Chapter 3

Fractional Total \(k\)-Domination in Graphs

3.1 Introduction

Meir and Moon [21] introduced the concept of \(k\)-packing set and distance \(k\)-domination in a graph \(G\) as a generalization of 2-packing set and domination. Fink and Jacobson [10, 11] introduced the concept of \(n\)-domination in graphs. In this chapter we investigate the fractional version of total distance \(k\)-domination and fractional total \(k\)-packing.

For any positive integer \(k\), Hattingh et al. [15] introduced the concept of distance \(k\)-dominating function. They defined upper dis-
tance fractional domination number $\Gamma_{kf}(G)$ and studied the computational complexity of $\Gamma_{kf}(G)$. Arumugam et al. [2] introduced the concept of fractional distance $k$-domination number $\gamma_{kf}(G)$. In this chapter we study the fractional version of total distance $k$-domination in graphs. We define fractional total distance $k$-domination number $\gamma^t_{kf}$ of a graph $G$ and determine the same for several families of graphs. We also obtain sharp bounds for $\gamma^t_{kf}(G)$.

3.2 Fractional Total $k$-Domination

Definition 3.2.1. Let $G = (V, E)$ be a graph without isolates. A function $f : V \rightarrow [0, 1]$ is called a total $k$-dominating function (TKDF) of $G$, if for every $u \in V$, $f(N_k(v)) = \sum_{u \in N_k(v)} f(u) \geq 1$.

The characteristic function $\chi_s$ defined by $\chi_s(v) = 1$ when $v \in S$ and 0 otherwise is a total $k$-dominating function if and only if $S$ is a total $k$-dominating set.

Definition 3.2.2. A total $k$-dominating function $f$ of a graph $G$ without isolates is called a minimal total $k$-dominating function (MTKDF), if any function $g : V \rightarrow [0, 1]$, with $g \leq f$ and
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$f(v) \neq g(v)$ for at least one $v \in V$, is not a total $k$-dominating function of $G$

**Definition 3.2.3.** The fractional total $k$-domination number $\gamma_{k_f}^t(G)$ and the upper fractional total $k$-domination number $\Gamma_{k_f}^t(G)$ are defined as follows:

$$\gamma_{k_f}^t(G) = \min \{|f| : f \text{ is an MTKDF of } G\} \text{ and }$$

$$\Gamma_{k_f}^t(G) = \max \{|f| : f \text{ is an MTKDF of } G\}.$$

Here we assume that $k \leq \text{diam}(G)$.

**Remark 3.2.4.** The characteristic function of a $\gamma_k^t$ set and that of a $\Gamma_k^t$-set of a graph $G$ are MTKDFs of $G$. Hence it follows that $1 \leq \gamma_{k_f}^t(G) \leq \gamma_k^t(G) \leq \Gamma_k^t(G) \leq \Gamma_{k_f}^t(G)$.

**Observation 3.2.5.** Let $G = (V, E)$ be a graph of order $n$ without isolated vertices. Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$. Then the problem of finding the fractional total $k$-domination number $\gamma_{k_f}^t(G)$ is equivalent to finding the optimal solution of the following linear programming problem.
Minimize \( z = \sum_{i=1}^{n} f(v_i) \)

Subject to \( \sum_{u \in N_k(v)} f(u) \geq 1 \) and
\( 0 \leq f(v) \leq 1 \) for all \( v \in V \).

**Definition 3.2.6.** Let \( G = (V, E) \) be a graph without isolated vertices. A function \( f : V \to [0,1] \) is called a total distance \( k \)-packing function or simply a total \( k \)-packing function of a graph \( G \) if for every \( v \in V, f(N_k(v)) \leq 1 \).

A total \( k \)-packing function \( f \) of a graph \( G \) without isolates is maximal (MTKDF) if \( g \) is not a total \( k \)-packing function of \( G \) for all functions \( g : V \to [0,1] \) with \( g > f \).

**Definition 3.2.7.** The lower fractional total \( k \)-packing number \( p_{tf}^k(G) \) and the upper total \( k \)-packing number \( p_{tf}^r(G) \) are defined as follows:

\[
p_{tf}^k(G) = \min\{|f| : f \text{ is a MTKPF of } G\} \quad \text{and} \\
p_{tf}^r(G) = \max\{|f| : f \text{ is a MTKPF of } G\}.
\]

**Observation 3.2.8.** Let \( G = (V, E) \) be a graph of order \( n \) without isolates where \( V = \{v_1, v_2, \ldots, v_n\} \). Then the problem of finding the
fractional total $k$-packing number $p_{kf}^t(G)$ is equivalent to finding the optimal solution of the following linear programming problem.

