Chapter-5

Convergence Results for Countable Family of Multivalued Quasi Non Expansive Operators Using Some New Multi Step Iterative Procedures.
Chapter 5.

Convergence Results for Countable Family of Multivalued Quasi Non Expansive Operators Using Some New Multi Step Iterative Procedures.

5.1. Introduction

5.1. In this chapter, we study strong and weak convergence results for countable family of quasi non expansive type mappings using some multi step iterative procedures with some conditions in real uniformly convex Banach space. Rate of convergence of some existing and these new multi step iterative schemes is also discussed with the help of a numerical example.

In section 5.2, we discuss the weak convergence results for countable family of quasi non expansive mappings using some new multi step iterative procedures with some conditions in real uniformly convex Banach space. The results proved in this section are generalization of the result proved by zhang et al. [278] and some other results in literature.

In section 5.3, we study the strong convergence results for two new multi step iterative schemes for countable family of quasi non expansive mappings with some conditions in real uniformly convex Banach space.

In section 5.4, we analyze the rate of convergence of above used new multi step iterative schemes with the help of a numerical example using C++ programs and show the convergence of these iterative schemes to common fixed point of countable family of quasi non expansive mappings.

Results Proved in this chapter are submitted in journal of Abstract and Applied Analysis.

5.2. Weak Convergence Results for Some New Multistep Iterative Schemes.
Let $X$ be real Banach space. The convex subset $K$ is called proximinal set if for each $x \in X$ there exists at least one $y \in K$ such that \[\|x - y\| = d(x, K) = \inf \{\|x - k\| : k \in K\}.\]

Every closed convex subset of uniformly convex Banach space is proximinal. We use following notations for multi valued mappings

$C(X)$: Collection of all nonempty compact subsets of $X$.

$P(X)$: Collection of all nonempty proximinal bounded subsets of $X$.

$CB(X)$: Collection of all nonempty bounded closed subsets of $X$.

Let $H$ be Hausdorff metric induced by $d$ of $K$ defined as

\[H(A, B) = \max \{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\},\] for every $A, B \in CB(K)$.

Let $T : K \rightarrow P(K)$, a multivalued mapping is said to be nonexpansive if $H(Tx, Ty) \leq \|x - y\|$, for all $x, y \in K$.

An element $p \in K$ is called fixed point of $T$, if $p \in T(p)$, where the set of all fixed point of $T$ is denoted by $F_T$. The mapping $T$ is said to be quasi-nonexpansive if $F_T \neq \emptyset$ and $H(Tx, Ty) \leq \|x - p\|$ for all $x \in K$ and $p \in F_T$. It is known that every nonexpansive multivalued mapping with $F_T \neq \emptyset$ is quasi-nonexpansive but there exists quasi-nonexpansive mappings which are not nonexpansive. It is well known that if $T$ is quasi-nonexpansive mapping, then $F_T$ is closed.

**Definition 5.2.1:** A map $T : K \rightarrow CB(K)$ is called hemicompact if, for any sequence $\{x_n\}$ in $K$ such that $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p \in K$. It is clear that if $K$ is compact, then every multivalued mapping $T$ is hemicompact.

**Definition 5.2.2:** A Banach space $X$ is said to satisfy Opial's condition, if $z \neq y$ imply that

\[
\limsup_{n \rightarrow \infty} \|x_n - z\| < \limsup_{n \rightarrow \infty} \|x_n - y\|. \tag{5.2.1}
\]
The study of multivalued non expansive mappings is harder than the corresponding theory of single valued non expansive mappings. In 1969, Nadler [146] proved the convergence theorem for multivalued contraction mappings. Then in 1973, Markin [138] studied the multivalued contraction and nonexpansive mappings in Hausdorff metric space. Later in 1997, Hu et al. [91] proved the convergence theorems for finding common fixed point of two multivalued nonexpansive mappings that satisfy certain contractive conditions. Sastry and Babu [232] proved the convergence of Mann and Ishikawa iterates to a fixed point $p$ of the multivalued mapping $T$ with fixed point $p$ under certain conditions. They proved with the help of example that limit of the sequence is different from the point of initial choice. Then Abbas et al. [1] introduced the new one step iterative process to compute the common fixed point of two multivalued nonexpansive mappings in a real uniformly convex Banach space. Let $S, T: K \to P(K)$ be two multivalued nonexpansive mappings. They introduced iteration as follows:

$$
\begin{cases}
  x_0 \in K \\
  x_{n+1} = a_n x_n + b_n y_n + c_n z_n, n \in \mathbb{N}
\end{cases}
$$

(5.2.2)

where, $y_n \in Tx_n$, $z_n \in Sx_n$, such that $\|y_n - p\| \leq d(p, Sx_n)$ and $\|z_n - p\| \leq d(p, Tx_n)$ for $p \in F(S) \cap F(T)$ and $a_n, b_n, c_n \in (0,1)$ satisfying $a_n + b_n + c_n \leq 1$. Then they obtained strong convergence theorems for the proposed process under some basic boundary conditions.

In 2012 Bunyawat and Suantai [30] introduced the one step iterative process as follows:

$$
\begin{cases}
  x_{n+1} = \alpha_{n,0} x_n + \sum_{i=1}^{\infty} \alpha_{n,i} x_{n,i},
\end{cases}
$$

(5.2.3)

where the sequence $\{\alpha_{n,i}\}_{i=0}^{\infty} \subset [0,1)$ satisfying $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ and $x_{n,i} \in T_i x_n$ such that $d(p, x_{n,i}) = d(p, T_i x_n)$ for $i \in \mathbb{N}$. They proved the convergence of iterative processes to common fixed point of countable family of multivalued quasi-nonexpansive mappings in uniformly convex Banach space.
Then Zhang et al. in 2013, introduce the two step iterative process as follows:

\[
\begin{align*}
    x_{n+1} &= \alpha_{n,0}x_n + \sum_{i=1}^{\infty} \alpha_{n,i}y_{n,i} \\
y_n &= \beta_{n,0}x_n + \sum_{i=1}^{\infty} \beta_{n,i}x_{n,i}
\end{align*}
\]

(5.2.4)

where the sequences \( \{\alpha_{n}\}_{i=0}^{\infty}, \{\beta_{n}\}_{i=0}^{\infty} \subseteq [0,1] \) satisfying \( \sum_{i=0}^{\infty} \alpha_{n,i} = 1, \sum_{i=0}^{\infty} \beta_{n,i} = 1 \),

\( y_{n,i} \in T_iy_n \) such that \( d(p, y_{n,i}) = d(p, T_iy_n) \) and \( x_{n,i} \in T_ix_n \) such that \( d(p, x_{n,i}) = d(p, T_ix_n) \) for \( i \in \mathbb{N} \). Zhang et al. extended the results of Bunyawat and Suantai from one countable family to two countable families and also give a new proof for the iteration used in the paper of Abbas et al. (2011).

