EUROPEAN OPTION AND AMERICAN OPTION

3.1. Introduction: Black-Scholes [18] initiated the modelling of some equity derivative by parabolic differential equations with appropriate boundary conditions. Options are certain kind of contracts, many of them have been named as European, American, Asian and Russian, but they have no meaning with the continent of origin, rather they refer to a technically in the option contract. Here we discuss briefly the modelling of the European and American options depending on history of asset price. The definition of each of the terms, underlying asset, equity, derivative, expiry, option, call option, put option, strike price or exercise price etc. can be found in chapter 1.

3.2 EUROPEAN OPTION: After derived the Black-Scholes equation for the value of an option we consider final and boundary condition, for otherwise the partial differential equation does not have a unique solution. For European call, with value denoted by $C(S,t)$, with exercise price $E$ and expiry date $T$.

The final condition, to be applied at $t = T$ the value of call is known
with certainty to be the payoff

\[ C(S, T) = \text{Max} (S - E, 0) \]  

(3.2.1)

This is the final condition for our partial differential equation.

*Our asset price boundary condition are applied at zero asset price, \( S = 0 \) and as \( S \rightarrow \infty \). We can see from the Stochastic differential equation i.e.*

\[ \frac{dS}{S} = \sigma dX + \mu dt \]

that is \( S \) is ever zero then \( dS \) is also zero and therefore \( S \) can never change. This is the only deterministic case of the Stochastic differential equation. If \( S = 0 \) at expiry the payoff is zero. Thus the call option is worthless on \( S = 0 \) even if there is a large time to expiry. Hence on \( S = 0 \) we have

\[ C(0, t) = 0. \]  

(3.2.2)

As the asset price increase without bound it becomes ever more likely that the option will be exercised and the magnitude of the exercise price becomes less and less important. Thus as \( S \rightarrow \infty \) the value of the option becomes that of the asset and we write

\[ C(S, t) \sim S \text{ as } S \rightarrow \infty \]  

(3.2.3)

For a put option, with value \( P(S, t) \) the final condition is the payoff

\[ P(S, t) = \text{Max} (E - S, 0) \]  

(3.2.4)
We have already mentioned that if \( S \) is ever zero then it must remain zero. The final payoff for a put is known with certainty to be \( E \). To determine \( P(0, t) \) we simply have to calculate the present value of an amount \( E \) received at time \( T \). Assuming that interest rate are constant we find the boundary condition at \( S = 0 \) to be

\[
P(0, t) = Ee^{-r(T-t)}
\]

(3.2.5)

More generally, for a time dependent interest rate we have

\[
P(0, t) = Ee^{-\int_0^t r(\tau)d\tau}
\]

As \( S \rightarrow \infty \) the option is unlikely to be exercised and so

\[
P(S, t) \rightarrow 0 \text{ as } S \rightarrow \infty
\]

(3.2.6)

### 3.3 EUROPEAN CALL AND PUT OPTION:

Here we solve the exact solution of the European call option and put option problem when the interest rate and volatility are constant. The Black-Scholes equation and boundary condition for a European call with value \( C(S, t) \) are

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0.
\]

(3.3.1)

with

\[
C(0, t) = 0, \quad C(S, t) \sim S \text{ as } S \rightarrow \infty
\]

and

\[
C(S, t) = \text{Max} (S - E, 0)
\]

equation (3.3.1) looks a little the diffusion equation but it has more terms and each time \( C \) is differentiated with respect to \( S \) it is multiplied by \( S \),
giving non constant coefficients, also the equation is clearly backward form, with final date given at \( t = T \).

The first thing to do is to get rid to the awkward \( S \) and \( S^2 \) terms multiplying \( \frac{\partial C}{\partial S} \) and \( \frac{\partial^2 C}{\partial S^2} \). At the same time we take the opportunity of making the equation dimensionless and we turn it into a forward equation. We set

\[
S = Ee^x, \quad t = T - \frac{\tau}{\frac{1}{2} \sigma^2}, \quad C = Ev(x, \tau).
\]

This results in the equation

\[
\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k_1 - 1) \frac{\partial v}{\partial x} - k_1 v. \tag{3.3.2}
\]

Where \( k_1 = \frac{\tau}{\frac{1}{2} \sigma^2} \). The initial condition becomes

\[
v(x, 0) = \text{Max}(e^x - 1, 0).
\]

Notice in particular that this equation contains only one parameter \( k_1 \). The only essential factor controlling the option value is \( \frac{\tau}{\frac{1}{2} \sigma^2} \), which is the only dimensionless parameter.

