CHAPTER-2

BLACK-SCHOLES MODEL

2.1. Introduction: In a research paper, Fischer Black and Myron Scholes [18] developed a precise model for determining the equilibrium value of an option. They then went on to observe that the option pricing concepts can be used to value other contingent claim. In particular, the model provides rich insight into the valuation of debt relative to equity. The Black-Scholes model has been extended and refined in major ways and new applications are continually unfolding. The model has both theoretical importance for valuing contingent claims and practical importance for identifying overvalued and undervalued options in the market.

2.2 BLACK-SCHOLES MODEL: A number of assumptions are in order before we can discuss the Black-Scholes model:

1. The rates of return on a share are lognormally distributed.
2. The value of the share (The underlying asset) and the risk free rate are constant during the life of the option.
3. The market is efficient and there are no transaction costs and taxes.
4. There is no dividend to be paid on the share during the life of the option.
5. There are no arbitrage possibilities. The absence of arbitrage opportunities means that all risk free portfolios must earn the same return.

6. Trading of the underlying asset can take place continuously.

7. Short selling is permitted and the asset are divisible.

Now suppose that $V$ is the value of an option and a function of the current value of underlying asset $S$ and time $t$ i.e. $V = V(S, t)$

$C(S, t)$ : denote a call option

$P(S, t)$ : denote a put option

The value of an option also depends on the following parameters : 

$\sigma$ : The volatility of the underlying asset

$E$ : The exercise price of the option

$T$ : The expiry time

$r$ : The short term annual interest rate continuously compounded

Suppose that we have an option whose value $V(S, t)$ depends only on $S$ and $t$. It is not necessary at this stage to specify whether $V$ is call or put; $V$ can be the value of a whole portfolio of different options although for simplicity we can think of a simple call or put. Using Ito's lemma

$$df = \sigma S \frac{\partial f}{\partial S} dx + (\mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}) dt. \quad (2.2.1)$$

We can write

$$dV = \sigma S \frac{\partial V}{\partial S} dx + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}) dt. \quad (2.2.2)$$
This gives the random walk followed by $V$. Note that we required $V$ to have at least one $t$ derivative and two $S$ derivatives.

Now construct a portfolio of one option and a number $-\Delta$ of the underlying asset. This number is as yet unspecified. The value of this portfolio is

$$\Pi = V - \Delta S$$  \hspace{1cm} (2.2.3)

The jump in the value of this portfolio is one step is

$$d\Pi = dV - \Delta dS$$  \hspace{1cm} (2.2.4)

Here $\Delta$ is held fixed during the time step. If it were not then $d\Pi$ would contain term is $dS$. To find $\Pi$ follows the the random walk from (2.2.4) i.e. $d\Pi = dV - \Delta dS$, from (2.2.2)

$$d\Pi = \sigma S \frac{\partial V}{\partial S} dx + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}) dt - \Delta dS.$$  \hspace{1cm} (2.2.5)

We also know that

$$\frac{dS}{S} = \sigma dX + \mu dt$$

$$dS = \sigma S dX + \mu S dt$$

(2.2.5) becomes

$$d\Pi = \sigma S \frac{\partial V}{\partial S} dx + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}) dt - \Delta (\sigma S dX + \mu S dt).$$

$$d\Pi = \sigma S \frac{\partial V}{\partial S} dx + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}) dt - \Delta \sigma S dX - \Delta \mu S dt.$$  \hspace{1cm} (2.2.6)

$$d\Pi = \sigma S (\frac{\partial V}{\partial S} - \Delta) dx + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \Delta \mu) dt.$$
Now we can eliminate the random component in this random by choos­ing
\[ \Delta = \frac{\partial V}{\partial S} \] (2.2.7)
Note that \( \Delta \) is the value of \( \frac{\partial V}{\partial S} \) at the start of the time step \( dt \).