In the same year Ahmed and Altwqi introduce the three step iterative process as follows:

\[
\begin{align*}
    x_{n+1} &= \alpha_{n,0}x_n + \sum_{i=1}^{p} \alpha_{n,i}y_{n,i} \\
y_n &= \beta_{n,0}x_n + \sum_{i=1}^{p} \beta_{n,i}z_{n,i} \\
z_n &= \gamma_{n,0}x_n + \sum_{i=1}^{p} \gamma_{n,i}x_{n,i}
\end{align*}
\]

(5.2.5)

where \( z_{n,i} \in T_i z_n \), \( y_{n,i} \in T_i y_n \) and \( x_{n,i} \in T_i x_n \) and the sequences \( \{\alpha_{n}\}_{i=1}^{p}, \{\beta_{n}\}_{i=1}^{p}, \{\gamma_{n}\}_{i=0}^{p} \subseteq [0,1] \) satisfying \( \sum_{i=0}^{p} \alpha_{n,i} = 1, \sum_{i=0}^{p} \beta_{n,i} = 1, \sum_{i=0}^{p} \gamma_{n,i} = 1 \), and proved the strong and weak convergence results for three finite family of multivalued nonexpansive mappings.

Different iterative processes have been used to approximate fixed points of multivalued mappings. Many authors have intensively studied the fixed point theorems and got some results. At the same time, they extended these results to many discipline branches, such as control theory, convex optimization, variational inequalities, differential inclusion and economics (see [118,190,255-59, 86,2,240,121,90,50]).
Motivated by [278, 1, 30-31], in this paper, we extend the result of Zhang et al. [278] from two countable family to k- countable family and prove weak and strong convergence results of two new multi-step iterative processes to common fixed point of countable family of multivalued quasi non expansive mappings in a uniformly convex Banach space. Also with the help of numerical example we compare the convergence step of two different multi step iterative processes. We use the following iteration process:

\[
\begin{align*}
x_{n+1} &= \alpha_{n,0}x_n + \sum_{i=1}^{\infty} \alpha_{n,i}u_{i,j}^i, \\
y_n &= \beta_{n,1}x_n + \sum_{i=1}^{\infty} \beta_{n,i}u_{i,j}^i, j = 1, 2, ..., k - 2 \\
\end{align*}
\]

\(5.2.6\)

\[
\begin{align*}
x_{n+1} &= \alpha_{n,0}y_n + \sum_{i=1}^{\infty} \alpha_{n,i}u_{i,j}^i, \\
y_n &= \beta_{n,1}y_n + \sum_{i=1}^{\infty} \beta_{n,i}u_{i,j}^i, j = 1, 2, ..., k - 2 \\
\end{align*}
\]

\(5.2.7\)

Where \(u_{i,j}^i \in T_i^j y_i^j, j = 1, 2, k - 1\) such that \(d(p, u_{i,j}^i) = d(p, T_i^j y_i^j), j = 1, 2, ..., k - 1\) and \(u_{n,i}^i \in T_i^k x_n\) such that \(d(p, u_{n,i}^i) = d(p, T_i^k x_n)\), \(\{\alpha_{n,i}\}_{i=0}^{\infty}\) and \(\{\beta_{n,i}\}_{i=0}^{\infty}\), \(j = 1, 2, ..., k - 1\) are sequences in \([0, 1]\) which satisfies \(\sum_{i=0}^{\infty} \alpha_{n,i} = 1\) and \(\sum_{i=0}^{\infty} \beta_{n,i} = 1, j = 1, 2, ..., k - 1\),

\[
\begin{align*}
\limsup_{n \to \infty} \alpha_{n,0} &< 1 \quad \text{and} \quad \limsup_{n \to \infty} \beta_{n,0}^j < 1, j = 1, 2, ..., k - 2, \\
\liminf_{n \to \infty} \alpha_{n,0} &> 0 \quad \text{and} \quad \liminf_{n \to \infty} \beta_{n,0}^j > 0, j = 1, 2, ..., k - 1.
\end{align*}
\]

**Remark 5.2.3:** If \(k = 1\) and 2 then multi step iteration \((5.2.6)\) reduces to one step, two step iteration \((5.2.3)\) and \((5.2.4)\) defined by Bunyawat and Suantai and Zhang et al.
whereas for $k=3$, $i=p$ multi step iteration (5.2.6) reduces to finite three step iteration (5.2.5) defined by Ahmed and Altwqi.

**Lemma 5.2.4** [31]: Let $X$ be a uniformly convex Banach space, $r > 0$ a positive number and $B_r(0)$ a closed ball of $X$. Then, for any given sequence $\{x_n\}_{n=1}^\infty \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^\infty$ of positive numbers with $\sum_{n=1}^\infty \lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g: [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that, for any positive integer $i, j$ with $i \neq j$

$$\left\| \sum_{n=1}^\infty \lambda_n x_n \right\|^2 \leq \sum_{n=1}^\infty \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|).$$

(5.2.8)

**Lemma 5.2.5** [235]: Suppose that $X$ is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers $n$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of $E$ such that $\limsup_{n \to \infty} \|x_n\| \leq r, \limsup_{n \to \infty} \|y_n\| \leq r$, and $\lim_{y \to 0} x_n + (1-t)y = r$, holds for some $r > 0$ then $\lim_{k \to \infty} \|x_k - y\| = 0$.

**Lemma 5.2.6** [129, 271]: Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following property:

$a_{n+1} \leq (1-t_n) a_n + b_n + t_n c_n$, where $\{t_n\}$, $\{b_n\}$ and $\{c_n\}$ satisfy the following restrictions:

(i) $\sum_{n=1}^\infty t_n = \infty$;

(ii) $\sum_{n=1}^\infty b_n < \infty$;

(iii) $\limsup_{n \to \infty} c_n \leq 0$;

Then, $\{a_n\}$ converges to zero as $n \to \infty$.
Theorem 5.2.7. Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $X$ with Opial's condition. For $i \in N$, let $\{T_i\}_{i=1}^k$ be $k$ sequences of multivalued quasi-nonexpansive mappings from $K$ into $P(K)$ with $F = \bigcap_{i=1}^k \{T_i\} \neq \emptyset$ and $p \in F$. Let $\{x_n\}$ be the sequence defined by (5.2.6) and then it converges weakly to a point $q \in F$.

Proof: Let $p \in F$, first we prove that $\{x_n\}$ is bounded and $\lim_{n \to \infty} \|x_n - p\|$ exists. Now from Lemma 5.2.4 and (5.2.6), we have

$$
\|x_{n+1} - p\| = \|\beta_{n+1}^* x_n + \sum_{i=1}^k \beta_{n+1}^* u_{n+i} - p\|
\leq \beta_{n+1}^* \|x_n - p\| + \sum_{i=1}^k \beta_{n+1}^* \|u_{n+i} - p\| - \beta_{n+1}^* \beta_{n+1}^* g(\|x_n - u_{n+i}\|)
\leq \beta_{n+1}^* \|x_n - p\| + \sum_{i=1}^k \beta_{n+1}^* (d(u_{n+i}, T_{n+i} p)) - \beta_{n+1}^* \beta_{n+1}^* g(\|x_n - u_{n+i}\|)
\leq \beta_{n+1}^* \|x_n - p\| + \sum_{i=1}^k \beta_{n+1}^* (H(T_{n+i} x_n, T_{n+i} p)) - \beta_{n+1}^* \beta_{n+1}^* g(\|x_n - u_{n+i}\|)
\leq \beta_{n+1}^* \|x_n - p\| + \sum_{i=1}^k \beta_{n+1}^* \|x_n - p\| - \beta_{n+1}^* \beta_{n+1}^* g(\|x_n - u_{n+i}\|)
\leq \|x_n - p\| - \beta_{n+1}^* \beta_{n+1}^* g(\|x_n - u_{n+i}\|)
$$