Maximize $z = \sum_{i=1}^{n} f(v_i)$

Subject to $\sum_{u \in N_k(v)} f(u) \geq 1$ and

$0 \leq f(v) \leq 1$, for all $v \in V$.

**Remark 3.2.9.** The Linear Programming Problem given in Observation 3.2.5 and that given in Observation 3.2.8 are duals of each other. Hence it follows from the Strong Duality Theorem that $p_{kf}^t(G) = \gamma_{kf}^t(G)$.

Hence if there exists a minimal total $k$-dominating function $f$ and a maximal total $k$-packing function $g$ with $|f| = |g|$, then $p_{kf}^t(G) = |g| = |f| = \gamma_{kf}^t(G)$.

**Example 3.2.10.** Consider the graph $G$ given in Figure 3.1

Then $D = \{v_1, v_5\}$ is a total 2-dominating set of $G$. Hence $\gamma_2^t(G) = 2$. We have $N_2(v_1) = \{v_2, v_3, v_7, v_8\}$, $N_2(v_2) = \{v_1, v_3, v_4, v_8\}$, $N_2(v_3) = \{v_1, v_2, v_4, v_5\}$, $N_2(v_4) = \{v_2, v_3, v_5, v_6\}$, $N_2(v_5) = \{v_3, v_4, v_6, v_7\}$, $N_2(v_6) = \{v_4, v_5, v_7, v_8\}$, $N_2(v_7) = \{v_1, v_5, v_6, v_8\}$ and $N_2(v_8) = \{v_1, v_2, v_6, v_7\}$. 
Let $g : V(G) \rightarrow [0, 1]$ be the constant function defined by $g(v) = \frac{1}{4}$ for all $v \in V$. Clearly $g$ is a total 2-dominating function of $G$ with $|g| = 2$ and hence $\gamma_{2f}^t(G) \leq 2$. Further $g$ is also total 2-packing function of $G$ and hence $\gamma_{2f}^t(G) = 2$.

The following lemma gives an upper bound for $\gamma_{kf}^t(G)$.

**Lemma 3.2.11.** For any graph of order $n$ without isolates we have $\gamma_{kf}^t(G) \leq \frac{n}{k+1}$ and the bound is sharp.

*Proof.* Since $|N_k(u)| \geq k+1$ for all $u \in V$ it follows that the constant function $f$ defined on $V$ by $f(v) = \frac{1}{k+1}$ for all $v \in V$ is a total $k$-dominating function with $|f| = \frac{n}{k+1}$. Hence $\gamma_{kf}^t(G) \leq \frac{n}{k+1}$.

For sharpness of the bound, consider the graph $G$ consisting of a cycle of length 4 with a path of length 2 attached to every vertex of the cycle. Clearly $n = 12$. 

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**Figure 3.1:** A graph with $\gamma_{2t}^f(G) = 2$. 

- $v_1$, $v_2$, $v_3$, $v_4$, $v_5$, $v_6$, $v_7$, $v_8$ are the vertices of the graph. 
- The cycle consists of vertices $v_1$, $v_2$, $v_3$, $v_4$. 
- The path consists of vertices $v_1$, $v_2$, $v_7$, $v_6$, $v_5$. 

- The edges are drawn between adjacent vertices in the cycle and between the cycle and the path.
Figure 3.2: A graph with $\gamma_{2f}(G) = \frac{n}{3}$.

Let $f$ be any total 2-dominating function of $G$. Then $\sum_{u \in N_2[v_6]} f(u) = f(v_6) + f(v_5) + f(v_1)$.

Hence $f(v_6) + f(v_5) + f(v_1) \geq 1$

Similarly, $f(v_8) + f(v_7) + f(v_2) \geq 1$, $f(v_{10}) + f(v_9) + f(v_3) \geq 1$ and

$f(v_{12}) + f(v_{11}) + f(v_4) \geq 1$.

Hence it follows that

$|f| = \sum_{u \in V} f(u) \geq 4$.

Thus, $\gamma_{2f}(G) \geq 4$. Also $\gamma_{2f}(G) \leq 4$ and hence $\gamma_{2f}(G) = 4 = \frac{n}{k+1}$.
We now proceed to determine fractional total $k$-domination number for some standard graphs.

