CHAPTER 3

SOME PROPERTIES OF COMPACTNESS AND CONNECTEDNESS IN CLOSURE SPACES

INTRODUCTION

In this chapter we firstly describe the fundamental properties of compactness. We define compact closure spaces and study some properties of compactness. Čech defined closure space $X$ to be compact if the intersection of the closures of sets belonging to any proper filter in $X$ is nonempty. He proved some properties of compactness in closure spaces [CE]. In section 1 of this chapter, we find the relation between compactness in $(X,c)$ and $(X,t)$ and prove some related results.

Čech described the concept of connectedness in $[CE]$ as "a subset $A$ of a closure space $X$ is said to be connected in $X$ if $A$ is not the union of two nonempty semi-separated subsets of $X$. That is $A = A_1 \cup A_2$, $(cA_1 \cap A_2) \cup (A_1 \cap cA_2) = \emptyset$ implies that $A_1 = \emptyset$ or $A_2 = \emptyset$". It can be easily seen that this is precisely the connectedness of the associated topological space. Plastria, F obtained certain conditions which imply the connectedness of simple extensions [P]; it has been proved that local connectedness of certain subspaces implies the local connectedness of simple extensions.

We define the concept of connectedness in section 3.2 in a slightly different and perhaps more reasonable way and prove some results in connectedness. We note that the image of a connected space under a $c$-$c_1$ morphism need not be connected.
In section 3.3 we introduce the concepts of local connectedness and path connectedness. We also define compactness and connectedness in monotone spaces in section 3.4.

3.1 SOME PROPERTIES OF COMPACTNESS

The following definitions and results are due to E. Čech.

Definitions 3.1.1

(i) Let \((X, c)\) be a closure space, \(\mathcal{F}\) be a proper filter on \(X\) and \(x\) be an element of \(X\). We shall say that \(x\) is a cluster point of \(\mathcal{F}\) in \((X, c)\) if \(x\) belongs to \(\cap \{cF : F \in \mathcal{F}\}\), that is if each neighbourhood of \(x\) intersects each \(F \in \mathcal{F}\).

(ii) A closure space \((X, c)\) is said to be compact, if every proper filter of sets on \(X\) has a cluster point in \(X\).

Results 3.1.2

(i) For a closure space \((X, c)\) to be compact, it is necessary and sufficient that every interior cover \(\mathcal{V}\) of \((X, c)\) has a finite subcover.

(ii) Any image under a \(c\)-morphism of a compact space \((X, c)\) is compact.

(iii) If \((Y, c)\) is a compact subspace of a Hausdorff closure space \((X, c)\), then \(Y\) is closed in \((X, c)\).

(iv) Every closed subspace of a compact space \((X, c)\) is compact.
Result 3.1.3

If \((X, c)\) is compact, then \((X, t)\) is compact.

**Proof**

Let \((X, c)\) be compact. Then every proper filter of sets on \(X\) has a cluster point in \(X\). Let \(\mathcal{F}\) be a proper filter of sets on \(X\) and \(x\) be a cluster point. Then \(x \in \cap (cF), F \in \mathcal{F}\).

That is \(\cap (cF) \neq \emptyset\) but \(cF \subseteq \text{cl}F\) for every \(F \in \mathcal{F}\). Then \(\cap (\text{cl}F) \neq \emptyset\). So \((X, t)\) is compact.

**Note 3.1.4**

The converse of the above result is not true.

**Example**

Consider \(X = \mathbb{N} \times \mathbb{N} \cup \{x, y\} \cup \{a_i : i \in \mathbb{N}\} \cup \{b_j : j \in \mathbb{N}\}\), \(a_i\)'s, \(b_j\)'s, \(x, y\) are all distinct and do not belong to \(\mathbb{N} \times \mathbb{N}\).

Let \(c\) be defined on \(X\) as in Example 2.1.11

Let \(A_k = \{(m, m) : m \geq k\}\) for \(k \in \mathbb{N}\).

