CHAPTER-5

MODELLING OF AMERICAN OPTIONS THROUGH VARIATIONAL INEQUALITY

5.1. Introduction: A study of option pricing depending on history of the asset price and caring for inflation and devaluation is proposed. Black and Scholes [18] initiated the modelling of some equity derivatives by the parabolic partial differential equation with appropriate boundary conditions.

Wilmott, Dewynne and Howison [69] have extensively studied the applications of numerical methods in problems of bank and finance. Nowadays the main priority of financial institution is to manage risk instead of dealing with cash and securities. In view of the unparalleled growth of financial derivatives in the last two decades, the proper modelling and studies of inter-element relationship is a challenging problem. The main task before successful financial institution is to understand these instruments and to develop risk-free strategies to yield maximum benefit.

5.2 BASIC CONCEPTS RELATED TO EUROPEAN AND AMERICAN OPTIONS: Before introducing European and American options, we briefly mention the commonly used terms like asset or underlying asset, equity, derivative, equity derivative, expiry, option pricing, call
option, put option, strike price (exercise price), risk management, volatility.

By underlying asset, often called underlying or asset, we mean commodity, exchange, shares, stocks and bonds etc. Equity is a share in the ownership of a company which usually guarantees the right to vote at meetings and a share in the dividends (payment to shareholders as return for investment in the co-operation). Derivatives refers to either a contract or a security whose payoff or final value is dependent on one or more features of the underlying equity. In many cases it is the price of the underlying equity which determines to a large extent the value of the equity derivative or derivative based on equity, although other factors like interest rates, time to maturity and strike price can also play a significant role. The termination time of derivatives contracts, usually when the final pay-off value is calculated and paid, is called expiry. Option pricing or options are some kind of contracts. The right to the holder (owner) and an obligation to the seller (writer) of a contract either to buy or to sell an underlying asset at a fixed price for a premium. In call options the holder has the right buy not the obligation to buy the underlying asset at the strike price. Options in which the right to sell for the holder and the obligation to buy for the writer at a strike price $E$ for the payment of a premium is guaranteed, are called put options. Strike price or exercise price is the price at which the underlying asset is bought in options. Risk management is the process of establishing the type and magnitude of risk in a business enterprise and using derivatives to control and shape that risk to maximize the business objective. Volatility is a measure of the standard deviation of returns. In
practice it is understood as the average daily range of the last few weeks or average absolute value of the daily net change of the last few weeks.

5.3 MODELLING OF EUROPEAN OPTIONS: A European call option is a contract with the following conditions: At a prescribed time in the future, known as the expiry date, the owner of the option may purchase a prescribed asset, called underlying asset, for a prescribed amount (strike price or exercise price). Similarly, a European put option is a contract in which at a prescribed time in the future the owner (holder) of the option may sell an asset for a prescribed amount.

Let $V(S,t)$ denote the value of an option which is the function of the underlying asset $S$ at time $t$. Black and Scholes [18] proved that $V$ is a solution of the parabolic partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \tag{5.3.1}$$

Where $\sigma$ and $r$ are volatility and interest rate, respectively.

Let $C(S,t)$ and $P(S,t)$ denote the value of a call option and put option. We know that a European call option $C(S,t)$ is a solution of the following boundary value problem:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0. \tag{5.3.2}$$

$$C(S,t) = \text{Max}(S - E, 0) \tag{5.3.3}$$

$$C(0,t) = 0 \tag{5.3.4}$$
where $S, \sigma, r$ are as above, and $E$ and $t$ are the exercise price and expiry time respectively.

On the other hand, a European put option $P(S, t)$ is a solution of the following boundary value problem:

\[ \frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0. \]  

\[ P(S, t) = \max (E - S, 0) \]  

\[ P(0, t) = Ee^{-r(t-t)} \]  

If $r$ is independent of time

\[ P(0, t) = E e^{-\int_{0}^{t} r(t) \, dt} \]  

If $r$ is time dependent

As $S \to \infty$, the option is unlikely to be exercised and so

\[ P(S, t) \to 0 \text{ as } S \to \infty \]  

Equation (5.3.2) - (5.3.5) and (5.3.6) - (5.3.10) are known as Black-Scholes model for call and put options respectively.

The Black-Scholes call option model can be transformed into the diffusion equation

\[ \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \text{ for } -\infty < x < \infty, \tau > 0 \]  

\[ C(S, t) \to S \text{ as } S \to \infty \]  

\[ \frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0. \]  

\[ P(S, t) = \max (E - S, 0) \]  

\[ P(0, t) = Ee^{-r(t-t)} \]  

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\[ C(S, t) \to S \text{ as } S \to \infty \]
with
\[ u(x, 0) = \text{Max}(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0) \] (5.3.12)

by putting \( S = Ee^x, t = T - \frac{T}{\sigma^2} \) and
\[ C(S, t) = Ee^{\frac{1}{2}(K-1)x - \frac{1}{2}(k+1)^2 \tau} \tilde{u}(x, \tau) \]

where \( k = \frac{1}{\sigma^2} \).

The Black-Scholes put option model can analogously be written in the form of the diffusion equation.