**Theorem 3.2.12.** For the corona $G = C_n \circ K_1$, we have $\gamma_{k_f}(G) = \frac{n}{2k-1}$.

*Proof.* Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Let $u_i$ be the pendant vertex adjacent to $v_i$. Clearly $|N_k(u_i) \cap V(C_n)| = 2k - 1$ and $N_k(u_i) \subset N_k(v_i)$, for $1 \leq i \leq n$. Hence the function $f : V(G) \to [0, 1]$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x = u_i \\ \frac{1}{2k-1}, & \text{if } x = v_i \end{cases}$$

is a minimal total $k$-dominating function of $G$ with $|f| = \frac{n}{2k-1}$. Also we have $|N_k(v_i) \cap \{u_i : 1 \leq j \leq n\}| = 2k - 1$ for $1 \leq i \leq n$. Hence the function $g : V \to [0, 1]$ defined by

$$g(x) = \begin{cases} \frac{1}{2k-1}, & \text{if } x = u_i \\ 0 & \text{if } x = v_i \end{cases}$$

is a maximal total $k$-packing function of $G$ with $|g| = \frac{n}{2k-1}$. Hence from the Strong Duality Theorem, we have $|f| = |g|$. Thus $\gamma_{k_f}(G) = \frac{n}{2k-1}$. □
In the following theorem we determine $\gamma_{kf}^t(G)$ for the friendship graph $G$. Since diameter of $G$ is 2, we take $k = 2$.

**Theorem 3.2.13.** For the friendship graph $G = F_n$ with $n$ blocks we have $\gamma_{kf}^t(G) = \frac{2n+1}{2n}$.

**Proof.** Let $V = \{v_1, v_2, v_3, \ldots, v_{2n}\}$ be the vertices of $F_n$ and let $E(F_n) = \{vv_i : 1 \leq i \leq 2n\} \cup \{v_iv_{i+1} : 1 \leq i \leq 2n - 1, i \text{ is odd}\}$.

Let $f : V \to [0, 1]$ be the constant function $f$ defined by $f(x) = \frac{1}{2n}$ for all $x \in V$. Then $\sum_{u \in N_k(x)} f(u) = 1$, for all $x \in V$.

Hence $\gamma_{kf}^t(F_n) \leq |f| \leq \frac{2n+1}{2n}$. (3.1)

Now, let $g$ be any total 2-dominating function of $G$. Then $g(N_2(v)) = |g| - g(v)$. Hence $|g| - g(v) \geq 1$. Also $g(N_2(v_i)) = |g| - g(v_i), 1 \leq i \leq 2n$. Hence $|g| - g(v_i) \geq 1, 1 \leq i \leq 2n$. Adding these $(2n + 1)$ inequalities we get

$$(2n + 1)|g| - |g| \geq 2n + 1$$

Hence $|g| \geq \frac{2n+1}{2n}$ and so

$$\gamma_{2f}^t(G) \geq \frac{2n+1}{2n}$$ (3.2)
From (3.1) and (3.2), we get

\[ \gamma_{2f}(F_n) = \frac{2n + 1}{2n} \]

\[ \square \]

**Theorem 3.2.14.** For the three dimensional hypercube \( G = Q_3 \) given in Figure 3.3 we have \( \gamma_{2f}(G) = \frac{4}{3} \)

![Figure 3.3: A graph \( G = Q_3 \).](image)

**Proof.** Let \( V = \{v_1, v_2, v_3, \ldots, v_8\} \) be the vertex set of \( Q_3 \). Let \( f : V \to [0, 1] \) be the constant function defined by \( f(v_i) = \frac{1}{6}, 1 \leq i \leq 8 \).

Then \( \sum_{u \in N_2(v)} f(u) = 1 \) for all \( v \in V \).

\[ \therefore \gamma_{k_f}(Q_3) \leq |f| \leq \frac{4}{3} \quad \text{(3.3)} \]

Now, let \( g \) be any total 2-dominating function of \( G \). Then \( g(N_2(v_1)) = g(v_2) + g(v_3) + g(v_4) + g(v_5) + g(v_6) + g(v_8) \geq 1 \).
Hence $|g| - (g(v_1) + g(v_7)) \geq 1$. Similarly considering the inequalities $g(N_2(v_i)) \geq 1$ and adding these eight inequalities we get $8|g| - 2|g| \geq 8$. Hence $6|g| \geq 8$, so that $|g| \geq \frac{4}{3}$