This results in a portfolio whose increment is wholly deterministic:
\[ d\Pi = \sigma S \left( \frac{\partial V}{\partial S} - \frac{\partial V}{\partial S} \right) dx + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - S \mu \frac{\partial V}{\partial S} \right) dt. \]
\[ d\Pi = \left( \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \] (2.2.8)

We now appeal to the concepts of arbitrage and supply and demand, with the assumption of no transaction costs. The return on an amount \( \Pi \) invested in riskless assets would see a growth of \( r \Pi dt \) in a time \( dt \). If the right hand side of (2.2.8) were greater than this amount; an arbitrage could make a guaranted riskless profit, by borrowing an amount \( \Pi \) to invest in the portfolio. The return of this strategy would be greater than the cost of borrowing conversly, if the right hand side of (2.2.8) were less than \( r \Pi dt \) then the arbitrage would short the portfolio and invest \( \Pi \) in the bank. Either way the arbitrager would make a riskless, no cost, instantaneous profit. The existence of such arbitragers with the ability to trade at low cost ensures that the return on the portfolio and on the riskless account are more or less equal. Thus we have
\[ r \Pi dt = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \] (2.2.9)
From (2.2.3) and (2.2.7) we get

\[ \Pi = V - \frac{\partial V}{\partial S} S. \]  

(2.2.10)

Put the value of \( \Pi \) from (2.2.10) in (2.2.9) we get

\[ r(V - S \frac{\partial V}{\partial S})dt = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right)dt \]

\[ (rV - rS \frac{\partial V}{\partial S})dt = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right)dt. \]  

(2.2.11)

Dividing (2.2.11) by \( dt \) we get

\[ rV - rS \frac{\partial V}{\partial S} = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \]

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \]  

(2.2.12)

This is the Black-Scholes partial differential equation.

It is hard to overemphasise the fact that, under the assumption that any derivative security whose price depends only on the current value of \( S \) and on \( t \) and which is paid for up front, must satisfy the Black-Scholes equations. It is also important to note, though that many options, for example American options, have values that depend on the history of the asset price as well as its present value. There are three remarks about the derivations.

Firstly, the delta given by

\[ \Delta = \frac{\partial V}{\partial S} \]

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the rate of change of the value of our option or portfolio of options with respect to $S$. It is of fundamental importance in both theory and practice, and we return to it repeatedly. It is the measure of the correlation between the movements of the option or options and those of the underlying asset.

Secondly the linear differential operator $L_{BS}$ given by

$$L_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS \frac{\partial}{\partial S} - r$$

has a financial interpretation as a measure of the difference between the return on a hedged option portfolio (the first two terms) and the return on a bank deposit (the last two terms). Although this return is identically zero for a European option and need not be so for an American option.

Thirdly we note that the Black-Scholes equation (2.2.12) does not contain the growth parameter $\mu$. In other words, the value of an option is independent of how rapidly or slowly an asset grows. The only parameter from the stochastic differential equation

$$\frac{dS}{S} = \sigma dX + \mu dt$$

for the asset price that affects the option price is the volatility $\sigma$. A consequence of this is that two people may differ in their estimates for $\mu$: yet still agree on the value of an option.

Black-Scholes equation is partial differential equation. By deriving the partial differential equation for a quantity such as an option price, we have
made an enormous step towards finding its value. However, a partial differential equation on its own generally has many solutions. The value of an option should be unique (otherwise arbitrage possibilities would arise) and so to pin down the solution, we must also impose boundary conditions.

The commonest type of partial differential equation in financial problem is the parabolic equation. In its simplest form a parabolic equation relates the partial derivatives of a function $V(S, t)$ say the highest derivative with respect to $S$ must be a second derivative and the highest derivative with respect to $t$ must only be first derivative.

The Black-Scholes option pricing model provides an exact formula for determining the value of an option based on the volatility of the stock. The price of the stock, the exercise price of the option, the time of expiration of the option, and the short term interest rate are calculated from Black-Scholes model. The model is based on the notion that investors are able to maintain reasonably hedged position over time and that arbitrage will derive the return on such position to the risk free rate. As a result, the option price will bear a precise relationship to the stock price. The Black-Scholes model provides considerable insight into the valuation of contingent claims.