Equation (3.3.2) now looks much more like a diffusion equation and we can turn it into one by a simple change of variable. If we try putting

\[
v = e^{\alpha x + \beta \tau} u(x, \tau)
\]

for some constant \( \alpha \) and \( \beta \) to be found, then differentiation gives

\[
\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k_1 - 1)(\alpha u + \frac{\partial u}{\partial x}) - k_1 v. \tag{3.3.3}
\]
We can obtain an equation with no \( u \) term by choosing
\[
\beta = \alpha^2 + (k_1 - 1)\alpha - k_1.
\]
while the choice
\[
0 = 2\alpha + (k_1 - 1)
\]
eliminates the \( \frac{\partial u}{\partial x} \) term as well as. These equation for \( \alpha \) and \( \beta \) give
\[
\alpha = \frac{-1}{2}(k_1 - 1)
\]
\[
\beta = \frac{-1}{4}(k_1 + 1)
\]
We then have
\[
v = e^{\frac{1}{4}(k_1-1)x-\frac{1}{4}(k_1+1)^2\tau} u(x, \tau)
\]
(3.3.4)

Where
\[
\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \text{ for } -\infty < x < \infty, \tau > 0
\]
with
\[
u(x,0) = u_0(x) = \max(e^{\frac{1}{2}(k_1+1)x} - e^{\frac{1}{2}(k_1-1)x}, 0)
\]
(3.3.5)
The solution of the diffusion equation problem is
\[
u(x, \tau) = \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{\infty} u_0(S) e^{\frac{-S^2}{4\tau}} dS
\]
(3.3.6)
where \( u_0(x) \) is given by (3.3.5)

It remains to evaluate the integral in (3.3.6). It is convenient to make
the change of variable \( x' = \frac{(x-S)^2}{\sqrt{\alpha}} \) so that
\[
u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(x'\sqrt{2\tau} + x) e^{\frac{-1}{2}x'^2} dx'
\]
$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2}(k_1+1)(x+x'\sqrt{2\tau})} e^{\frac{1}{2}x'^2} dx' - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2}(k_1+1)(x+x'\sqrt{2\tau})} e^{\frac{1}{2}x'^2} dx'$$

$$u(x, \tau) = I_1 - I_2 \text{ (say).}$$

Here

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2}(k_1+1)(x+x'\sqrt{2\tau})} e^{\frac{1}{2}x'^2} dx'$$

$$= e^{\frac{1}{2}(k_1+1)x} \int_{-\infty}^{\infty} e^{\frac{1}{2}(k_1+1)^2\tau} e^{\frac{1}{2}(x'-\frac{1}{2}(k_1+1)\sqrt{2\tau})^2} dx'$$

$$= e^{\frac{1}{2}(k_1+1)x+\frac{1}{4}(k_1+1)^2\tau} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2 \rho^2} d\rho$$

$$= e^{\frac{1}{2}(k_1+1)x+\frac{1}{4}(k_1+1)^2\tau} N(d_1)$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k_1+1)\sqrt{2\tau}$$

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{\frac{1}{2}S^2} dS$$

is the cumulative distribution function for the normal distribution. The calculation of $I_2$ is identical to that of $I_1$, except that $(k_1 + 1)$ is replaced by $(k_1 - 1)$ throughout. Lastly, we retrace our steps, writing

$$v(x, \tau) = e^{\frac{1}{2}(k_1-1)x-\frac{1}{4}(k_1+1)^2\tau} u(x, \tau)$$

and then putting $x = \log \left( \frac{S}{E} \right)$, $\tau = \frac{1}{2}\sigma^2(T-t)$ and $C = Ev(x, \tau)$, to recover

$$C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$
where

\[ d_1 = \frac{\log\left(\frac{S}{F}\right) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}} \]

\[ d_2 = \frac{\log\left(\frac{S}{F}\right) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}} \]