Thus, $\gamma_{t2}^f(G) \geq \frac{4}{3} \quad (3.4)$

From (3.3) and (3.4), we get $\gamma_{t}^r(G) \geq \frac{4}{3}$. \qed

**Theorem 3.2.15.** For the complete bipartite graph $G = K_{m,n}$, we have $\gamma_{t2}^f(G) \geq \frac{m+n}{m+n-1}$.

*Proof.* Let $X$ and $Y$ be the bipartition of $G$. Let $X = \{v_1, v_2, v_3, \ldots, v_m\}$ and $Y = \{u_1, u_2, u_3, \ldots, u_n\}$. Let $f : V \to [0,1]$ be the constant function defined by $f(v) = \frac{1}{m+n-1}$ for all $v \in V$. Then

$$\sum_{u \in N_2(x_i)} f(v) = \frac{m+n-1}{m+n-1} = 1.$$ Similarly $\sum_{v \in N_2(y_i)} f(v) = 1$. Hence $f$ is a total 2-dominating function of $G$.

Thus $\gamma_{t2}^f(K_{m,n}) \leq |f| = \frac{m+n}{m+n-1}. \quad (3.5)$

Now, let $g$ be any total 2-dominating function of $G$. Then for any $v \in V(G)$, $\sum_{u \in N_2(v)} g(u) = |g| - g(v)$. Hence $|g| - g(v) \geq 1$ for all $v \in V$.  

Adding these \((m+n)\) inequalities, we get \((m+n)|g| - |g| \geq m+n\).

Thus \(|g| \geq \frac{m+n}{m+n-1}.

Hence \(\gamma_{2f}(K_{m,n}) \geq \frac{m+n}{m+n-1}. \tag{3.6}\)

From (3.5) and (3.6), we get \(\gamma_{2f}(K_{m,n}) \geq \frac{m+n}{m+n-1}. \tag*{□}\)

**Theorem 3.2.16.** If \(G\) is a graph on \(n\) vertices with \(\text{diam}(G) = k\), then \(\gamma_{k_f}(G) = \frac{n}{n-1}.

**Proof.** Consider the constant function \(f : V \to [0,1]\) defined by \(f(v) = \frac{1}{n-1}\) for all \(v \in G\). Since \(k = \text{diam}(G)\), we have \(N_k(G) = V - v\) for all \(v \in V\). Hence \(\sum_{u \in N_k(v)} f(u) = |f| - \frac{1}{n-1} = 1\).

\[\gamma_{r_f}(G) \leq |f| = n \left(\frac{1}{n-1}\right). \tag{3.7}\]

Now, let \(g\) any total \(k\)-dominating function of \(G\). Let \(v \in V\). Then \(\sum_{u \in N_k(v)} g(u) \geq 1\) and hence \(|g| - g(v) \geq 1\). Adding these \(n\) inequalities, we get \(n|g| - |g| \geq n\), Hence \(|g| \geq \frac{n}{n-1}\).

Thus \(\gamma_{k_f}(G) \geq \frac{n}{n-1}. \tag{3.8}\)

From (3.7) and (3.8), we get \(\gamma_{k_f}(G) \geq \frac{n}{n-1}. \tag*{□}\)
3.3 Convexity of Total $k$-Domination Functions

Cockayne et al. [7] have studied the concept of convex combinations of dominating functions of a graph. In this section, we investigate convex combinations of total $k$-dominating functions and minimal total $k$-dominating function.

We know that a total $k$-dominating function $f$ of a graph $G$ is minimal if any function $g$ with $g \leq f$ and $f(v) \neq g(v)$ for at least one $v \in V$ is not a total $k$-dominating function.

Lemma 3.3.1. Let $f$ be a total $k$-dominating function of a graph $G = (V,E)$. Then $f$ is a minimal total $k$-dominating function if and only if whenever $f(v) > 0$, there exists $u \in N_k(v)$ such that $f(N_k(v)) = 1$

Proof. Let $f$ be a minimal total $k$-dominating function of $G$. Let $v \in V$ and $f(v) > 0$. Suppose $f(N_k(v)) > 1$ for all $u \in N_k(v)$. Let $\epsilon_1 = \min\{f(N_k(u)) - 1 : u \in N_k(u)\}$. Let $\epsilon = \frac{1}{2}\min\{f(v), \epsilon_1\}$. Clearly $\epsilon > 0$
Now, define $g: V \to [0, 1]$ by

$$
g(x) = \begin{cases} 
  f(x) - \epsilon & \text{if } x = v \\
  f(x) & \text{otherwise}
\end{cases}
$$

Then $g(N_k(u)) = f(N_k(u)) - \epsilon \geq 1$ for all $u \in N_k(v)$. Also $g(N_k(u)) = f(N_k(u)) \geq 1$ for all $u \notin N_k(v)$. Thus $g$ is a total $k$-dominating function of $G$, $g(v) < f(v)$ and $g(x) = f(x)$ for all $x \neq v$. Hence $f$ is not a minimal total $k$-dominating function, which is a contradiction. Thus there exists $u \in N_k(v)$ such that $f(N_k(u)) = 1$.

