F(t) \ast G(t)] = f^{*}(s) \ast g^{*}(s) \)

is known as Stieltjes convolution of \( F(t) \) and \( G(t) \)