(5.2.9)

$$
\|x_{n+1} - p\| = \|\beta_{n+1}^* x_n + \sum_{i=1}^k \beta_{n+1}^* u_{n+i} - p\|
\leq \beta_{n+1}^* \|x_n - p\| + \sum_{i=1}^k \beta_{n+1}^* \|u_{n+i} - p\| - \beta_{n+1}^* \beta_{n+1}^* g(\|x_n - u_{n+i}\|)
\leq \beta_{n+1}^* \|x_n - p\| + \sum_{i=1}^k \beta_{n+1}^* d(u_{n+i}, T_{n+i} p) - \beta_{n+1}^* \beta_{n+1}^* g(\|x_n - u_{n+i}\|)
\leq \beta_{n+1}^* \|x_n - p\| + \sum_{i=1}^k \beta_{n+1}^* H(T_{n+i} x_n, T_{n+i} p) - \beta_{n+1}^* \beta_{n+1}^* g(\|x_n - u_{n+i}\|)
\leq \beta_{n+1}^* \|x_n - p\| + \sum_{i=1}^k \beta_{n+1}^* \|x_n - p\| - \beta_{n+1}^* \beta_{n+1}^* g(\|x_n - u_{n+i}\|)
\leq \beta_{n+1}^* \|x_n - p\| - \beta_{n+1}^* \beta_{n+1}^* g(\|x_n - u_{n+i}\|)
\leq \|x_n - p\| - (1 - \beta_{n+1}^*) \beta_{n+1}^* \beta_{n+1}^* g(\|x_n - u_{n+i}\|)
$$

(5.2.10)
Similarly, we get

\[
\|v_i^* - p\| \leq \beta_i^2 \|x - p\|^2 + \sum_{j=1}^{\infty} \beta_j^2 \|u_{i,j}^* - p\|^2 - \beta_i^2 \beta_j^2 \alpha_j \|y_{i,j} - u_{i,j}^*\|
\]

\[
\leq \beta_i^2 \|x - p\|^2 + \sum_{j=1}^{\infty} \beta_j^2 \|u_{i,j}^* - p\|^2 - \beta_i^2 \beta_j^2 \alpha_j \|y_{i,j} - u_{i,j}^*\|
\]

\[
\leq \beta_i^2 \|x - p\|^2 + \sum_{j=1}^{\infty} \beta_j^2 H(T_{i,j}^*, y_{i,j}^*) \beta_i^2 \beta_j^2 \alpha_j \|y_{i,j} - u_{i,j}^*\|
\]

\[
\leq \beta_i^2 \|x - p\|^2 + \sum_{j=1}^{\infty} \beta_j^2 \beta_i^2 \beta_j^2 \alpha_j \|y_{i,j} - u_{i,j}^*\|
\]

\[
\|v_i^* - p\| \leq \beta_i^2 \|x - p\|^2 + \sum_{j=1}^{\infty} \beta_j^2 \|u_{i,j}^* - p\|^2 - \beta_i^2 \beta_j^2 \alpha_j \|y_{i,j} - u_{i,j}^*\|
\]

Now continuing like this in last, we have

\[
\|v_i^* - p\| \leq \|x - p\| - (1 - \beta_i^2) ... (1 - \beta_i^2) \beta_i^2 \beta_j^2 \alpha_j \|y_{i,j} - u_{i,j}^*\|
\]

\[
\|x_{i,n} - p\| = \|x_{i,n} - p\|^2 + \sum_{j=1}^{\infty} \|u_{i,j}^* - p\|^2
\]

\[
\leq \alpha_{i,n} \|x - p\|^2 + \sum_{j=1}^{\infty} \alpha_{i,n} \|u_{i,j}^* - p\|^2
\]

\[
\leq \alpha_{i,n} \|x - p\|^2 + \sum_{j=1}^{\infty} \alpha_{i,n} \|y_{i,j} - u_{i,j}^*\|^2
\]

By putting (5.2.12) in (5.2.13), we get

\[
\|x_{i,n} - p\| \leq \|x - p\| - (1 - \alpha_{i,n}) ... (1 - \alpha_{i,n}) \beta_i^2 \beta_j^2 \alpha_j \|y_{i,j} - u_{i,j}^*\|
\]
So from (5.2.14), we say that \( \{||x_n - p||\} \) is nondecreasing and bounded and hence, \( \{x_n\} \) is bounded and \( \lim_{n \to \infty} ||x_n - p|| \) exists.

Now we prove that \( \lim_{n \to \infty} ||x_n - u_{n,m}^k|| = 0 \) and \( \lim_{n \to \infty} ||y_{n,j}^k - u_{n,m}^j|| = 0 \) where \( j = 1, 2 \ldots k-1 \).

From (2.6), we can write as

\[
(1-\alpha_{n,0})(1-\beta_{n,0}^1)\ldots(1-\beta_{n,0}^{k-2})(1-\beta_{n,0}^{k-1})\beta_{n,0}^{k-1}\beta_{n,m}^{k,m}g(\|x_n - u_{n,m}^k\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \tag{5.2.15}
\]

Since, we assume that

\[
\begin{align*}
\limsup_{n \to \infty} \alpha_{n,0} < 1, \limsup_{n \to \infty} \beta_{n,0}^j < 1, j = 1, 2 \ldots k-2, \liminf_{n \to \infty} \alpha_{n,0}\alpha_{n,m} > 0, \\
\liminf_{n \to \infty} \beta_{n,0}^j\beta_{n,0}^i > 0, j = 1, 2 \ldots k-1 \text{ and } \lim\sup_{n \to \infty} g(\|x_n - u_{n,m}^k\|) = 0
\end{align*}
\]

and from the continuity of \( g \), we have

\[
\lim_{n \to \infty} ||x_n - u_{n,m}^k|| = 0, \text{ for each } m \in N \tag{5.2.16}
\]

Now we will prove that \( \lim_{n \to \infty} ||x_n - u_{n,m}^j|| = 0 \), for each \( j = 1, 2 \ldots k-1 \)

From (5.2.9) and (5.2.10), we have

\[
\|y_{n}^{k-2} - p\|^2 \leq \|x_n - p\|^2 - \beta_{n,0}^{k-2} \beta_{n,m}^{k,m}g(\|x_n - u_{n,m}^{k-1}\|)
\]

Now by putting (5.2.10) in (5.2.11), we have

\[
\|y_{n}^{k-3} - p\|^2 \leq \|x_n - p\|^2 - (1 - \beta_{n,0}^{k-3}) \beta_{n,0}^{k-2} \beta_{n,m}^{k,m}g(\|x_n - u_{n,m}^{k-1}\|) \tag{5.2.17}
\]

Now continuing like this in last we have

\[
\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - (1 - \alpha_{n,0})(1 - \beta_{n,0}^1)\ldots(1 - \beta_{n,0}^{k-1})\beta_{n,0}^{k-2} \beta_{n,m}^{k,m}g(\|x_n - u_{n,m}^{k-1}\|) \tag{5.2.18}
\]

we can write it as
\[(1 - \alpha_{n,0})(1 - \beta_{n,0}) \cdots (1 - \beta_{n,k-2}^{*})\beta_{n,k-2}^{*2} g(\|x_{n} - u_{n,m}^{k}\|) \leq \|x_{n} - p\|^{2} - \|x_{n+1} - p\|^{2} \quad (5.2.19)\]