The family \(\mathcal{F} = \{A_k : k \in \mathbb{N}\}\) is a filter base.

\(cA_k = A_k\), for every \(A_k \in \mathcal{F}\) but \(\cap_{k=1}^{\infty} cA_k = \emptyset\).

So \((X, c)\) is not compact. But \((X, t)\) is compact as can be proved easily.
Result 3.1.5

Any image under a \(c - c'\) morphism of a compact closure space \((X, c)\) is compact in the associated topology of \(c'\).

Using the Result 3.1.2 (ii) and the Result 3.1.3, we get this result.

Note 3.1.6

If \((X, cl)\) is compact and \(f: (X, cl) \rightarrow (Y, c')\) is a surjective \(c - c'\) morphism, then \((Y, c')\) need not necessarily be compact.

Result 3.1.7

The associated space \((Y, t')\) of a compact closure space \((Y, c')\) is closed as a subspace of the Hausdorff space \((X, c)\)

Definition 3.1.11

A closure space \((X, c)\) is locally compact if and only if each point in \(X\) has a neighbourhood base consisting of compact sets.

Using the Result 3.1.2 (iii) and \(cA=X \Rightarrow clA=X\), we get the above result.

Result 3.1.8

Every closed closure subspace of an associated topological space \((X, t)\) of a compact closure space \((X, c)\) is compact.

Proof

Let \((Y, c')\) be a closed subspace of a compact space \((X, t)\). Let \(\mathcal{F}\) be a proper filter on \((Y, c')\). Let us consider the smallest filter \(\mathcal{G}\) on \(X\) containing \(\mathcal{F}\). \(\mathcal{F}\) is a filter base for \(\mathcal{G}\). Since \(cl Y = Y\), we have \(c'A = clA\) for each \(A \subseteq Y\) and hence \(\cap (c'F) = \cap (cl F)\).

Therefore \(\cap (cl F) = \cap (cl G)\). Since \((X, t)\) is compact \(\cap (cl G) \neq \emptyset\). That is \(\cap (c' F) \neq \emptyset\).
Corollary 3.1.9

Closed subspace \((Y,t')\) of compact space \((X,c)\) is compact.

Result 3.1.10

\((X,c)\) is compact. \(Y \subset X\). Then \(cY\) is compact.

Proof

Let \(c'\) be the closure on \(cY\) induced by \(c\). Let \(F\) be a filter on \(cY\). We have to prove that \(\cap (c' F), F \in J\) is nonempty. \(\{cF \cap cY : F \in J\}\) is a filter base on \(X\). Since \(X\) is compact, \(\cap (cF \cap cY)\) is nonempty. So \(\cap c' F = \cap (cF \cap cY) \neq \emptyset\).

Definition 3.1.11

A closure space \((X,c)\) is locally compact if and only if each point in \(X\) has a neighbourhood base consisting of compact sets.

Note 3.1.12

\((X,c)\) is locally compact does not imply that \((X,t)\) is locally compact and vice-versa.

Result 3.1.13

Let \((X,c)\) be locally compact. If \(f\) is an open \(c\)-\(c'\) morphism from \((X,c)\) onto \((Y,c')\), then \(Y\) is locally compact.
Proof

Suppose \( y \in Y \). Let \( V \) be a neighbourhood of \( y \). Take \( x \in f^{-1}(y) \). Since \( f \) is c-c' morphism and \( X \) is locally compact, we can find a compact neighbourhood \( U \) such that \( f(U) \subseteq V \). \( x \in \text{Int}_x U \) so \( y \in f(\text{Int}_x U) \subseteq f(U) \). Since \( f \) is open, \( f(\text{Int}_x U) \) is a neighbourhood of \( y \). Hence \( f(U) \) is a compact neighbourhood of \( y \) contained in \( V \).

3.2 CONNECTEDNESS IN CLOSURE SPACES.

In this section we introduce and study connectedness.

