Chapter 3

FUZZIFICATION, DEFUZZIFICATION AND I-FUZZIFICATION METHODS

Contents

3.1 Fuzzification

By fuzzification, we mean the process of converting an object into a fuzzy object. We may distinguish between two types of fuzzifications – fuzzification of crisp sets and fuzzification of IFSs. A crisp set may be fuzzified by attaching a membership grade to each element of the set. An IFS may be fuzzified by removing the hesitancy part.

Defuzzification is the process of converting a FS into a crisp set. Several methods of fuzzification are available in the literature.

We review some of these methods and introduce several methods for fuzzification of IFS.

3.1.1 RSM Fuzzifier [11]

Consider the operation of restricted scalar multiplication defined in 1.5. When A is crisp set, the restricted scalar multiplication produces a fuzzy set αA. Hence, RSM is a fuzzification operator.

3.1.2 Remark

The RSM fuzzifier is an example for fuzzification method of the first type, viz., crisp-to-fuzzy conversion. In the next section we consider the reverse process, viz., fuzzy-to-crisp conversions.

3.2 Defuzzification

This is an important concept in FS theory. In a fuzzy controller, the final option cannot be fuzzy. To communicate the option to the machine, it must be crisp. i.e., if the final conclusion is a fuzzy set B, then the control command will be given as defuzz(B), where defuzz(B) is a function mapping FSs to crisp ones.

α-cuts and strong α-cuts described in 1.2.14 and 1.2.15 produce crisp sets out of FSs. We make an overview of some of the other important defuzzification techniques available.

3.2.1 Definition [5]

Let A be the FS given by A = {(x_i, A(x_i)), i = 1, 2, …., n} where $x_i \in \mathbf{R}$. Then the *centroid defuzzifier* δ is given by

$$\delta = \frac{\sum_{i=1}^{n} x_i A(x_i)}{\sum_{i=1}^{n} A(x_i)}$$

It may be observed that the centroid defuzzifier can be used only when $X \subseteq \mathbf{R}$, the set of real numbers. It is the weighted average of the given values, weighted by the respective membership values.

3.2.2 Example

Let X = {1, 2, 3, 4, 5} and A = 0.8/1 + 0.6/2 + 0.8/3 + 0.3/4 + 0.1/5

Then, $\delta = \frac{0.8 \times 1 + 0.6 \times 2 + 0.8 \times 3 + 0.3 \times 4 + 0.1 \times 5}{0.8 + 0.6 + 0.8 + 0.3 + 0.1} = 2.35$

Here $\delta \notin X$. Hence the value may be rounded off to a near value of elements of X. So, we take $\delta = 2$.

3.2.3 Definition [5]

Let A be a fuzzy set over **R** and let $C_m = \{x \in X / A(x) = h(A)\}$, where h(A) denotes the height of A. Then the *center of maxima defuzzifier*, denoted by γ, is given by $\gamma = (\min C_m + \max C_m)/2$.

The center of maxima defuzzifier is the average of lowest and highest of the elements with the largest membership grade. It identifies the central value among the values with the highest membership grade.

3.2.4 Example

Let $A = 0.5/a_1 + 1/a_2 + 0.5/a_3$. Then, $\gamma = a_2$, since $C_m = \{a_2\}$.

3.2.5 Example

Let $A = 0.5/a_1 + 1/a_2 + 1/a_3 + 0.2/a_4$. Then, $\gamma = (a_2 + a_3)/2$, since $C_m = \{a_2, a_3\}$.

3.2.6 Example

Let A = 0.8/1 + 0.6/2 + 0.8/3 + 0.3/4 + 0.1/5. Then $\gamma = (1+3)/2 = 2$.

3.2.7 Definition [5]

Let C_m = {x∈X/A(x)=h(A)}, where h(A) is the height of A. Then, the *mean of the maxima defuzzifier*, denoted by η, is defined by $\eta = (1/k) \sum_{i=1}^{k} x_i$, where $x_i \in C_m$ and $k = |C_m|$.

The mean of the maxima is the average of the values with the highest membership grade. For a normal fuzzy set, it gives the average of the core of A.

3.2.8 Example

Let A = 0.8/1 + 0.6/2 + 0.8/3 + 0.8/4 + 0.2/5.

Then $\eta = (1+3+4)/3 = 2.6 \approx 3$.

3.2.9 Remark

The center of maxima and the mean of the maxima defuzzifiers may not give a value in the universe. In this case we may approximate to an appropriate value as in the case of the centroid defuzzifier.