Since, we assume that
\[
\limsup_{n \to \infty} \alpha_{n,0} < 1, \limsup_{n \to \infty} \beta_{n,j}^{*} < 1, j = 1, 2, \ldots, k - 2, \liminf_{n \to \infty} \alpha_{n,m} > 0,
\]

\[
\liminf_{n \to \infty} \beta_{n,j}^{*} > 0, j = 1, 2, \ldots, k - 1 \quad \text{and} \quad \lim \|x_{n} - p\| \quad \text{exists}, \quad \text{we have} \quad \lim g(\|x_{n} - u_{n,m}^{k}\|) = 0
\]

and from the continuity of \(g\), we have
\[
\lim_{n \to \infty} \|x_{n} - u_{n,m}^{k}\| = 0, \text{ for each } m \in N \quad (5.2.20)
\]

So repeating these steps for different values of \(j = 1, 2, \ldots, k-2\), we have
\[
\lim_{n \to \infty} \|x_{n} - u_{n,m}^{k}\| = 0, \text{ for each } m \in N, \text{ where } j = 1, 2, \ldots, k - 1. \quad (5.2.21)
\]

Next, we prove that \(\lim_{n \to \infty} \|y_{n}^{j} - u_{n,m}^{j}\| = 0\), for each \(m \in N\), where \(j = 1, 2, \ldots, k - 1\).

From (1.2), we have
\[
\|x_{n} - y_{n}^{j}\| = \|\beta_{n,j}^{*}x_{n} + \sum_{i=1}^{\infty} \beta_{n,j}^{*} u_{n,j}^{i+1} - x_{n}\| \
\leq \sum_{i=1}^{\infty} \beta_{n,j}^{*} \|x_{n} - u_{n,j}^{i+1}\|
\]

From (5.2.21), we have
\[
\lim_{n \to \infty} \|x_{n} - y_{n}^{j}\| = 0, \text{ where } j = 1, 2, \ldots, k - 1. \quad (5.2.22)
\]

By using triangle inequality, we have
\[
\|y_{n}^{j} - u_{n,m}^{j}\| \leq \|y_{n}^{j} - x_{n}\| + \|x_{n} - u_{n,m}^{j}\|
\]

Together with (5.2.21) and (5.2.22), we have
\[
\lim_{n \to \infty} \|y_{n}^{j} - u_{n,m}^{j}\| = 0, \text{ where } j = 1, 2, \ldots, k - 1, \text{ for each } m \in N. \quad (5.2.23)
\]
Now, we prove that \( \{x_n\} \) converges weakly to a point \( q \in F \). Since, we have proved that, \( \{x_n\} \) is bounded, so there exists a subsequence \( \{x_{n_k}\}_{k=1}^\infty \) of \( \{x_n\} \) such that \( x_{n_k} \) converges weakly to \( q \in K \), using (5.2.22), we can say \( y_n^j \) converges weakly to \( q \in K \), for \( j=1, 2, \ldots k-1 \), Now suppose that there exists \( i \in N \), such that \( T_i/q \neq q \), for \( j=1,2,\ldots k \), then by Opial condition we have,

\[
\limsup_{n \to \infty} \|x_n - q\| < \limsup_{n \to \infty} \|x_n - T_i^j q\| \tag{5.2.24}
\]

\[
\limsup_{n \to \infty} \|y_n^j - q\| < \limsup_{n \to \infty} \|y_n^j - T_i^j q\| \tag{5.2.25}
\]

As, \( \{T_i^j\}_{j=1}^k \) be \( k \)-multivalued quasi nonexpansive mappings, we have

\[
\|x_n - T_i^k q\| \leq \|x_n - u_n^k\| + \|u_n^k - T_i^k q\|
\leq \|x_n - u_n^k\| + d(T_i^k x_n, T_i^k q)
\leq \|x_n - u_n^k\| + H(T_i^k x_n, T_i^k q) \tag{5.2.26}
\]

\[
\|y_n^j - T_i^j q\| \leq \|y_n^j - u_n^j\| + \|u_n^j - T_i^j q\|
\leq \|y_n^j - u_n^j\| + d(T_i^j y_n^j, T_i^j q)
\leq \|y_n^j - u_n^j\| + H(T_i^j y_n^j, T_i^j q) \tag{5.2.27}
\]

where \( j=1,2,\ldots k-1 \).

Taking \( \limsup \) both sides of (2.26) and (2.27) and from (5.2.20) and (5.2.23), we have

\[
\limsup_{k \to \infty} \|x_n - T_i^k q\| \leq \limsup_{k \to \infty} \|x_n - q\| \tag{5.2.28}
\]

\[
\limsup_{k \to \infty} \|y_n^j - T_i^j q\| \leq \limsup_{k \to \infty} \|y_n^j - q\| \tag{5.2.29}
\]
Now combining (5.2.26) with (5.2.28) and (5.2.27) with (5.2.29), we have

$$\limsup_{k \to \infty} \|x_{n_k} - q\| < \limsup_{k \to \infty} \|x_{n_k} - q\|$$

$$\limsup_{k \to \infty} \|y_{n'_k} - q\| < \limsup_{k \to \infty} \|y_{n'_k} - q\|$$

Which gives contradiction so we have \( T'_i q = q \), for \( j=1,2,...,k \) and \( i \in N \), this implies \( q \in F \). Now we prove that \( \{x_n\} \) converges weakly to \( q \). Let \( \{x_{n_k}\} \) be another subsequence of \( \{x_n\} \) converges weakly to some \( r \in K \). Again as above, we conclude \( r \in F \). We show that \( q=r \). Let \( q \neq r \), since \( \lim \|x_n - p\| \) exists for every \( p \in F \). From (1.1), we have

$$\lim_{n \to \infty} \|x_n - q\| = \limsup_{k \to \infty} \|x_{n_k} - q\| < \limsup_{k \to \infty} \|x_{n_k} - r\|$$

$$= \limsup_{k \to \infty} \|x_{n_k} - r\| < \limsup_{k \to \infty} \|x_{n_k} - q\|$$

$$= \lim_{k \to \infty} \|x_{n_k} - q\|$$

(5.2.30)

It implies that \( \lim_{n \to \infty} \|x_n - q\| < \lim_{n \to \infty} \|x_n - q\| \), a contradiction. So we have \( q=r \). It means \( \{x_n\} \) converges weakly to \( q \) as \( n \to \infty \).

For \( T'_i = T \), \( T'_j = S \), \( j=1,2,...,k-1 \) and \( T'_k = W \), Theorem 2.1 reduces to following corollary.

**Corollary 5.2.8:** Let \( K \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \) with Opial's condition. Let \( T \), \( S \), and \( W \) be three multivalued quasi-nonexpansive mappings from \( K \) into \( P(K) \) with \( F = F(T) \cap F(S) \cap F(W) \neq \emptyset \) and \( p \in F \). Let \( \{x_n\} \) be the iteration defined as

\[
\begin{align*}
x_{n+1} &= \alpha_n x_n + (1-\alpha_n)Ty_n \\
y_n &= \beta_n x_n + (1-\beta_n)Sz_n \\
z_n &= \gamma_n x_n + (1-\gamma_n)Wx_n
\end{align*}
\]
where \( y_n \in T y_n \) such that \( d(p,y_n) = d(p,T y_n) \), \( z_n \in S z_n \) such that \( d(p,z_n) = d(p,S z_n) \) and \( x_n \in W x_n \) such that \( d(p,x_n) = d(p,W x_n) \), \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are sequences in \([0,1]\)

which satisfies \( \limsup_{n \to \infty} \alpha_n < 1 \) and \( \limsup_{n \to \infty} \beta_n < 1 \),

\[ \liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0, \liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0 \] and \( \liminf_{n \to \infty} \gamma_n (1 - \gamma_n) > 0 \), then \( \{x_n\} \) converges weakly to a point \( q \in F \).