Conversely, suppose that for any $v \in V$ with $f(v) > 0$ there exists $u \in N_k(v)$ such that $f(N_k(u)) = 1$. Let $g: V \to [0, 1]$ be any function such that $g \leq f$ and $g(v) < f(v)$ for some $v \in V$. Then $f(v) > 0$. Hence there exists $u \in N_k(v)$ such that $f(N_k(u)) = 1$. Since $g(v) < f(v)$, it follows that $g(N_k(v)) < 1$, so that $g$ is not a total $k$-dominating function of $G$. Hence $f$ is a minimal total $k$-dominating function of $G$. \hfill \Box

**Definition 3.3.2.** Let $G$ be a graph and let $A, B \subseteq V$. We say that $A$ totally $k$-dominates $B$, if $N_k(v) \cap A \neq \emptyset$ for all $u \in B$ and we write $A \rightarrow_{tk} B$
**Definition 3.3.3.** Let $f$ be a total $k$-dominating function of a graph $G$. The boundary set $B_f$ and the positive set $P_f$ of $f$ are defined by

$$B_f = \{ u \in V(G) : f(N_k(u)) = 1 \} \text{ and } P_f = \{ u \in V(G) : f(u) > 0 \}.$$

The following theorem gives a necessary and sufficient condition for a total $k$-dominating function $f$ to be minimal.

**Theorem 3.3.4.** A total $k$-dominating function $f$ of $G$ is a minimal total $k$-dominating function if and only if $B_f \rightarrow_{tk} P_f$.

*Proof.* Let $f$ be a minimal total $k$-dominating function of $G$. Let $v \in P_f$ so that $f(v) > 0$. It follows from Lemma 3.3.1 that there exists $u \in N_k(v)$ such that $f(N_k(u)) = 1$. Hence $u \in B_f$ and $u$ totally $k$-dominates $v$. Thus $B_f \rightarrow_{tk} P_f$. Conversely, suppose $B_f \rightarrow_{tk} P_f$. Let $v \in V$ and $f(v) > 0$. Then $v \in P_f$. Since $B_f \rightarrow_{tk} P_f$, there exists $u \in B_f$ such that $u$ totally $k$-dominates $v$. Hence $u \in N_k(v)$ and $f(N_k(v)) = 1$. Thus if follows from Lemma 3.3.1 that $f$ is a minimal total $k$-dominating function of $G$. □
Let $f$ and $g$ be two total $k$-dominating functions and let $0 < \lambda < 1$. Then $h_\lambda = \lambda f + (1 - \lambda)g$ is called a convex combination of $f$ and $g$.

It follows from the definition that any convex combination of two total $k$-dominating functions is again a total $k$-dominating function of $G$. However a convex combination of two minimal total $k$-dominating functions need not be an minimal total $k$-dominating function, as shown in the following example.

**Example 3.3.5.** Consider the cycle $G = C_5 = (u_1, u_2, u_3, u_4, u_5, u_1)$. Let $k = 2$.

The function $f : V(G) \to [0, 1]$ defined by

$$f(x) = \begin{cases} 
1 & \text{if } x \in \{u_1, u_4\} \\
0 & \text{otherwise}
\end{cases}$$

is a minimal total 2-dominating function of $G$ with $P_f = \{u_1, u_4\}$. 
and $B_f = \{u_1, u_4\}$. Also the function $g : V(G) \rightarrow [0, 1]$ defined by

$$g(x) = \begin{cases} 
1 & \text{if } x \in \{u_3, u_5\} \\
0 & \text{otherwise}
\end{cases}$$

is a minimal total 2-dominating function of $G$ with $P_g = \{u_3, u_5\}$ and $B_g = \{u_3, u_5\}$.