3.2.10 Definition [12]

Let $D = \{d_1, d_2, \ldots, d_k\}$ with membership function $D^+: D \to [0, 1]$. The *maximum value defuzzification operator*, denoted by Defuzz^{MV}, is defined by

$$\text{Defuzz}^{MV}(D) = \{d_j : D^+(d_j) \geq D^+(d_i)\ \forall\ i \in \{1, 2, \ldots, k\}\}.$$

This measure identifies the elements with the highest membership grade. Also, this may produce a subset rather than a single point of the universe.

In the next section we discuss IFS-to-FS conversions.

3.3 Fuzzification of IFS

Fuzzification of IFSs is the process of converting an IFS into a FS by eliminating the hesitancy grade. Here we propose four methods of fuzzification of IFSs and a generalization. In this section, we denote IFSs by A, B, ... and the corresponding fuzzified sets by $\widetilde{A}, \widetilde{B}$, ... First, we consider singleton IFSs and then extend it to finite IFSs. In what follows, a singleton IFS A ={<x, $A^+(x)$, $A^-(x)$>} will be denoted as A = <x, $A^+(x)$, $A^-(x)$>.

3.3.1 Method 1 (Arbitrary Allocation of Hesitancy)

Here, the hesitancy part is assigned to any of the grades A^+ or A^-.

3.3.2 Method 2 (Assigning Hesitancy to the Major Grade)

In this method, the hesitancy membership A^0 is added to the larger of the two grades A^+ and A^-.

3.3.3 Example

Let A = <x, 0.5, 0.3> be a given IFS. Then, the corresponding FS, fuzzified by the above method is, $\tilde{A}$ = <x, 0.7>.

3.3.4 Example

Let A = <x, 0.4, 0.5>. Then, $\tilde{A}$ = <x, 0.4>.

3.3.5 Remarks

i. When $A^+ = A^-$, A^0 may be merged with any of the grades.

ii. When the hesitancy is added to the non-membership grade (as in example 3.3.4), we get the necessity operator □.

iii. When the hesitancy is added to the membership grade (as in example 3.3.3), we get the possibility operator ◊.

3.3.6 Method 3 (Equal Distribution of Hesitancy)

Here, the hesitancy grade is divided equally among the membership and non-membership grades.

3.3.7 Example

Let $A = <x, 0.5, 0.3>$. Then $\tilde{A} = <x, 0.6, 0.4>$

3.3.8 Example

Let $A = <x, 0.6, 0.3>$. Then, $\tilde{A} = <x, 0.65, 0.35>$

3.3.9 Method 4 (Proportionate Allocation)

Here, A^0 is divided in the proportion of A^+ and A^-.

i.e., $\tilde{A}^+ = A^+ + A^0 (A^+/ (A^+ + A^-))$ and $\tilde{A}^- = A^- + A^0 (A^-/(A^+ + A^-))$.

3.3.10 Example

Let A = $<x, 0.4, 0.2>$.

Then, $\tilde{A}^+$ = $0.4 + 0.4\ (0.4/(0.4+0.2)) = 0.666$, and

$\tilde{A}^-$ = $0.2 + 0.4\ (0.2/(0.4+0.2)) = 0.334$

i.e., $\tilde{A}$ = $<x, 0.666, 0.334>$

We can generalize the above methods to obtain a (continuous) class of fuzzification functions as follows.

3.3.11 Method 5 (Weighted Proportionate Allocation)

Here, the hesitancy is allocated to A^+ and A^- in the proportion λ: $(1-\lambda)$, where $\lambda \in [0, 1]$.

i.e., If $A = <x, A^+, A^->$, then, $\tilde{A} = <x, A^+ + \lambda A^0, A^- + (1-\lambda)\ A^0>$

3.3.12 Example

Suppose, A^+ denotes the proportion that favours a particular concept and A^-, the proportion that opposes it. Then, A^0 denotes the neutral or undecided segment. Now, suppose that 70% of the undecided segment may later adopt or made to accept the concept, then we may take $\lambda = 0.7$. Consequently, $\tilde{A} = \langle x, A^{+} + 0.7A^{0}, A^{-} + 0.3A^{0})\rangle$.

3.3.13 Remarks

Method 5 proposed above generalises the previous methods.

i. When λ switches between 0 and 1 we get method 1.

ii. When $\lambda = 1$, the hesitancy is added to membership totally and when $\lambda = 0$, the hesitancy is added to non membership totally.

These two cases correspond to method 2.

iii. When $\lambda = 1/2$, the hesitancy is distributed to both the grades equally. This corresponds to method 3.

iv. Taking $\lambda = A^{+}/ (A^{+} + A^{-})$ we get method 4.

The methods discussed above relate to singleton IFSs only. They may be extended to arbitrary IFSs, by point-wise fuzzification. We illustrate the method by an example.