**Theorem 5.2.9.** Let \( K \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \) with Opial’s condition. For \( i \in \mathbb{N} \), let \( \{T_i^j\}_{i=1}^k \) be \( k \) sequences of multivalued quasi-nonexpansive mappings from \( K \) into \( (K) \) with \( F = \bigcap_{j=1}^k T_i^j \neq \emptyset \) and \( p \in F \). Let \( \{x_n\} \) be the sequence defined by (1.7) and then it converges weakly to a point \( q \in F \).

Proof: Let \( p \in F \), first we prove that \( \{x_n\} \) is bounded and \( \lim_{n \to \infty} \|x_n - p\| \) exists. Now from Lemma 1.2 and (5.2.7), we have

\[
\|y_{n+1} - p\|^2 = \|\beta_{n+1} x_n + \sum_{i=1}^n \beta_n u_{n,i} - p\|^2 \\
\leq \beta_{n+1}^2 \|x_n - p\|^2 + \sum_{i=1}^n \beta_{n,i}^2 \|u_{n,i} - p\|^2 - \beta_{n+1} \beta_{n,0}^2 g(\|x_n - u_{n,n}\|) \\
\leq \beta_{n+1} \|x_n - p\|^2 + \sum_{i=1}^n \beta_{n,i} (d(u_{n,i}, T_i p))^2 - \beta_{n,0} \beta_{n,0}^2 g(\|x_n - u_{n,n}\|) \\
\leq \beta_{n,0}^2 \|x_n - p\|^2 + \sum_{i=1}^n \beta_{n,i} \|H(T_i x_n, T_i p)\|^2 - \beta_{n,0} \beta_{n,0} g(\|x_n - u_{n,n}\|) \\
\leq \|x_n - p\|^2 - \beta_{n,0} \beta_{n,0}^2 g(\|x_n - u_{n,n}\|) \\
\leq \|x_n - p\|^2 - \beta_{n,0} \beta_{n,n}^2 g(\|x_n - u_{n,n}\|)
\]

(5.2.32)
Similarly, we get

\[ \| y_{k+1} - p \| \leq \beta_{n,0} \| y_{k+1} - p \| + \sum_{i=1}^{n} \beta_{n,i} \| u_{k+1} - p \| - \beta_{n,0} g(\| y_{k+1} - u_{k+1} \|) \]

\[ \leq \beta_{n,0} \| y_{k+1} - p \| + \sum_{i=1}^{n} \beta_{n,i} d(u_{k+1}, T_{k+1} p) - \beta_{n,0} g(\| y_{k+1} - u_{k+1} \|) \]

\[ \leq \beta_{n,0} \| y_{k+1} - p \| + \sum_{i=1}^{n} \beta_{n,i} H(T_{k+1} y_{k+1}, T_{k+1} p) - \beta_{n,0} \beta_{n,0} g(\| y_{k+1} - u_{k+1} \|) \]

\[ \leq \beta_{n,0} \| y_{k+1} - p \| + \sum_{i=1}^{n} \beta_{n,i} \| y_{k+1} - p \| - \beta_{n,0} \beta_{n,0} g(\| y_{k+1} - u_{k+1} \|) \]

\[ \leq \| y_{k+1} - p \| - \beta_{n,0} \beta_{n,0} g(\| y_{k+1} - u_{k+1} \|) \]

\[ \leq \| x_{k} - p \| - \beta_{n,0} \beta_{n,0} g(\| x_{k} - u_{k} \|) \]

(5.2.33)

Similarly, we get

\[ \| y_{n+1} - p \| = \| \beta_{n,0} y_{n+2} + \sum_{i=1}^{n} \beta_{n,i} u_{n+2} - p \| \]

\[ \leq \beta_{n,0} \| y_{n+2} - p \| + \sum_{i=1}^{n} \beta_{n,i} \| u_{n+2} - p \| - \beta_{n,0} \beta_{n,0} g(\| y_{n+2} - u_{n+2} \|) \]

\[ \leq \beta_{n,0} \| y_{n+2} - p \| + \sum_{i=1}^{n} \beta_{n,i} d(u_{n+2}, T_{n+2} p) - \beta_{n,0} \beta_{n,0} g(\| y_{n+2} - u_{n+2} \|) \]

\[ \leq \beta_{n,0} \| y_{n+2} - p \| + \sum_{i=1}^{n} \beta_{n,i} H(T_{n+2} y_{n+2}, T_{n+2} p) - \beta_{n,0} \beta_{n,0} \beta_{n,0} g(\| y_{n+2} - u_{n+2} \|) \]

\[ \leq \beta_{n,0} \| y_{n+2} - p \| + \sum_{i=1}^{n} \beta_{n,i} \| y_{n+2} - p \| - \beta_{n,0} \beta_{n,0} \beta_{n,0} g(\| y_{n+2} - u_{n+2} \|) \]

\[ \leq \| y_{n+2} - p \| - \beta_{n,0} \beta_{n,0} \beta_{n,0} g(\| y_{n+2} - u_{n+2} \|) \]

\[ \leq \| x_{n} - p \| - \beta_{n,0} \beta_{n,0} \beta_{n,0} g(\| x_{n} - u_{n} \|) \]

(5.2.34)

Now continuing like this, we get

\[ \| y_{n} - p \| \leq \| x_{n} - p \| - \beta_{n,0} \beta_{n,0} \beta_{n,0} g(\| x_{n} - u_{n} \|) \]

(5.2.35)
By putting (5.2.35) in (5.2.36), we get
\[ \| x_{n+1} - p \|^2 = \| \alpha_{n,0} y_n^i + \sum_{j=1}^{\infty} \alpha_{n,j} u_{n,j} - p \|^2 \]
\[ \leq \alpha_{n,0} \| y_n^i - p \|^2 + \sum_{j=1}^{\infty} \alpha_{n,j} \| u_{n,j} - p \|^2 \]
\[ \leq \alpha_{n,0} \| y_n^i - p \|^2 + \sum_{j=1}^{\infty} \alpha_{n,j} (d(u_{n,j}, T_p))^2 \]
\[ \leq \alpha_{n,0} \| y_n^i - p \|^2 + \sum_{j=1}^{\infty} \alpha_{n,j} \| y_n^i - p \|^2 \]
\[ \leq \| y_n^i - p \|^2 \] (5.2.36)

So from (5.2.37), we say that \( \| x_n - p \| \) is nondecreasing and bounded and hence, \( \{ x_n \} \) is bounded and \( \lim \| x_n - p \| \) exists.

Now we prove that \( \lim \| x_n - u_{n,m} \| = 0 \), \( \lim \| x_n - y_n^i \| = 0 \) and \( \lim \| y_n^i - u_{n,m} \| = 0 \) where \( j = 1, 2, \ldots, k-1 \).