**Definition 3.2.1**

\((X,c)\) is said to be disconnected if it can be written as two disjoint nonempty subsets \( A \) and \( B \) such that \( cA \cup cB = X \), \( cA \cap cB = \emptyset \) and \( cA \) and \( cB \) are nonempty. A space which is not disconnected is said to be connected.

**Example 3.2.2**

\( X = \{a,b,c\} \)

\( c \) can be defined on \( X \) such that

\[
c \{a\} = \{a,b\}, \ c \{b\} = \{c\} = \{b,c\}, \ c \{a,b\} = c \{a,c\} = cX = X, \ c\emptyset = \emptyset
\]

Then \( c \) is a closure operation on \( X \).

Here \((X,c)\) is connected because we can not find nonempty subsets \( A \) and \( B \) such that \( cA \cup cB = X \) and \( cA \cap cB = \emptyset \).
Definition 3.2.3

(X, c) is said to be feebly disconnected if it can be written as two disjoint nonempty subsets A and B such that \( A \cup cB = cA \cup B = X \) and \( cA \cap B = \emptyset = A \cap cB \).

Note 3.2.4

It is clear that \((X, c)\) is disconnected implies \((X, c)\) is feebly disconnected. The following example shows that the converse is not true.

Example 3.2.5

\[ X = \{a, b, c\} \]
\[ c\{a\} = \{a, c\}, c\{b\} = c\{c\} = \{b, c\}, c\{a, b\} = c\{a, c\} = cX = X, c\emptyset = \emptyset \]

Here \((X, c)\) is feebly disconnected, but not disconnected.

Result 3.2.6

\((X, t)\) is disconnected \(\implies\) \((X, c)\) is disconnected.

Proof

\((X, t)\) is disconnected implies that it is the union of two disjoint nonempty subsets A and B such that \( clA \cup clB = X, clA \cap clB = \emptyset \) and \( clA, clB \) are nonempty. \( clA \cap clB = \emptyset \). So \( cA \cap cB = \emptyset \). That is \((X, c)\) is disconnected.

Note 3.2.7

\((X, t)\) is connected need not imply that \((X, c)\) is connected.
Example

\[ X = \{a, b, c\} \]. Let \( c \) be a closure operation defined on \( X \) in such a way that

\[
\begin{align*}
    c(a) &= \{a\}, c(b) = \{b, c\}, c(c) = c(a, b) = c(b, c) = cX = X, c\emptyset = \emptyset \\
(X, \emptyset) &= \{X, \emptyset, \{b, c\}\}
\end{align*}
\]

Here \((X, c)\) is disconnected, but \((X, t)\) is connected.

Remark

Connectedness of a subspace \( Y \) of \((X, c)\) can be defined in the same manner.

Note 3.2.8

Let \( c \) be a closure operation defined on \( X \) such that

\[
\begin{align*}
    c(a) &= \{a\}, c(b) = \{a, b, c\}, c(c) = \{b, c\}, c(d) = \{b, c, d\}, \\
    c(a, b) &= c(a, c) = c(b, c) = c(a, b, c) = \{a, b, c\}, \\
    c(c, d) &= \{b, c, d\}, c(a, d) = c(b, d) = c(a, b, d) = c(a, c, d) = c(b, c, d) = \{a, b, c, d\}, \\
    c(e) &= c(a, e) = c(b, e) = c(c, e) = c(d, e) = c(a, b, e) = c(a, c, e) = c(a, d, e) = c(b, c, e) = c(c, d, e) = cX = X, c\emptyset = \emptyset
\end{align*}
\]

Here \( Y = \{b, c\} \) is connected.
cY=\{a,b,c\}; if c' isthe induced closure operation on cY, then
\[ c'(a)=\{a\}, c'(b)=\{b,c\}, c'(c)=c'(a,b)=c'(b,c)=c'(a,c)=c'cY=cY. \]
cY is disconnected.

**Note 3.2.10**

If cA and cB form a separation of X and if Y is a connected subset of X, then Y need not be entirely within either cA or cB.