Now, let $h = \frac{1}{2}f + \frac{1}{2}g$. Then $h(u_1) = h(u_3) = h(u_4) = h(u_5) = \frac{1}{2}$ and $h(u_2) = 0$. Hence $P_h = \{u_1, u_3, u_4, u_5\}$ and $B_h = \emptyset$ which shows that $B_h$ does not 2-dominate $P_h$. Thus the total 2-dominating function $h$ is not minimal

**Theorem 3.3.6.** Let $f$ and $g$ be minimal total $k$-dominating functions of a graph $G$ and let $0 < \lambda < 1$. Then $h_\lambda = \lambda f + (1 - \lambda)g$ is a minimal total $k$-dominating function of $G$, if and only if

$$B_f \cap B_g \xrightarrow{tk} P_f \cup P_g.$$

**Proof.** Suppose $h_\lambda$ is a minimal total dominating function of $G$. We claim that $B_{h_\lambda} = B_f \cap B_g$ and $P_{h_\lambda} = P_f \cap P_g$. Let $v \in B_{h_\lambda}$. Then $h_\lambda(N_k(v)) = 1$. Hence $(\lambda f + (1 - \lambda)g)(N_k(v)) = 1$, so that $\lambda f(N_k(v)) + (1 - \lambda)g(N_k(v)) = 1$. Since $f(N_k(v)) \geq 1$ and $g(N_k(v)) \geq 1$, it follows that $f(N_k(v)) = g(N_k(v)) = 1$. Thus $v \in B_f \cap B_g$ and
hence $B_{h\lambda} \subseteq B_f \cap B_g$. Now, let $v \in B_f \cap B_g$. Then $f(N_k(v)) = g(N_k(v)) = 1$. Hence $h_{\lambda}(N_k(v)) = \lambda f(N_k(v)) + (1 - \lambda) g(N_k(v)) = 1$. Thus, $v \in B(h_{\lambda})$ and hence $B_f \cap B_g \subseteq B_{h\lambda}$.

Therefore, $B_{h\lambda} = B_f \cap B_g$.

We now claim that $P_{h\lambda} = P_f \cup P_g$. Let $v \in P_{h\lambda}$. Then $h_{\lambda}(v) = \lambda f(v) + (1 - \lambda) g(v) > 0$. Hence either $f(v) > 0$ or $g(v) > 0$, so that $v \in P_f \cup P_g$. Now if $v \in P_f \cup P_g$, then $f(v) > 0$ or $g(v) > 0$. Hence $h_{\lambda}(v) = \lambda f(v) + (1 - \lambda) g(v) > 0$. Thus $v \in P_{h\lambda}$, so that $P_f \cup P_g \subseteq P_{h\lambda}$.

Now, let $v \in P_{h\lambda}$. Then $h_{\lambda}(v) = \lambda f(v) + (1 - \lambda) g(v) > 0$. Hence $f(v) > 0$ or $g(v) > 0$, so that $v \in P_f \cup P_g$. Therefore, $P_{h\lambda} \subseteq P_f \cup P_g$ and hence $P_{h\lambda} = P_f \cup P_g$. Since $h_{\lambda}$ is a minimal total $k$-dominating function of $G$, it follows from Theorem 3.3.4 that $B_{h\lambda} \rightarrow_{tk} P_{h\lambda}$. Hence $B_f \cap B_g \rightarrow_{tk} P_f \cup P_g$. Since $B_f \subseteq B_f \cap B_g$ and $P_f \subseteq P_f \cup P_g$, it follows that $B_f \rightarrow_{tk} P_f$. Similarly $B_g \rightarrow_{tk} P_g$. Thus $f$ and $g$ are minimal total $k$-dominating functions of $G$.

Conversely, let $f$ and $g$ be minimal total $k$-dominating functions of $G$. Then $B_f \rightarrow_{tk} P_f$ and $B_g \rightarrow_{tk} P_g$. Hence $B_f \cap B_g \rightarrow_{tk} P_f \cup P_g$, so that $B_{h\lambda} \rightarrow_{tk} P_{h\lambda}$. Thus $h_{\lambda}$ is a minimal total $k$-dominating function of $G$. $\square$
Observation 3.3.7. The above theorem shows that if $f$ and $g$ are minimal total $k$-dominating functions of $G$ then either all convex combinations of $f$ and $g$ are minimal total $k$-dominating function or no convex combination is minimal total $k$-dominating function of $G$. We shall use this observation to define and investigate the concept of total $k$-convexity graph of a graph.

3.4 Conclusion

In this chapter we have determined the fractional total $k$-domination number of some families of graphs. We have also investigated the minimality of a convex combination of two minimal total $k$-dominating functions. A study of this parameter for graph products and for other graph operations will be taken up and results in this direction will be reported.