From (5.2.37), we can write as
\[ \beta_{n,0}^k \beta_{n,n}^i g(\| x_n - u_{n,m}^i \|) \leq \| x_n - p \|^2 - \| x_{n+1} - p \|^2 \] (5.2.38)

Since, we assume that
\[ \limsup_{n \to \infty} \alpha_{n,0} < 1, \quad \limsup_{n \to \infty} \beta_{n,0}^j < 1, \quad j = 1, 2, \ldots, k-2, \quad \liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,m} > 0, \]
\[ \liminf_{n \to \infty} \beta_{n,0}^j \beta_{n,j}^i > 0, \quad j = 1, 2, \ldots, k-1 \]
and \( \lim \| x_n - p \| \) exists, we have \( \lim g(\| x_n - u_{n,m}^i \|) = 0 \) and from the continuity of \( g \), we have
\[ \lim_{n \to \infty} \| x_n - u_{n,m}^i \| = 0, \quad \text{for each} \quad m \in \mathbb{N} \] (5.2.39)
From (5.2.32) and (5.2.33), we have

\[ \| y_n - p \|^2 \leq \| x_n - p \|^2 - \beta_n^{k-2} \frac{1}{\alpha_n} \| y_n^{k-1} - u_{n,m} \| \]

Now by putting (5.2.33) in (5.2.34), we have

\[ \| y_n^{k-3} - p \|^2 \leq \| x_n - p \|^2 - \beta_n^{k-2} \beta_n^{k-3} \frac{1}{\alpha_n} g(\| y_n^{k-1} - u_{n,m} \|) \]  \hspace{1cm} (5.2.40)

Now continuing in this way, we have

\[ \| x_{n+1} - p \|^2 \leq \| x_n - p \|^2 - \beta_n^{k-2} \beta_n^{k-3} \frac{1}{\alpha_n} g(\| y_n^{k-1} - u_{n,m} \|) \]  \hspace{1cm} (5.2.41)

we can write it as

\[ \beta_n^{k-2} \beta_n^{k-3} \frac{1}{\alpha_n} g(\| y_n^{k-1} - u_{n,m} \|) \leq \| x_n - p \|^2 - \| x_{n+1} - p \|^2 \]  \hspace{1cm} (5.2.42)

Since, we assume that

\[ \lim_{n \to \infty} \beta_n^{j} \beta_n^{j} > 0, j = 1, 2, \ldots, k - 1 \quad \text{and} \quad \lim_{n \to \infty} \| x_n - p \| \text{ exists, we have} \quad \lim_{n \to \infty} g(\| y_n^{k-1} - u_{n,m} \|) = 0 \]

and from the continuity of \( g \), we have

\[ \lim_{n \to \infty} \| y_n^{k-1} - u_{n,m} \| = 0, \quad \text{for each} \quad m \in N \]  \hspace{1cm} (5.2.43)

So repeating these steps for different values of \( j = 1, 2, \ldots, k - 2 \), we have

\[ \lim_{n \to \infty} \| y_n^{j} - u_{n,m} \| = 0, \quad \text{for each} \quad m \in N, \quad \text{where} \quad j = 1, 2, \ldots, k - 1 \]  \hspace{1cm} (5.2.44)

Now, we prove that \( \lim_{n \to \infty} \| x_n - y_n^{j} \| = 0 \), from (1.7), we have

\[ \| x_n - y_n^{k-1} \| = \left\| x_n - \beta_n^{k-1} x_{n+1} + \sum_{j=1}^{k-1} \beta_n^{k-1} u_{n,j} \right\|, \quad \text{where} \quad u_{n,j} \in T_x^n x_n \]
Taking limit \( n \to \infty \), both side of (5.2.45) and using (5.2.39), we have

\[
\lim \| x_{B} - 1 \| = 0 \quad (5.2.46)
\]

Again using (5.2.7), we have

\[
\left\| x_{n} - y_{n}^{j-2} \right\| = \left\| x_{n} - \beta_{n,0}^{j-2} x_{n} - \sum_{i=1}^{n} \beta_{n,i}^{j-2} u_{n,i}^{j-2} \right\| 
\leq \beta_{n,0}^{j-2} \left\| x_{n} - y_{n}^{j-1} \right\| + \beta_{n,i}^{j-2} \left\| x_{n} - u_{n,i}^{j-2} \right\|
\]

\[
\left\| x_{n} - u_{n,i}^{j-2} \right\| \leq \left\| x_{n} - y_{n}^{j-1} \right\| + \left\| y_{n}^{j-1} - u_{n,i}^{j-1} \right\|
\]

From (5.2.44) and (5.2.46), we have

\[
\lim_{n \to \infty} \left\| x_{n} - y_{n}^{j-2} \right\| = 0 , \text{ Now repeating these steps for all values of } j=1, 2...k-3, \text{ we get,}
\]

\[
\lim_{n \to \infty} \left\| x_{n} - y_{n}^{j} \right\| = 0 , \text{ where } j=1,2...k-1. \quad (5.2.47)
\]

Now, we prove that \( \{x_{n}\} \) converges weakly to a point \( q \in F \). Since, we have proved that, \( \{x_{n}\} \) is bounded, so there exists a subsequence \( \{x_{n_{k}}\}_{k=1}^{\infty} \) of \( \{x_{n}\} \) such that \( x_{n_{k}} \) converges weakly to \( q \in K \), using (2.39), we can say \( y_{n_{k}}^{j} \) converges weakly to \( q \in K \), for \( j=1, 2...k-1 \), Now suppose that there exists \( i \in N \), such that \( T_{i}^{j}/q \neq q \), for \( j=1,2,...,k \), then by Opial condition we have,

\[
\lim_{n \to \infty} \sup \left\| x_{n} - q \right\| < \lim_{n \to \infty} \sup \left\| x_{n} - T_{i}^{j} q \right\| \quad (5.2.48)
\]

\[
\lim_{n \to \infty} \sup \left\| y_{n}^{j} - q \right\| < \lim_{n \to \infty} \sup \left\| y_{n}^{j} - T_{i}^{j} q \right\| , \text{ where } j=1,2,...k-1 \quad (5.2.49)
\]

As, \( \{T_{i}^{j}\}_{j=1}^{k} \) be \( k \)- multivalued quasi nonexpansive mappings, we have
\[
\|x_n - T_i^i q\| \leq \|x_n - u_{n,j}\| + \|u_{n,j} - T_i^i q\|
\]
\[
\leq \|x_n - u_{n,j}\| + d(T_i^i x_n, T_i^i q)
\]
\[
\leq \|x_n - u_{n,j}\| + H(T_i^i x_n, T_i^i q)
\]
\[
\leq \|x_n - u_{n,j}\| + \|x_n - q\|
\]  
\[\text{(5.2.50)}\]

\[
\|y_{n}' - T_i'^i q\| \leq \|y_{n}' - u_{n,j}'\| + \|u_{n,j}' - T_i'^i q\|
\]
\[
\leq \|y_{n}' - u_{n,j}'\| + d(T_i'^i y_{n}', T_i'^i q)
\]
\[
\leq \|y_{n}' - u_{n,j}'\| + H(T_i'^i y_{n}', T_i'^i q)
\]
\[
\leq \|y_{n}' - u_{n,j}'\| + \|y_{n}' - q\|
\]  
\[\text{(5.2.51)}\]

where \(j = 1, 2, \ldots, k-1\).