**Example 3.2.11**

Let c be a closure operation defined on X such that
\[ c(a)=\{a\}, c(b)=\{b,c\}, c(c)=\{a,c\}, c(a,b)=c(b,c)=cX=X, c(a,c)=\{a,c\}. \]
Y=\{a,c\} is connected.

**Note 3.2.12**

The image of a connected space under a c-c' morphism need not be connected.

**Example**

Let X=\{a,b,c,d,e\}. A closure operation c is defined on X as in Example 3.2.9

Let Y=\{a,b,c\}
c' be defined on Y such that
\[ c'(a)=\{a\}, c'(b)=\{b,c\}, c'(c)=c'(a,b)=c'(b,c)=c'(a,c)=c'X=X, c'\phi =\phi. \]
Let \( f \) be a map from \((X, c)\) into \((Y, c')\) defined in such a way that \( f(a) = a, f(b) = c, f(c) = b, f(d) = c, f(e) = c \). Here \( f \) is a \( c\)-\( c' \) morphism. But \( f(X) \) is disconnected.

Result 3.2.13

Suppose \( c_i \) is a closure operator on \( Y \) with degree \( k \) and \( f \) is a \( c\)-\( c_i \) morphism from \((X, c)\) to \((Y, c_i)\). If \( c_i(A) \) and \( c_i(B) \) form a separation of \( Y \), then \( c(f^{-1}(c_i(A))) \) and \( c(f^{-1}(c_i(B))) \) form a separation on \( X \).

Proof

Let \( c_i(A) \cup c_i(B) = Y \) and \( c_i(A) \cap c_i(B) = \emptyset \).

Then \( f^{-1}(c_i(A)) \cup f^{-1}(c_i(B)) = X \).

That is \( c(f^{-1}(c_i(A))) \cup c(f^{-1}(c_i(B))) = X \), since \( f^{-1}(c_i(A)) \subset c(f^{-1}(c_i(A))) \).

Hence \( c(f^{-1}(c_i(A))) \cap c(f^{-1}(c_i(B))) = \emptyset \).

In similar manner \( c(f^{-1}(c_i(B))) \cap c(f^{-1}(c_i(A))) = \emptyset \).

Hence \( c(f^{-1}(c_i(A))) \) and \( c(f^{-1}(c_i(B))) \) form a separation on \( X \).

Result 3.2.14

Let \((X, c)\) be connected and \( f \) is a \( c\)-\( c_i \) morphism from \((X, c)\) on to \((Y, c_i)\). Then \((Y, t_i)\) is connected.
Proof

Since \( f(cA) \subseteq c_1 f(A) \subseteq c_{11} f(A) \), \( f \) being \( c - c_1 \) morphism and we get \( f \) is \( c - c_{11} \) morphism. Suppose \( cl_1 A \) and \( cl_1 B \) form a separation on \( Y \) . Then \( cl_1 A \cup cl_1 B = Y \) and \( cl_1 A \cap cl_1 B = \emptyset \). \( f^{-1}(cl_1 A) \cup f^{-1}(cl_1 B) = X \) and \( f^{-1}(cl_1 A) \cap f^{-1}(cl_1 B) = \emptyset \). By the above result \( c(f^{-1}(cl_1 A)) \) and \( c(f^{-1}(cl_1 B)) \) form a separation on \( X \). This is a contradiction.

Hence \( (Y, t) \) is connected.

3.3 PATHWISE AND LOCAL CONNECTEDNESS

In this section we define and study pathwise connectedness and local connectedness.

Definition 3.3.1

A space \((X, c)\) is pathwise connected if and only if for any two points \( x \) and \( y \) in \( X \), there is a \( cl_1 - c \) morphism \( f : I \rightarrow X \) such that \( f(0) = x \) and \( f(1) = y \) where \( cl_1 \) is the usual closure on \( I \), \( f \) is called a path from \( x \) to \( y \).