3.3.14 Example

Let A be a finite IFS consisting of three elements, given by $A = \{<a_1, 0.6, 0.3>, <a_2, 0.5, 0.3>, <a_3, 0.4, 0.4>\}$.

Then, applying the second method, viz. equal distribution of hesitancy, we get the FS $\tilde{A} = \{<a_1, 0.65, 0.35>, < a_2, 0.6, 0.4>, < a_3, 0.5, 0.5>\}$.

i.e., $\tilde{A} = \{<a_1, 0.65>, < a_2, 0.6>, < a_3, 0.5, >\}$.

3.4 I-fuzzification

We have discussed methods to convert a FS to a crisp set and an IFS to a FS. M. Ganesh [11] has suggested restricted scalar multiplication to convert a crisp set to a FS, which is described in section 1.5. We give below an extension of the method for converting a FS to IFS. We call the new method *i-fuzzification*.

In what follows we consider two methods for converting FSs to IFSs – *restricted scalar multiplication by α* (or, *α-multiplication*) and *restricted scalar multiplication by* (*α,β*) (or, *(α,β)- multiplication*).

Both the methods are used to produce an IFS associated with a given IFS. But, they may be conveniently used to construct IFSs from given FSs.

3.4.1 Definition

Let $A = \{\langle x, A^{+}(x), A^{-}(x)\rangle / x \in X\}$, be an IFS and let $\alpha \in (0, 1]$. Then by αA, we mean an IFS on X, given by $\alpha A = \{\langle x, \alpha A^{+}(x), \alpha A^{-}(x)\rangle / x \in X\}$. This process is called *restricted scalar multiplication by* α (or, *α-multiplication* for short).

3.4.2 Example

Let $A = \{\langle x_1, 0.5\rangle, \langle x_2, 0.6\rangle, \langle x_3, 0.4\rangle\}$ and let $\alpha = 0.5$.

We can write the FS A as

$$A = \{\langle x_1, 0.5, 0.5\rangle, \langle x_2, 0.6, 0.4\rangle, \langle x_3, 0.4, 0.6\rangle\}.$$

Then, $\alpha A = \{\langle x_1, 0.25, 0.25\rangle, \langle x_2, 0.3, 0.2\rangle, \langle x_3, 0.2, 0.3\rangle\}$

3.4.3 Remarks

From the above definition, we can observe the following:

i. $\alpha A = A$ if and only if $\alpha = 1$.

ii. If $\alpha \neq 1$, then, αA produces an IFS from an IFS A.

iii. When A is FS, we can write it as an IFS in the form $A = \{<x, A^+(x), 1\text{-}A^+(x)>\}$ and consequently αA produces an IFS.

3.4.4 Definition

Let $A = \{<x, A^+(x), A^-(x)> /x \epsilon X\}$, be an IFS on X and let $\alpha, \beta \; \epsilon \; (0, 1]$. Then by (α,β)A, we mean an IFS on X, given by $(\alpha,\beta)A = \{<x, \alpha A^+(x), \beta A^-(x)> x \epsilon X\}$. This process is called *restricted (α-β) scalar multiplication* (or, *(α,β)multiplication* for short).

3.4.5 Example

Let A = $\{<x_1, 0.6>, <x_2, 0.8>, <x_3, 0.4>\}$ and let $\alpha = 0.5$, $\beta = 0.4$.

First we write A as

A = $\{<x_1, 0.6, 0.4>, <x_2, 0.8, 0.2>,<x_3, 0.4, 0.6>\}$.

Then, (α,β)A = $\{<x_1, 0.3, 0.16>, <x_2, 0.4, 0.08>, <x_3, 0.2, 0.24>\}$.

3.4.6 Remarks

The following are direct consequences of the above definition.

i. (α,β)A = A if and only if $\alpha = 1$ and $\beta = 1$.

ii. If $\alpha \neq 1$, or $\beta \neq 1$ then, $(\alpha,\beta)A$ produces an associated IFS from a given IFS A.

iii. When $\alpha = \beta$, (α,β)-multiplication reduces to α-multiplication.

iv. (α,β)-multiplication produces an IFS even when A is a FS.

3.4.7 Remark

Even though the actual life situations are fuzzy, in any decision making, the final option cannot be fuzzy. Hence, we need to convert fuzzy sets to crisp sets. Similarly, when an IFS is obtained as a decision, we need to convert it into a crisp one. First, a fuzzification process may be used to obtain a FS and then a defuzzification process may be applied to the result to obtain a crisp set as the solution.

The concepts and results presented in this chapter form the content of three research papers ([2], [47] & [48]).