Taking \(\limsup\) both sides of (5.2.50) and (5.2.51) and from (5.2.39) and (5.2.44), we have

\[
\limsup_{k \to \infty} \|x_n - T_i^i q\| \leq \limsup_{k \to \infty} \|x_n - q\|
\]  
\[\text{(5.2.52)}\]

\[
\limsup_{k \to \infty} \|y_{n}' - T_i'^i q\| \leq \limsup_{k \to \infty} \|y_{n}' - q\|
\]  
\[\text{(5.2.53)}\]

Now combining (5.2.48) with (5.2.50) and (5.2.49) with (5.2.51), we have

\[
\limsup_{k \to \infty} \|x_n - q\| < \limsup_{k \to \infty} \|x_n - q\|
\]

\[
\limsup_{k \to \infty} \|y_{n}' - q\| < \limsup_{k \to \infty} \|y_{n}' - q\|
\]

Which gives contradiction so we have \(T_i^i q = q\), for \(j = 1, 2, \ldots, k\) and \(i \in N\), this implies \(q \in F\). Now we prove that \(\{x_n\}\) converges weakly to \(q\). Let \(\{x_{n_i}\}\) be another subsequence of \(\{x_n\}\) converges weakly to some \(r \in K\). Again as above we conclude \(r \in F\). We show that \(q = r\). Let \(q \neq r\), since \(\lim_{n \to \infty} \|x_n - p\|\) exists for every \(p \in F\). From (1.1), we have
\[
\lim_{n \to \infty} \|x_n - q\| = \limsup_{k \to \infty} \|x_{n_k} - q\| < \limsup_{k \to \infty} \|x_{n_k} - r\| \\
= \limsup_{k \to \infty} \|x_{n_k} - r\| < \limsup_{k \to \infty} \|x_{n_k} - q\| \\
= \lim_{k \to \infty} \|x_n - q\|
\]

(5.2.54)

It implies that \( \lim_{n \to \infty} \|x_n - q\| < \lim_{n \to \infty} \|x_n - q\| \), a contradiction. So we have \( q = r \). It means \( \{x_n\} \) converges weakly to \( q \) as \( n \to \infty \).

5.3. Strong and Weak Convergence Results for Some New Multistep Iterative Schemes.

**Theorem 5.3.1:** Let \( X \) be a uniformly convex real Banach space and \( K \) be a bounded and closed convex subset of \( X \). For \( i \in N \), let \( \{T_i^{(j)}\}_{j=1}^k \) be a sequence of multivalued quasi-nonexpansive and continuous mappings from \( K \) into \( P(K) \) with

\( F := \cap_{j=1}^m F(T_j') \neq \emptyset, j = 1,2...k \) and \( p \in F \). Let \( \{x_n\} \) be a sequence defined by (5.2.2) with \( \{\alpha_{n,i}\}_{i=0}^m \) and \( \{\beta_{n,j}\}_{i=0}^m, j = 1,2...k-1 \) are sequences in \([0, 1]\) which satisfies

\[
\sum_{i=0}^m \alpha_{n,i} = 1 \text{ and } \sum_{i=0}^m \beta_{n,j} = 1, j = 1,2,...k-1, \quad \limsup_{n \to \infty} \alpha_{n,0} < 1 \text{ and } \\
\limsup_{n \to \infty} \beta_{n,0} < 1, j = 1,2,...k-2, \liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,m} > 0 \text{ and } \liminf_{n \to \infty} \beta_{n,0} \beta_{n,m} > 0, j = 1,2...k-1,
\]

for all \( i \in N \). Assume that one of \( T_i' \) is hemicompact. Then \( \{x_n\} \) converges strongly to a common fixed point of \( \{T_i'\} \).

**Proof:** Let \( T_i' \) is hemicompact for some \( i, j \in N \) then from (5.2.20) and (5.2.23), we have, \( \lim d(x_n, T_i x_n) = 0, \forall i \in N \) and \( \lim d(y_n', T_i' y_n') = 0, \forall i \in N, j=1,2...k-1 \). So there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \lim x_{n_k} = q \in K \) using (5.2.22), we can say that \( \lim y_{n_k}' = q \in K \). From continuity of \( \{T_i'\}_{j=1}^k, i \in N \), we have

\[
\lim d(x_{n_k}, T_i x_{n_k}) \to d(q, T_i q) \text{ and } \lim d(y_{n_k}', T_i' y_{n_k}') \to d(q, T_i' q) \text{ This implies that}
\]
\[d(q, T_j^k q) = 0, \quad j = 1, 2 \ldots k \text{ and } q \in F.\] Since \(\lim_{n \to \infty} \|x_n - q\|\) exists, it follows that \(\{x_n\}\) converges strongly to \(q\).

**Theorem 5.3.2:** Let \(X\) be a uniformly convex real Banach space and \(K\) be a compact convex subset of \(X\). For \(i \in \mathbb{N}\), let \(\{T_j^i\}_{j=1}^k\) be a sequence of multivalued quasi-nonexpansive mappings from \(K\) into \(P(K)\) with \(F := \cap_{j=1}^k F(T_j^i) \neq \emptyset, j = 1, 2 \ldots k\) and \(p \in F\). Let \(\{x_n\}\) be a sequence defined by (5.2.2) with \(\alpha_{n,j} \to 0, j = 1, 2 \ldots k\) and \(\beta_{n,j} \to 0, j = 1, 2 \ldots k\)

\[\beta_{n,j} = \frac{1}{j}, \quad j = 1, 2 \ldots k - 1\] are sequences in \([0, 1]\) which satisfies \(\sum_{i=0}^{\infty} \alpha_{n,i} = 1\) and

\[\sum_{j=0}^{\infty} \beta_{n,j} = 1, \quad j = 1, 2 \ldots k - 1, \quad \limsup_{n \to \infty} \alpha_{n,0} < 1 \text{ and } \limsup_{n \to \infty} \beta_{n,0} < 1, \quad j = 1, 2 \ldots k - 2,\]

\[\liminf_{n \to \infty} \alpha_{n,0} \alpha_{n,m} > 0 \text{ and } \liminf_{n \to \infty} \beta_{n,0} \beta_{n,j} > 0, \quad j = 1, 2 \ldots k - 1, \text{ for all } i \in \mathbb{N}.

Then \(\{x_n\}\) converges strongly to common fixed point of \(\{T_j^i\}\).

**Proof:** Since \(K\) is compact so there exists a subsequence \(\{x_{n_j}\}\) of \(\{x_n\}\) such that \(\lim_{k \to \infty} \|x_{n_j} - q\| = 0\) for some \(q \in K\), also from (5.2.22), we can say that

\[\lim_{k \to \infty} \|y_{n_j} - q\| = 0, \quad j = 1, 2 \ldots k - 1\]

Now, we have

\[d(q, T_j^k q) \leq d(q, x_{n_j}) + d(x_{n_j}, T_j^k x_{n_j}) + H(T_j^k x_{n_j}, T_j^k q)\]

\[\leq 2 \|x_{n_j} - q\| + d(x_{n_j}, T_j^k x_{n_j}) \to 0 \text{ as } k \to \infty\]

\[d(q, T_j^k q) \leq d(q, y_{n_j}) + d(x_{n_j}, T_j^k y_{n_j}) + H(T_j^k y_{n_j}, T_j^k q)\]

\[\leq 2 \|y_{n_j} - q\| + d(y_{n_j}, T_j^k y_{n_j}) \to 0 \text{ as } k \to \infty, \text{ where } j = 1, 2 \ldots k - 1\]

Hence this implies \(q \in F\) and \(\{x_n\}\) converges strongly to common fixed point of \(\{T_j^i\}\).