The corresponding calculation for a European put option follows similar lines, but having evaluated the call, a simpler way is to use the put-call parity formula

\[ C - P = S - E e^{-r(T-t)} \]

i.e.

\[ P(S, t) = E e^{-r(T-t)} N(d_2) - S N(d_1) \]

3.4 AMERICAN OPTION: An American option has the additional feature that exercise is permitted at any time during the life of the option. American option gives its holder greater rights than the European option via the right to early exercise, potentially it has a higher value.

We consider two valuation problems for American type options as free boundary problems. The first is the American put problem and second is an American call option on an asset which pays dividends at a continuous rate.

The dividing price between exercise and non-exercise is called the optimal exercise price \( S_f(t) \). It depends on the time remaining to expiry as well as the other parameters of the problem such as volatility.
3.4.1 **AMERICAN PUT OPTION:** Consider the Black-Scholes partial differential equation for the valuation of a European put,

\[ \frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0. \]  

(3.4.1)

with payoff

\[ P(S, T) = \text{Max}(E - S, 0) \]  

(3.4.2)

and boundary condition

\[ P(0, t) = E e^{-r(T-t)}, \quad P(S, t) \to 0 \text{ as } S \to \infty \]  

(3.4.3)

As is well-known, the value of the European put falls below its intrinsic value for some value of \( S \). This is easily seen by considering the value of the put option at \( S = 0 \). Here the intrinsic value of the option is \( E \) but, from the boundary condition (3.4.3)

\[ P(0, t) = E e^{-r(T-t)} \leq E \]

Thus the value of the option is less than its intrinsic value for \( t < T \). If the American put option were valued according to the European put option formula then there would be arbitrage possibilities. The absence of these means that we must impose the condition

\[ P(S, T) \geq \text{Max}(E - S, 0) \]  

(3.4.4)

for the American put. Also an American option does not satisfy an equality but an inequality. For an American option instead of (3.4.1) we have

\[ \frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP \leq 0. \]
We know that a free boundary must exist since the European put option formula does not satisfy the constraint (3.4.4). Suppose further that

\[ P = E - S \text{ for some } S < E. \]

If this is the case then \( P \) most certainly does not satisfy the Black-Scholes equation (unless \( r = 0 \)) since

\[ \frac{\partial}{\partial t}(E - S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}(E - S) + rS \frac{\partial}{\partial S}(E - S) - r(E - S) = -rE < 0. \]

But \( P \) does not satisfy the inequality. When \( P = E - S \) the return from the portfolio is less than the return from an equivalent bank deposit, and hence it is optimal to exercise the option. At any given time \( t \), we must divide the \( S \) axis into two distinct regions, one where early exercise is optimal and

\[ P = E - S, \quad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP < 0. \]

and the other where early exercise is not optimal and

\[ P > E - S, \quad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0. \]

3.4.2 AMERICAN CALL OPTION: We consider some analytical aspects of the model for an American call option on a dividend paying asset, the value \( C(S, t) \) of the call satisfies

\[
\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0)S \frac{\partial C}{\partial S} - rC = 0. \tag{3.4.5}
\]

So long as exercise is not optimal. The payoff condition is

\[ C(S, t) = \text{Max}(S - E, 0) \tag{3.4.6} \]
and because the option can be exercised at any time, we always have

\[ C(S, t) \geq \text{Max}(S - E, 0). \]  \hspace{1cm} (3.4.7)

If there is an optimal exercise boundary \( S = S_f(t) \) then at \( S = S_f(t) \)

\[ C(S_f(t), t) = S_f(t) - E, \quad \frac{\partial C}{\partial S}(S_f(t), t) = 1. \] \hspace{1cm} (3.4.8)

If an optimal exercise boundary does exist, then (3.4.5) is only valid while

\[ C(S, t) > \text{Max}(S - E, 0) \]