5.4 MODELLING OF AMERICAN OPTIONS: American options are those options which can be exercised by any time prior to expiry time. American call and put options are related to buying and selling respectively. The valuation of American options leads to a free boundary problem. Typically, at each time \( T \) there is a valuation of \( S \) which marks the boundary between two regions, namely to one side one should hold the option and to the other side one should exercise it. Let us denote this boundary by \( S_f(t) \) (generally, this critical asset value varies with time). Since we do not know \( S_f(t) \) a priori, we are lacking one piece of information compared with the corresponding European option problem. Thus with American options we do not know a priori where to apply boundary conditions. This situations resembles the obstacle problem and can be effectively tackled by methods of variational inequalities. Wilmott, Dewynne and Howison [69] have shown that American call option and American put option can be formulated as the following boundary value problems and equivalent variational inequalities.
American call option is modelled by the following boundary value problem

\[
\begin{align*}
\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} & \geq 0 \\
u(x, \tau) - g(x, \tau) & \geq 0 \\
(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}) (u(x, \tau) - g(x, \tau)) & = 0 \\
u(x, 0) & = g(x, 0) \\
u(a, \tau) & = g(a, \tau) = 0 \\
u(b, \tau) & = g(b, \tau)
\end{align*}
\]

where

\[g(x, \tau) = e^{\frac{1}{2}(k+1)^2 \tau} \max(e^{\frac{1}{2}(k+1)^2 x} - e^{\frac{1}{2}(k-1)^2 x}, 0)\]

The financial variables \(S, t\) and the option value \(C\) are again computed by putting

\[S = E e^x, \quad t = T - \frac{\tau}{\frac{1}{2} \sigma^2}\]

and

\[C(S, t) = E e^{\frac{1}{2}(k-1)x - \frac{1}{2}(k+1)^2 \tau} u(x, \tau)\]

In order to avoid technical complications, the problem is restricted to a finite interval \((a, b)\) with \(a\) and \(b\) large enough. In financial terms, we assume that we can replace the exact boundary conditions by the approximation that for small values of \(S, P = E - S\), while for large value, \(P = 0\).
Let us denote by \( u_r \) the function \( x \rightarrow u(x, \tau) \). The equivalent parabolic variational inequality is as follows:

Find \( u = u_r \in k_r(\tau \text{ runs over } [0, \frac{T}{2\sigma^2}]) \) such that

\[
\left( \frac{\partial u}{\partial \tau}, \varphi - u \right)_2 + a(u, \varphi - u) \geq 0
\]

(5.4.8)

for all \( \varphi \in K_r, \text{ a.e. } \tau \in (0, \frac{1}{2}\sigma^2 T) \),

\[
u(x, 0) = g(x, 0)
\]

where

\[
k_r : \{ v \in H'(a, b) \mid v(a) = g(a, \tau), v(b) = g(b, \tau), v(x) \geq g(x, \tau) \}
\]

and \((\cdot, \cdot)_2\) denotes the inner product on \( L^2(a, b) \) with

\[
W(0, \frac{1}{2}\sigma^2 T) = \{ v \mid v \in L^2(0, \frac{1}{2}\sigma^2 T; H'(a, b)), \frac{\partial u}{\partial \tau} \in L^2(0, \frac{1}{2}\sigma^2 T; H'(a, b)) \}
\]

\[
W_0(0, \frac{1}{2}\sigma^2 T) = \{ v \mid v \in W(0, \frac{1}{2}\sigma^2 T), v(0) = g(\cdot, 0) \}
\]

and

\[
K = \{ v \mid v \in W(0, \frac{1}{2}\sigma^2 T), v_r \in K_r \text{ for a. e. } \tau \in [0, \frac{1}{2}\sigma^2 T] \}
\]

\[
K_0 = \{ v \mid v \in W_0(0, \frac{1}{2}\sigma^2 T), v_r \in K_r \text{ for a. e. } \tau \in [0, \frac{1}{2}\sigma^2 T] \}
\]

we can formulate an equivalent variational inequality:

Find \( u \in K_0 \) such that

\[
\int_0^{\frac{1}{2}\sigma^2 T} \left( \frac{\partial u}{\partial \tau}, \varphi - u \right)_2 \, d\tau + \int_0^{\frac{1}{2}\sigma^2 T} a(u, \varphi - u) \, d\tau \geq 0
\]

(5.4.9)
for all $\varphi \in K$.

**5.5 AMERICAN PUT OPTION:** American put option is modelled by a boundary value problem that only differs in the boundary conditions and the (transformed) pay-off function $g$:

\[
\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0
\]

(5.5.1)

\[
u(x, \tau) - g(x, \tau) \geq 0
\]

(5.5.2)

\[
(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2})(u(x, \tau) - g(x, \tau)) = 0
\]

(5.5.3)

\[
u(x, 0) = g(x, 0)
\]

(5.5.4)

\[
u(a, \tau) = g(a, \tau)
\]

(5.5.5)

\[
u(b, \tau) = g(b, \tau) = 0
\]

(5.5.6)

where

\[
g(x, \tau) = e^{\frac{1}{2}(k-1)^2 \tau} \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0)
\]

(5.5.7)

The equivalent variational inequality is formulated to (5.4.8) or (5.4.9) with the only change in the boundary conditions and the function $g$.

It may be remarked that in American call option $C(S, t)$ lies above the payoff $\max(S - E, 0)$; in the transformed variables, this conditions takes the form $u(x, \tau) - g(x, \tau) \geq 0$.

The condition $\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0$ means that the return from the risk delta-hedged portfolio is less than the risk free interest rate $r$. 

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References