**Remark 5.3.3:** If in iterative process defined by (5.2.6) we use \(k=1\) and 2, Then
Theorem 5.2.7, Theorem 5.3.1 and Theorem 5.3.2 reduces to convergence results proved in A Bunyawat and S. Suantai[30-31] and Zhang et al. [278].

For k=3 and i= p (any finite no.), Theorem 5.2.7, Theorem 5.3.2 and Theorem 5.3.2 reduces to the result proved by Ahmed and Altwqi.

**Theorem 5.3.4.** Let $X$ be a uniformly convex real Banach space and $K$ be a bounded and closed convex subset of $X$. For $i \in N$, let $\{T_i^j\}_{j=0}^k$ be a sequence of multivalued quasi-nonexpansive and continuous mappings from $K$ into $P(K)$ with $F := \cap_{i=0}^k F(T_i^j) \neq \emptyset, j = 1, 2, \ldots, k$ and $p \in F$. Let $\{x_n\}$ be a sequence defined by (5.2.7) with $\alpha_{n,j} \geq 0$ and $\beta_{n,j}^j \geq 0, j = 1, 2, \ldots, k - 1$ are sequences in $[0, 1]$ which satisfies

$$\sum_{i=0}^k \alpha_{n,i} = 1 \quad \text{and} \quad \sum_{j=0}^{k-1} \beta_{n,j}^j = 1, j = 1, 2, \ldots, k - 1, \liminf_{n \to \infty} \beta_{n,j}^j > 0, j = 1, 2, \ldots, k - 1, \text{ for all } i \in N.$$

Assume that one of $T_i^j$ is hemicompact. Then $\{x_n\}$ converges strongly to common fixed point of $\{T_i^j\}$.

Proof: Let $T_i^j$ is hemicompact for some $i, j \in N$ then from (5.2.39) and (5.2.44), we have, $\lim_{n \to \infty} d(x_n, T_i^k x_n) = 0, \forall i \in N$ and $\lim_{n \to \infty} d(y_n^j, T_i^k y_n^j) = 0, \forall i \in N, j = 1, 2, \ldots, k - 1$. So there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $\lim_{m \to \infty} x_{n_m} = q \in K$ using (5.2.47), we can say that $\lim_{k \to \infty} y_{n_m}^j = q \in K$. From continuity of $\{T_i^j\}_{j=0}^k, i \in N$, we have

$$\lim_{m \to \infty} d(x_{n_m}, T_i^k x_{n_m}) \to d(q, T_i^k q) \quad \text{and} \quad \lim_{m \to \infty} d(y_{n_m}^j, T_i^k y_{n_m}^j) \to d(q, T_i^k q).$$

This implies that $d(q, T_i^k q) = 0, j = 1, 2, \ldots, k$ and $q \in F$. Since $\lim_{n \to \infty} \|x_n - q\|$ exists, it follows that $\{x_n\}$ converges strongly to $q$.

**Theorem 5.3.5.** Let $X$ be a uniformly convex real Banach space and $K$ be a compact convex subset of $X$. For $i \in N$, let $\{T_i^j\}_{j=0}^k$ be a sequence of multivalued quasi-nonexpansive mappings from $K$ into $P(K)$ with $F := \cap_{i=0}^k F(T_i^j) \neq \emptyset, j = 1, 2, \ldots, k$ and $p \in F$. Let $\{x_n\}$ be a sequence defined by (1.7) with $\alpha_{n,j} \geq 0$ and $\beta_{n,j}^j \geq 0, j = 1, 2, \ldots, k - 1$.

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are sequences in \([0, 1]\) which satisfies \(\sum_{i=0}^{\infty} \alpha_{n,i} = 1\) and \(\sum_{i=0}^{\infty} \beta_{n,i} = 1, j = 1, 2, \ldots k - 1,\)
\[
\lim \inf_{n \to \infty} \beta_{n,i} > 0, j = 1, 2, \ldots k - 1, \text{ for all } i \in N.\text{ Then } \{x_n\} \text{ converges strongly to common fixed point of } \{T_i^j\}.
\]

Proof: Since \(K\) is compact, so there exists a subsequence \(\{x_{n_j}\}\) of \(\{x_n\},\)
such that \(\lim_{k \to \infty} \|x_{n_j} - q\| = 0\) for some \(q \in K.\) Also from (5.2.47), we can say that
\[
\lim_{k \to \infty} \|y_{n_j} - q\| = 0, j = 1, 2, \ldots k - 1.
\]

Now, we have
\[
d(q, T_i^k q) \leq d(q, x_{n_j}) + d(x_{n_j}, T_i^k x_{n_j}) + H(T_i^k x_{n_j}, T_i^k q) \\
\leq 2 \|x_{n_j} - q\| + d(x_{n_j}, T_i^k x_{n_j}) \to 0 \text{ as } k \to \infty
\]
\[
d(q, T_i^j q) \leq d(q, y_{n_j}) + d(x_{n_j}, T_i^j y_{n_j}) + H(T_i^j y_{n_j}, T_i^j q) \\
\leq 2 \|y_{n_j} - q\| + d(y_{n_j}, T_i^j y_{n_j}) \to 0 \text{ as } k \to \infty,
\]
\(\text{where } j = 1, 2, \ldots k - 1.
\]

Hence this implies \(q \in F\) and \(\{x_n\}\) converges strongly to common fixed point of \(\{T_i^j\}\).

\[\text{Remark 5.3.6: Since iteration used in \([278, 232, 1-2, 30]\) are special cases of iterative scheme (5.2.6) so motivated from these, we generalize in the following sense.}\]

1. Since we prove our result for quasi non expansive mappings, so it is a generalization from non expansive to quasi non expansive.

2. We generalize results of single valued maps to multivalued mappings.

3. Our results extend from one, two countable family to \(k\) no. of countable family of multivalued quasi nonexpansive mappings.

4. We prove weak and strong convergence results for new multistep iterative scheme (5.2.7). With the help of numerical example of multivalued quasi non expansive
mappings and computational program in C++, we prove fast rate of convergence of new multistep iterative scheme (5.2.7).

5.4. Rate of Convergence of Some Existing and Some New Multi Step Iterative Schemes.

We use the following numerical example of finite family of multivalued quasi nonexpansive mappings to compare the converging steps of one step, two steps and two new multistep steps iterative procedures. Let \( \{T_i^j\}: [0,1] \to [0,1] \) be k-countable family of multivalued quasi nonexpansive mappings defined as

\[
T_i^j x = \left[ 0, \frac{x}{i + j + 1} \right], \quad \text{where } x \in [0,1], i \in \mathbb{N} \text{ and } j = 1, 2, \ldots k
\]

Now using the initial value \( x_0 = 0.5 \) in C++ program, we get the following observation for different iterations.
Table 5.4.1. Shows the convergence of different iterative sequences to common fixed point \{0\} of countable family of multivalued quasi nonexpansive mappings. From table, we can compare the converging step of one step, two step, Multi step (5.2.6) and Multi step (5.2.7) Iterative schemes and conclude that Multi step iterative scheme (5.2.7) converge faster than one step, two step and Multi step iterative scheme (5.2.6)

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