Result 3.3.2

\((X, c)\) is pathwise connected implies \((X, t)\) is pathwise connected.

Proof

If \((X, c)\) is pathwise connected, then for any two points \( x \) and \( y \) in \( X \) there is a \( cl_1 - c \) morphism \( f : I \rightarrow X \) such that \( f(0) = x \) and \( f(1) = y \). If \( f \) is \( cl_1 - c \) morphism, then \( f \) is \( cl_1 - cl \) morphism. Therefore \((X, t)\) is pathwise connected.
Note 3.3.3

The converse of the above result is not true.

Note 3.3.4

Pathwise connected space need not be a connected space.

Definition 3.3.5

A space $X$ is said to be locally connected at $x$ if for every neighbourhood $U$ of $x$, there is a connected neighbourhood $V$ of $x$ contained in $U$. If $X$ is locally connected at each of its points, then $X$ is said to be locally connected.

Definition 3.3.6

A space $X$ is said to be locally path connected at $x$ if for every neighbourhood $U$ of $x$, there is a path connected neighbourhood $V$ of $x$ contained in $U$. If $X$ is locally path connected at each of its points, then it is said to be locally path connected.

Note 3.3.7

A space $(X,c)$ is locally connected need not imply that $(X,t)$ is locally connected and vice-versa.

A parallel study of the above concepts in the set up of closure spaces is interesting; however we are not attempting it in this thesis.
3.4. COMPACTNESS AND CONNECTEDNESS IN MONOTONE SPACES

**Definition 3.4.1**

Let \((X, c^*)\) be a monotone space. \(\mathcal{F}\) be a proper filter on \(X\) and \(x\) be an element of \(X\). We shall say that \(x\) is a cluster point of \(\mathcal{F}\) in \((X, c^*)\) if \(x\) belongs to \(\bigcap\{c^*F : F \in \mathcal{F}\}\). That is each neighbourhood of \(x\) intersects each \(F \in \mathcal{F}\).

**Definition 3.4.2**

A monotone space \((X, c^*)\) is said to be compact, if every proper filter of sets on \(X\) has a cluster point in \(X\).

**Remark 3.4.3**

It is clear that if \((X, c^*)\) is compact, then \((X, c)\) is compact but the converse is not true.

**Result 3.4.4**

Any image under a \(c-c^*\) morphism of a compact monotone space \((X, c^*)\) onto a monotone space \((Y, c^{'})\) is compact.

The proof is similar to the Proof of 41 A-15 in [CE₂].

**Result 3.4.5**

Every closed subspace of a compact monotone space is compact.

The proof is similar to the Proof of 41 A-10 in [CE₂].
Result 3.4.6

If \((Y, c')\) is a compact subspace of a Hausdorff monotone space \((X, c_\ast)\), then \(Y\) is closed in \(X\).

The proof is similar to the Proof of 41 A-II in \([CE_2]\).

Definition 3.4.7

A monotone space \((X, c_\ast)\) is said to be disconnected if it can be written as two disjoint nonempty subsets \(A\) and \(B\) such that \(c_\ast A \cup c_\ast B = X\), \(c_\ast A \cap c_\ast B = \emptyset\). A space which is not disconnected is said to be connected.

Remark 3.4.8

\((X, c)\) is disconnected implies \((X, c_\ast)\) is disconnected, and the converse is not true.

Example 3.4.9

\[ X = \{a, b, c\} \]

\(c_\ast\) be defined on \(X\) such that

\[ c_\ast\{a\} = \{a\}, c_\ast\{b\} = \{b, c\}, c_\ast\{c\} = \{b, c\}, c_\ast\{a, b\} = c_\ast\{b, c\} = c\{a, c\} = c_\ast X = X, c_\ast \emptyset = \emptyset \]

(2) The mapping \(x \mapsto x_\ast\): \((G, u) \rightarrow (G, u)\) is "continuous"

\(c_\ast\) is a monotone operator.

\((X, c_\ast)\) is disconnected. But \((X, c)\) is connected.